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THE PACKING DIMENSION OF A TYPICAL CONTINUOUS FUNCTION IS 2

In this paper we consider typical behavior of continuous functions $f:[0,1] \to [0,1]$. Typical properties are derived by counting the number of squares of the regular nxn grid in $[0,1]^2$ which intersect the graph of f and dividing by a power, α, of n. The (necessarily unique) α for which the $\limsup_{n\to\infty}$ of this quotient is not infinite is a rarefaction index for f and we call this index the grid dimension of f. First we show that α is the packing dimension for f and that typically, a continuous function has a grid dimension, and hence a packing dimension of 2.

Let C denote the space of all continuous functions $f:[0,1] \to [0,1]$ with the sup norm. A property of continuous functions (e.g. nondifferentiability) is said to be typical if the set of functions enjoying this property is a residual subset of C. There has been a lively typical continuous function literature in the past decade and both [B] and [T] contain retinues of results and bibliographies. In the next several paragraphs we define three distinct notions of the dimension of a planar set and then apply these to graphs of functions in C. If

$\mathcal{U} = \{U_n \subseteq [0,1]\}$ we let $\cup\, \mathcal{U} = \bigcup_{n=1}^{\infty} U_n$.

For each $n \in \mathbb{N}$ we define

$$I^n_{m,k} = \begin{cases} [\frac{m-1}{n}, \frac{m}{n})\times[\frac{k-1}{n}, \frac{k}{n}); & 1 \le m,\ k < n \\ [\frac{n-1}{n}, 1]\times[\frac{k-1}{n}, \frac{k}{n}); & k < n \\ \\ [\frac{m-1}{n}, \frac{m}{n})\times[\frac{n-1}{n}, 1]; & m < n \\ [\frac{n-1}{n}, 1]\times[\frac{n-1}{n}, 1]; & m = k = n. \end{cases}$$

Then, we let $G_n \equiv \{I^n_{m,k} : 1 \le m,k \le n\}$ and refer to G_n as the regular grid of index n in $[0,1]^2$. If $E \subseteq [0,1]^2$ we denote by $N(E,n)$ the number of squares in G_n which intersect E. It is evident that $1 \le N(E,n) \le n^2$ for $E \ne \emptyset$ and hence for α small enough,

$$\limsup_{n\to\infty} \frac{N(E,n)}{n^\alpha} = +\infty. \tag{1}$$

We define the grid dimension of E to be

$$\alpha(E) = \sup\{\alpha : \limsup_{n\to\infty} \frac{N(E,n)}{n^\alpha} = \infty\}. \tag{2}$$

If ν is any increasing sequence of natural numbers, we define the grid dimension of E relative to ν to be

$$\alpha_\nu(E) = \sup\{\alpha : \limsup_{n\to\infty} \frac{N(E,\nu(n))}{\nu(n)^\alpha} = \infty\}. \tag{3}$$

Obviously, $\alpha(E) \geqq 0$, $\alpha_\nu(E) \geqq 0$ for any $E \neq \emptyset$. If E is the graph of a continuous function, f, it is easy to see that $1 \leqq \alpha_\nu(f) \leqq \alpha(f) \leqq 2$ whenever ν is an increasing sequence. The various grids do not, in general, support a concomitant measure theory but the dimensions are related to the usual notions of Hausdorff dimension and packing dimension. These definitions are given below.

Let B_ϵ denote the family of all open balls in $\mathbb{R}^2$ of diameter less than ϵ and let $\alpha \in \mathbb{R}^+$. We denote the diameter of a set E by $\delta(E)$ and define

$$(4) \qquad \alpha\text{-}m(E) = \liminf_{\epsilon \to 0+} \{ \sum_{B \in F} \delta^\alpha(B) : \text{a and b hold (see below)} \}$$

$$(5) \qquad \alpha\text{-}P(E) = \limsup_{\epsilon \to 0+} \{ \sum_{B \in F} \delta^\alpha(B) : \text{a, c and d hold} \}, \text{ where}$$

a. $F \subseteq B_\epsilon$

b. $E \subseteq \cup F$

c. F is a pairwise disjoint family

d. $B \in F$ implies the center of B is in E

The Hausdorff dimension, $\beta(E)$, $\gamma(E)$, and the packing dimension, $\hat{\gamma}(E)$, are then defined as

$$(6) \qquad \beta(E) = \sup\{\alpha : \alpha\text{-}m(E) = \infty\}$$

$$(7) \qquad \gamma(E) = \sup\{\alpha : \alpha\text{-}P(E) = \infty\}.$$

$$(8) \qquad \tilde{\gamma}(E) = \inf\{\sup_{\cup E_i} \gamma(E_i): E \subseteq \cup E_i\}.$$

