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Locally Increasing Points of Nowhere Differentiable Functions

1 Introduction

Let H_f denote the set of points $x \in [0,1]$ for which $f \in C[0,1]$ is locally increasing at x. It is easy to see that H_f is an F_σ set for any $f \in C[0,1]$. Since $\underline{D}f(x)$ (the lower derivative of f at x, see [2]) is non-negative for every $x\ in H_f$, it follows from the Denjoy-Young-Saks Theorem that f is differentiable at a.e. point of H_f. In particular, if f is nowhere differentiable, then H_f is a set of measure zero. In this paper we prove that if $H \subset [0,1]$ is an arbitrary F_σ set of measure 0, then there exists a nowhere differentiable function $f \in C[0,1]$ such that $x \in H$ iff f is locally increasing at x.

2 Preliminaries

We denote by $\underline{D}\varphi$ ($\overline{D}\varphi$) the lower (upper) derivative of φ at x and we put $D_+\varphi(x)$ ($D^+\varphi(x)$) for the right hand lower (upper) derivative of φ at x. We define $D_-\varphi(x)$, $D^-\varphi(x)$ similarly. Lebesgue measure on the interval $[0,1]$ is denoted by m.

Now we need the following lemmas.

Lemma 1 *Let $f,g \in C[0,1]$, $f \leq g$ and let $F \subset [0,1]$ be a nonempty, nowhere dense, closed set such that $f(x) < g(x)$ for every $x \in F$. Then there exist functions $u,v \in C[0,1]$ such that:*

1. *$f \leq u \leq v \leq g$,*
2. *$u(x) = v(x)$, and $u'(x) = v'(x) = 0$ for every $x \in F$,*
3. *if $f(x) < g(x)$ and $x \notin F$, then $u(x) < v(x)$.*

Proof. Define the set $P = \{x : f(x) = g(x)\}$. We will define u and v so that $u|_P = v|_P = f|_P$. Let (a,b) be an arbitrary interval contiguous to P. Let $\{x_i\}_{i=-\infty}^{\infty} \subset \Re$, $x_i < x_{i-1}$ be a sequence such that:

(i) $\lim_{i\to-\infty} x_i = a$, and $\lim_{i\to\infty} = b$,

(ii) $\max\{f(x) : x \in [x_{i-1}, x_i]\} < \min\{g(x) : x \in [x_{i-1}, x_i]\}$,

(iii) for each i, $x_i \notin F$.

Put $\max\{f(x) : x \in c_i < \min\{g(x) : x \in [x_{i-1}, x_i]\}$, and choose y_i, z_i such that $F \cap [x_{i-1}, x_i] \subset [y_i, z_i] \subset (x_{i-1}, x_i)$. In addition select t_i, s_i so that $x_i < t_i < y_i < z_i < s_i < x_i$ and define

$$u(x) = c_i - \epsilon \operatorname{dist}^2(x, F)$$
$$v(x) = c_i + \epsilon \operatorname{dist}^2(x, F)$$

for every $x \in [t_i, s_i]$, where ϵ is chosen such that $f|_{[t_i,s_i]} < u|_{[t_i,s_i]} < v|_{[t_i,s_i]} < g|_{[t_i,s_i]}$. We can now extend u, v so that they are continuous and satisfy the requirements.

Lemma 2 *Let $u, v \in C[0,1]$, $u \leq v$, let $H \subset [0,1]$ be a set of the first category, and let $n \in N$ be fixed. Then there exist $f, g \in C[0,1]$ such that:*

1. *$u \leq f \leq g \leq v$, $g \leq f + \frac{1}{n}$.*
2. *For every x with $u(x) < v(x)$ and for every $h \in C[0,1]$ with $f \leq h \leq g$ there exist $y, z \in \Re$ such that $|y - x| < \frac{1}{n}$, $|z - x| < \frac{1}{n}$ and $\frac{h(z)-h(x)}{z-x} > n$, $\frac{h(y)-h(x)}{y-x} < -n$.*
3. *If $x \in H$ and $u(x) < v(x)$, then $f(x) < g(x)$.*

Proof. Let $P = \{x : u(x) = v(x)\}$. We define $f(x) = g(x) = u(x)$ for every $x \in P$. Let (a, b) be an arbitrary interval contiguous to P. Select a monotone increasing sequence $\{x_i\}_{i=-\infty}^{\infty} \subset (a, b) \setminus H$ such that:

(a) $\lim_{i\to-\infty} x_i = a$, and $\lim_{i\to\infty} = b$,

(b) $0 < x_{i+3} - x_i < \frac{1}{n}$, $\frac{v(x_{i+3})-u(x_i)}{x_{i+3}-x_i} > n$, $\frac{u(x_{i+3})-v(x_i)}{x_{i+3}-x_i} < -n$ for every $i \in N$.

We put

$$f(x_i) = g(x_i) = u(x_i) \text{ if } i \text{ is even}$$
$$f(x_i) = g(x_i) = v(x_i) \text{ if } i \text{ is odd },$$

and we define f, g on (a, b) such that $u \leq f \leq g \leq f + \frac{1}{n} \leq v$, f and g are continuous and $f(x) < g(x)$ for every $x \in (a, b) \setminus \{x_i\}_{i=-\infty}^{\infty}$. We prove that f and g satisfy the requirements. Properties 1 and 3 are obviously fulfiled. To prove 2 let x be such that $u(x) < v(x)$. Suppose $x \in (a, b)$, where (a, b) is an interval contiguous to P and let $x_i < x < x_{i+1}$. Suppose that i is even. Since $\frac{v(x_{i+3})-u(x_i)}{x_{i+3}-x_i} > n$, it follows that for every function h satisfying $h(x_j) = f(x_j) = g(x_j)$ for every j, we have either $\frac{h(x_{i+3})-h(x)}{x_{i+3}-x} > n$, or $\frac{h(x_i)-h(x)}{x_i-x} > n$. Similarly since $\frac{u(x_{i+2})-v(x_{i-1})}{x_{i+2}-x_{i-1}} < -n$, we have either $\frac{h(x_{i+2})-h(x)}{x_{i+2}-x} < -n$ or $\frac{h(x_{i-1})-h(x)}{x_{i-1}-x} < -n$.