Once again, it is easy to check that if $f \in C$ then $1 \leq \beta(f), \gamma(f) \leq 2$. Before proceeding we remark that although α-m is the classical Hausdorff measure, α-P is, in general, not a measure. It is, however, a premeasure and as such can be extended to a measure in the usual manner. The resulting measure is called packing measure and the theory of packing measures parallels but is distinct from the theory of Hausdorff measures. J. Taylor and C. Tricot prove the following result [TT1, Lemma 1 and Corollary 3.9] in which ν_1 denotes the diatic sequence, $\nu(n) = 2^{-n}$.

<u>Theorem TT</u>. If $E \subseteq \mathbb{R}^2$ then $0 \leq \beta(E) \leq \alpha_{\nu_1}(E) = \tilde{\gamma}(E) \leq 2$.

The next result implies that $\alpha(E) = \tilde{\gamma}(E)$ as well.

<u>Lemma 1</u>. Let ν and ν^* be two increasing sequences of natural numbers. If there is a constant $c > 0$ and a function $h: \mathbb{N} \to \mathbb{N}$ satisfying

$$c\nu(h(n)) \leq \nu^*(n) \leq \nu(h(n))$$

for every n, then for every $E \subseteq [0,1]^2$, $\alpha_\nu(E) \geq \alpha_{\nu^*}(E)$.

<u>Proof</u>. Let $n \in \mathbb{N}$ be fixed. If $\nu(h(n)) > \nu^*(n)$, and S is any square in $G_{\nu(h(n))}$, then S intersects at most four squares in

$G_{\nu^*(n)}$. Hence whenever $\nu^*(n) \leqq \nu(h(n))$ we have $4N(E,\nu(h(n))) \geqq N(E,\nu^*(n))$. Then, for each $\alpha > 0$ we have

$$\frac{N(E,\nu(h(n)))}{\nu(h(n))^\alpha} \geqq \frac{N(E,\nu^*(n))}{4\nu(h(n))^\alpha} \geqq \frac{\nu^*(n)^\alpha}{4\nu(h(n))^\alpha} \frac{N(E,\nu^*(n))}{\nu^*(n)^\alpha} \geqq$$

$$\geq \frac{c^\alpha}{4} \frac{N(E,\nu^*(n))}{\nu^*(n)^\alpha} .$$

Hence,

$$\limsup_{n\to\infty} \frac{N(E,\nu^*(n))}{\nu^*(n)^\alpha} \leqq \limsup_{n\to\infty} K \frac{N(E,\nu(h(n)))}{\nu(h(n))^\alpha}$$

where $K = \frac{4}{c^\alpha}$. The result follows directly from this last inequality.

It is an immediate consequence of Lemma 1 that if $E \subseteq [0,1]^2$, then $\alpha(E) = \alpha_{\nu_1}(E)$ and hence, by Theorem TT, we obtain that $\alpha(E) = \gamma(E)$ for any set $E \subseteq [0,1]^2$.

<u>Proposition 2</u>. Let $s(n)$ be any sequence with $\lim_{n\to\infty} s(n) = \infty$. Then, there is a residual set of functions, f, such that

$$\limsup_{n\to\infty} \frac{N(f,n)}{n^2} s(n) = \infty.$$

<u>Proof</u>. It suffices to show that for every open set of functions, $U \subseteq C$ and every integer k, there is an open set $V \subseteq U$ and an $m > k$ such that whenever $g \in V$, then

$$\frac{N(g,m)}{m^2} s(m) \geqq k.$$

Let $U \subseteq C$ be open, $k \in \mathbb{N}$, and suppose $f \in U$. There is an $\epsilon > 0$ such that $U_1 = \{g : \|f-g\| < \epsilon\} \subseteq U$. Let $m > k$ be so large that $s(m) > \frac{2k}{\epsilon}$ and if $J = \{S \in G_m : S \subseteq U_1\}$, then $\cup J$ contains an element of C and has an area which exceeds $\frac{\epsilon}{2}$, half the area of U_1. It is easy to see that there is a $g_o \in C$ and $\delta > 0$ such that

(i) $g_o \subseteq \cup J$

(ii) $g_o \cap S \neq \emptyset$ for every $S \in J$, and

(iii) if $\|g-g_o\| < \delta$ then $N(g_o,m) = N(g,m)$.

Therefore, if $\|g-g_o\| < \delta$ then

$$\frac{N(g,m)}{m^2} s(m) = \frac{N(g_o,m)}{m^2} s(m) \geqq \frac{\epsilon}{2} s(m) > k.$$

<u>Theorem 3</u>. There is a residual set, F, of continuous functions such that

$$\bar{\gamma}(f) = 2 \quad (f \in F),$$

i.e. typical continuous functions have a packing dimension of 2.

<u>Proof</u>. It is immediate that Proposition 2 holds relative to any open subinterval $I \subset [0,1]$. Letting I run through the intervals with rational endpoints we obtain that for typical continuous functions f, $\gamma(f_I) = 2$ for every rational interval, I. (Here f_I

denotes the restriction of f to I.) If, for such a function f, f $\subset \cup E_i$, then by the Baire Category Theorem there is a set E_i and a rational interval I such that E_i is dense in f_I. But, by Lemma 3.2 of [TT2], $\gamma(E) = \gamma(\mathrm{Cl}E)$ for any E and hence, $\gamma(E_i) = \gamma(\mathrm{Cl}E_i) \geq \gamma(f_I) = 2$. Hence, $\gamma(E_i) = 2$ and the theorem follows.

The following proposition is well known, but its proof is included for completeness.

Proposition 4. Let $\varphi:\mathbb{R}^+ \to \mathbb{R}^+$ be such that $\lim_{x\to 0+} \frac{\varphi(x)}{x} = 0$. Then there is a residual set of functions, f, such that

$$\liminf_{\epsilon\to 0+} \left\{ \sum_{B\in F} \varphi(\delta(B)) : \ F \subseteq B_\epsilon \text{ and } f \subseteq \cup F \right\} = 0.$$

Proof. It suffices to show that for each open set $U \subseteq C$ and every $\epsilon > 0$ there is an open set $V \subseteq U$ such that if $g \in V$ then there is an $F \subseteq B_{\epsilon_o}$ where

(i) $g \subseteq \cup F$, and

(ii) $\sum_{B\in F} \varphi(\delta(B)) < \epsilon_o$.

As $U \subseteq C$ is open, there is a rectifiable $g_o \in U$. Let $\epsilon_1 < \epsilon_o$ be sufficiently small that $0 < \frac{\varphi(X)}{x} < \frac{\epsilon_o}{L+1}$ whenever $0 < x < \epsilon_1$ where L is the length of g_o. Let $F \subseteq B_{\epsilon_1}$ be a cover of g_o such that

(i) $B \subseteq \cup U$ for every $B \in F$,

(ii) $\sum_{B \in F} \delta(B) - L < 1$.

We set $V = \{g \in C: g \subseteq \cup F\}$. Then if $g \in V$, $g \in \cup F$ and

$$\sum_{B \in F} \varphi(\delta(B)) = \sum_{B \in F} \delta(B) \left(\frac{\varphi(\delta(B))}{\delta(B)}\right) \leqq \left(\frac{\epsilon_o}{L+1}\right) \sum_{B \in F} \delta(B) < \epsilon_o.$$

Theorem 3 and Proposition 4 are summarized below.

Theorem 5. A typical continuous function has a Hausdorff dimension of 1 and a packing dimension of 2.

We return, now, to Lemma 1 which shows that if two sequences are comparable (in the sense of Lemma 1) then the grid dimension computed with the first sequence will coincide with the grid dimension computed with the second sequence. It is natural to inquire how different two sequences must be in order that the respective grid dimensions differ. There are several ways to formulate a response to this question; the following seems to us most concise.

Theorem 6. Let $\nu(n)$ $(n=1,2,\ldots)$ be an strictly increasing sequence of integers such that for any $\epsilon > 0$

$$\inf_n \frac{\nu(n+1)^\epsilon}{\nu(n)} > 0.$$

Then, there exists a continuous function f such that

$$\alpha_{\nu(2n)}(f) = 1,$$
$$\alpha_{\nu(2n+1)}(f) = 2.$$

<u>Remarks</u>. The proof is a bit simpler if the additional divisibility condition $2\cdot\nu(n)|\nu(n+1)$ holds for our sequence. We suppose this because obvious modifications in the construction easily yield the general case.

A sequence satisfying all conditions is, for example,

$$\nu(n) = 2^{(n+1)!} \quad (n=1,2,\ldots).$$

Indeed, $2^{\epsilon(n+1)!} = 2^{n![\epsilon(n+1)]} > 2^{n!}$, if $\epsilon(n+1) > 1$.