Lemma 3 *For each n let F_n be a nowhere dense, closed subset of $[0,1]$ with the property $F_n \subset F_{n+1}$. Put $H = \cup_{n=1}^{\infty} F_n$. Then there exist sequences of functions $\{f_n\}_{n=1}^{\infty}$, $\{g_n\}_{n=1}^{\infty} \subset C[0,1]$ such that for every $n \in N$*

1. $f_n \leq f_{n+1} \leq g_{n+1} \leq g_n$
2. $f_n(x) = g_n(x)$ *for every* $x \in F_n$
3. $f_n(x) < g_n(x) \leq f_n(x) + \frac{1}{n}$ *for every* $x \in (H \setminus F_n)$
4. $f_n'(x) = g_n'(x) = 0$ *for every* $x \in F_n$
5. *For each $x \notin F_n$ and for every $h \in C[0,1]$ with $f_n \leq h \leq g_n$ there exist y, z with $|y - x| < \frac{1}{n}$, $|z - x| < \frac{1}{n}$ such that $\frac{h(z)-h(x)}{z-x} > n$, and $\frac{h(y)-h(x)}{y-x} < n$.*

Proof. We define f_n, g_n by induction. Without loss of generality we can take

$$F_1 = \emptyset,\ f_1(x) = \left\{ \begin{array}{lll} 8x & \text{if} & x \in \left[0, \frac{1}{8}\right] \\ -8x+2 & \text{if} & x \in \left[\frac{1}{8}, \frac{3}{8}\right) \\ 8x-4 & \text{if} & x \in \left[\frac{3}{8}, \frac{5}{8}\right), \\ -8x+6 & \text{if} & x \in \left[\frac{5}{8}, \frac{7}{8}\right), \\ 8x-8 & \text{if} & x \in \left[\frac{7}{8}, 1\right] \end{array} \right\} \text{ and } g_1(x) = f_1(x) + \frac{1}{10}$$

Suppose that $n \in N$ and f_n, g_n fulfil the conditions 2 through 5. We can use Lemma 1 with $f = f_n$, $g = g_n$, $F = F_{n+1}$. Thus from Lemma 1 we get two functions u, v. Now we use Lemma 2 for these functions and for H, and we get f_{n+1}, g_{n+1} from Lemma 2 as f, g. It is easy to see that the sequence of functions f_n, g_n constructed in this was satisfy the requiremnets of Lemma 3.

3 Main Results

Theorem 1 *For every F_σ set of the first caregory $H \subset [0,1]$, there exists a function $h \in C[0,1]$ such that*

(i) $\underline{D}h(x) = -\infty$ *and* $\overline{D}h(x) = \infty$ *for every* $x \in [0,1] \setminus H$

(ii) $h'(x) = 0$ *for every* $x \in H$.

Proof. Let $\{f_n\}_{n=1}^{\infty}$, $\{g_n\}_{n=1}^{\infty}$ be as in Lemma 3. We define $h = \lim_{n\to\infty} f_n = \lim_{n\to\infty} g_n$. Let $x_0 \in [0,1] \setminus H$. Then for each $n \in N$, $x_0 \notin F_n$. Thus it follows from Lemma 3 that there are y, z such that $|y - x_0| < \frac{1}{n}$, $|z - x_0| < \frac{1}{n}$ and $\frac{h(z)-h(x_0)}{z-x_0} > n$, $\frac{h(y)-h(x_0)}{y-x_0} < -n$ because $f_n \leq h \leq g_n$. So we get (i).

We take $x_0 \in H$. Then there exists $n \in N$ such that $x_0 \in F_{n+1}$ but $x_0 \notin F_j$ if $j < n+1$. It follows from properties 2 and 4 that $h'(x_0) = 0$. So we get (ii).

Casper Goffman proved in [1] that if $H \subset [0,1]$ is a set of measure 0, then there exists a measurable set S whose metric density does not exist at any point of H. If for $x \in [0,1]$ we define $\varphi_H(x) = m(S \cap [0,x])$. Then $\varphi_H \in C[0,1]$, and

(i) $0 \leq \underline{D}\varphi_H(x) \leq \overline{D}\varphi_H(x) \leq 1$ for every $x \in [0,1]$

(ii) $\underline{D}\varphi_H(x) < \overline{D}\varphi_H(x)$ for every $x \in H$.

Using this function φ_H we will prove our main result.

Theorem 2 *For every $H \subset [0,1]$ the following are equivalent:*

(a) H is an F_σ set of measure zero

(b) There exists a function $f \in C[0,1]$ such that f does not have a finite or infinite derivative at any point and $H = H_f$ (the set of x for which f is locally increasing at x).

Proof. If $H \subset [0,1]$ is an F_σ set of measure zero, then let $f(x) = h(x) + \varphi_H(x) + x$ for every $x \in [0,1]$ where h was constructed in Theorem 1 and φ_H was constructed by C. Goffman in [1]. It is easy to see that:

1.
$$\underline{D}f(x) = -\infty, \ \overline{D}f(x) = \infty \text{ for every } x \in [0,1] \setminus H$$

2.
$$0 < \underline{D}f(x) < \overline{D}f(x) \text{ for every } x \in H.$$

Thus we have (a) $\Longrightarrow$ (b). As we saw in the introduction, (b) implies (a).

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References

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