It is also clear that the ϵ-condition actually implies

$$\frac{\nu(n+1)^{\epsilon}}{\nu(n)} \to \infty$$

for any $\epsilon > 0$ (and we shall make use of this fact in our proof):

$$\nu(n+1) = [\nu(n+1)^{\frac{\epsilon}{2}}]^2 > [c\nu(n)]^2$$

and hence

$$\frac{\nu(n+1)^{\epsilon}}{\nu(n)} > c^2\nu(n) \to \infty.$$

In particular,

$$r(n+1) = \frac{\nu(n+1)}{\nu(n)} \to \infty \qquad (n \to \infty),$$

which again will be needed in the proof.

Proof. For each n we select a subset $\mathscr{H}_n \subset G_{\nu(n)}$. Let $H_n = \cup\{I : I \in \mathscr{H}_n\}$, $r(n) = \nu(n)/\nu(n-1)$. We need the following properties of $\mathscr{H}_n$

(i) $H_{n+1} \subset H_n$, $H_{n+1} \cap I \neq \emptyset$ for any $I \in \mathscr{H}_n$,

(ii) $|\mathscr{H}_{2n+1}| = \frac{1}{2} r(2n+1)^2 |\mathscr{H}_{2n}|$ and
$r(2n)|\mathscr{H}_{2n-1}| \leqq |\mathscr{H}_{2n}| \leqq 3r(2n)|\mathscr{H}_{2n-1}|$ $(n=1,2,\ldots)$,

(iii) $\bigcap_{n=1}^{\infty} \mathrm{cl} H_n$ is the graph of a continuous function f, defined on $[0,1]$.

We show first that these properties really imply our statement. Identifying f with its graph we have $f \subset \mathrm{cl} H_n$ for any n and $f \cap I \neq \emptyset$ for any $I \in \mathscr{H}_n$ $(n=1,2,\ldots)$. Then

$$N(f,\nu(n)) = |\mathscr{H}_{\nu(n)}|,$$

and hence by (ii),

$$N(f,\nu(2n+1)) \geqq \prod_{k=1}^{n} [\tfrac{1}{2} r(2k+1)^2 r(2k)] \cdot |\mathscr{H}_{\nu(1)}|.$$

Therefore

$$\frac{N(f,\nu(2n+1)}{\nu(2n+1)^{2-\epsilon}} \geqq \frac{\nu(2n+1)^{\epsilon}}{\nu(2n)} \cdot [\prod_{k=1}^{n-1} \frac{r(2k+1)}{2} \cdot \frac{|\mathscr{H}_{\nu(1)}|}{2\cdot\nu(1)} \geqq C\cdot\frac{\nu(2n+1)^{\epsilon}}{\nu(2n)}.$$

Thus we get

$$\lim_{n\to\infty} \frac{N(f,\nu(2n+1))}{\nu(2n+1)^{2-\epsilon}} = \infty \qquad \text{for any } \epsilon > 0,$$

i.e. $\alpha_{\nu(2n+1)}(f) = 2$. Similarly, making use of the obvious estimate $|\mathscr{H}_{2k+1}| \leqq r(2k+1)^2 |\mathscr{H}_{2k}|$ we obtain

$$\frac{N(f,\nu(2n))}{\nu(2n)^{1+\epsilon}} \leqq 3^n \frac{r(2n)r(2n-1)^2\cdots r(2)|\mathscr{H}_1|}{\nu(2n)^{1+\epsilon}} =$$

$$= \frac{\nu(2n-1)}{\nu(2n)^{\epsilon}} \left(3\frac{\nu(2n-3)}{\nu(2n-2)}\right) \cdots \left(3\frac{\nu(3)}{\nu(4)}\right) \frac{|\mathscr{H}_1|}{\nu(1)\cdot\nu(2)}$$

$$\leqq C\cdot\frac{\nu(2n-1)}{\nu(2n)^{\epsilon}} \to 0 \qquad (n \to \infty)$$

and this gives $\alpha_{\nu(2n)}(f) = 1$.

To ensure (iii), H_n shall be made to satisfy

(iiia) H_n is connected;

(iiib) any vertical line $\ell_{x_0} = \{(x,y): x=x_0,\ 0 \leqq x_0 \leqq 1\}$ meets H_n in a linear segment of length $2^{-(n-1)/2}$

(iiic) the projection of H_n onto the x-axis is [0,1].

Then, as $\ell_{x_0} \cap (\bigcap_{n=1}^{\infty} H_n) = \bigcap_{n=1}^{\infty} (\ell_{x_0} \cap H_n)$ we get a graph $f = \bigcap_{n=1}^{\infty} \mathrm{cl} H_n$ and $f \subset \mathrm{cl}H_n$ implies that the oscillation of f at any point is less than $2^{-\frac{(n-2)}{2}}$; that is, f is a continuous function. Now turning to the construction of $\mathscr{H}_n$ we apply induction. Let $\mathscr{H}_1 = \{I : I \in G_{\nu(1)},\ I \cap ([0,1]\times\{0\}) \neq \emptyset\}$. Suppose that H_n has already been defined and (i), (ii), (iiia), (iiib), (iiic) all hold for indices $\leqq n$. Consider the squares $I_1, \ldots, I_{p_n}$ of $\mathscr{H}_n (p_n = |\mathscr{H}_n|)$ arranged in a sequence which corresponds to the increasing lexicographic order of their midpoints. The sequence

$I_1, \ldots, I_{p_n}$ splits into blocks

$$I_1, \ldots, I_{k_1}, I_{k_1+1}, \ldots, I_{k_2}, \ldots$$

where the midpoints of any two of the squares in $\mathcal{H}_n$ have equal x coordinates if any only if they belong to the same block. These blocks are called the columns of $\mathcal{H}_n$. This term is justified by (iiib) because the squares $I_{k_j+1}, \ldots, I_{k_{j+1}}$ themselves form a connected union. The grid $G_{\nu(n+1)}$ induces a subdivision of each $I \in \mathcal{H}_n$ into $r(n+1)^2$ smaller squares, and half of these squares will be selected for $\mathcal{H}_{n+1}$ if n is even but only $r(n+1)$ will be taken to $\mathcal{H}_{n+1}$ if n is odd. Let $n = 2k$ and consider a column $I_{k_j+1}, \ldots, I_{k_{j+1}}$ in $\mathcal{H}_n$. The grid $G_{\nu(n+1)}$ induces a vertical subdivision of $I_{k_j+1} \cup \ldots \cup I_{k_{j+1}}$ into $r(n+1)$ thin strips. We cut each of these horizontally into two equal parts and we select the upper or lower part of each strip. At the first and last position we choose the upper or lower section to make sure it will be connected to the previous column and the successor column of $\mathcal{H}_n$, respectively. Otherwise the selection of the upper or lower part is arbitrary (see the shaded area on Figure 1).

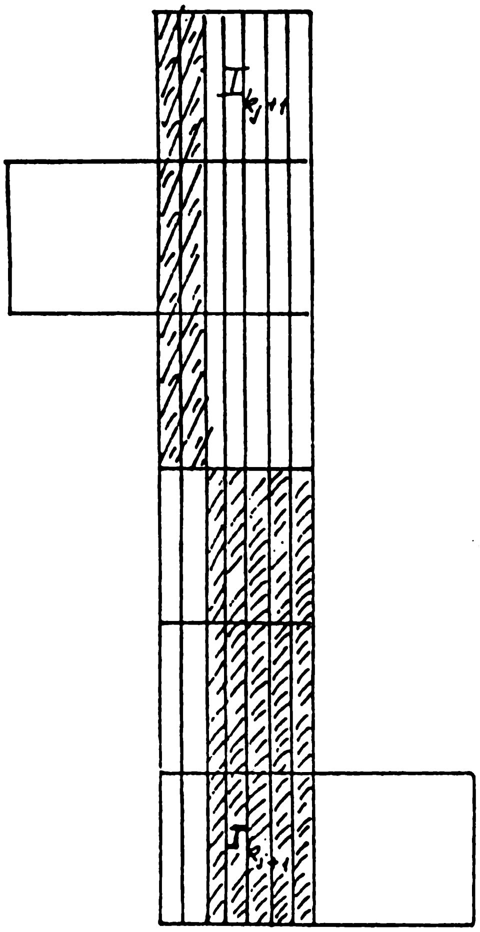

Figure 1

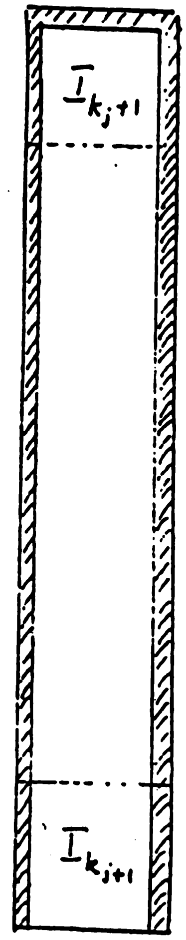

Figure 2

If $n = 2k-1$, we simply take the shaded part (as shown on Figure 2) of the induced subdivision of the given column.

It is trivial that (iiia), (iiib), (iiic) all hold by induction and hence our proof is complete.

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