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INTERPOLATED FOURIER TRANSFORMS

§0. Introduction. In this paper, the general form of transform T on $L^2(0, \infty)$ defined by a kernel φ is given by

$$Tf(t) = \underset{U\to\infty}{\mathrm{l.i.m.}} \int_0^U f(u)\varphi(tu)du \quad \text{in} \quad L^2(0, \infty),$$

for $f \in L^2(0, \infty)$. Let C and S denote the cosine and sine transforms on $L^2(0, \infty)$, i.e.

$$\left.\begin{aligned} Cf(t) &= \underset{U\to\infty}{\mathrm{l.i.m.}} \sqrt{2/\pi} \int_0^U f(u)\cos tu\, du \\ Sf(t) &= \underset{U\to\infty}{\mathrm{l.i.m.}} \sqrt{2/\pi} \int_0^U f(u)\sin tu\, du \end{aligned}\right\} \quad \text{in} \quad L^2(0, \infty)$$

for $f \in L^2(0, \infty)$. The Plancherel's Theorem states that

$$S^2 = C^2 = \mathrm{Id}.$$

In this paper, we consider the transform C_α defined by the kernel

$\sqrt{2/\pi}\ \cos(\theta - \frac{\pi}{2}\alpha)\quad (\alpha \in \mathbb{R})$:

$$(0.1)\qquad C_\alpha f(t) = \underset{U\to\infty}{\mathrm{l.i.m.}}\ \sqrt{2/\pi} \int_0^U f(u)\cos(tu - \tfrac{\pi}{2}\alpha)du \quad \text{in} \quad L^2(0, \infty)$$

for $f \in L^2(0, \infty)$. Note first that $C_0 = C$ and $C_1 = S$; and C_α is a bounded transform on $L^2(0, \infty)$, since it is a linear combination of two bounded transforms on $L^2(0, \infty)$. The object of this paper is to construct the inverse of C_α, and to show the inverse is a bounded transform on $L^2(0, \infty)$ for suitable α; see Theorem 1 in §2 below. For the kernel defining the inverse, see (0.5). We prove this with the help of the Riemann–Liouville and Weyl fractional integral operators, which are defined respectively as follows:

$$I_\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^x (x-y)^{\alpha-1} f(y)dy & (\alpha > 0,\ x > 0) \\ f(x) & (\alpha = 0,\ x > 0) \end{cases}$$

$$J_\alpha f(x) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_x^\infty (y-x)^{\alpha-1} f(y)dy & (\alpha > 0,\ x > 0) \\ f(x) & (\alpha = 0,\ x > 0). \end{cases}$$

There is a basic property for I_α and J_α:

$$(0.2)\qquad I_{\alpha+\beta} f(x) = I_\alpha I_\beta f(x), \quad J_{\alpha+\beta} f(x) = J_\alpha J_\beta f(x)$$

for $\alpha \geq 0$, $\beta \geq 0$.

Consider the Mellin transforms of $\cos\theta$ and $\sin\theta$. It is a consequence of (7.9.5) and (7.9.6) of [2] that

$$(0.3) \qquad J_\alpha \cos\theta = \cos(\theta + \tfrac{\pi}{2}\alpha), \qquad J_\alpha \sin\theta = \sin(\theta + \tfrac{\pi}{2}\alpha)$$

for $0 < \alpha < 1$. On the other hand, by considering the Taylor series expansion, we have

$$(0.4) \qquad I_\alpha \cos\theta = \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(2n+1+\alpha)} \theta^{2n+\alpha} \qquad (\alpha \geq 0);$$

the series on the right–hand side of (0.4) is defined for $\forall \alpha \in \mathbb{R}$.

Note that the kernel in (0.1) is given by (0.3) for suitable α.

Denote

$$(0.5) \qquad k_\alpha(\theta) = \sqrt{2/\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(2n+1+\alpha)} \theta^{2n+\alpha} \qquad (\alpha \in \mathbb{R}),$$

for negative odd integer α, the term with Γ–factor is defined to be zero for integer $2n+1+\alpha \leq 0$.

We show first in Lemma 3 that $k_\alpha(\theta)$ defines a bounded transform D_α for $-\frac{1}{2} < \alpha < \frac{3}{2}$, and then show in Theorem 1 that

$$C_\alpha D_\alpha f = D_\alpha C_\alpha f = f \qquad (-\tfrac{1}{2} < \alpha < \tfrac{3}{2}) \tag{0.6}$$

for $f \in L^2(0, \infty)$.

We introduce here the averaging operators which are needed in this paper. For $f \in L^2(0, \infty)$, we define the transforms A_α, B_β as follows:

$$A_\alpha f(x) = x^{-\alpha} \int_0^x y^{\alpha-1} f(y)dy \qquad (\alpha > \tfrac{1}{2})$$

$$B_\beta f(x) = x^{-\beta} \int_x^\infty y^{\beta-1} f(y)dy \qquad (\beta < \tfrac{1}{2})$$

§1. Some lemmas. To pursue the object of this paper, we need several lemmas.

Lemma 1. The A_α and B_β are bounded operators on $L^2(0, \infty)$, and

$$\|A_\alpha\|_2 \le (\alpha - \tfrac{1}{2})^{-1}, \qquad \|B_\beta\|_2 \le (\tfrac{1}{2} - \beta)^{-1}.$$

Proof. By (9.9.8) and (9.9.9) of [1], the results follow immediately.

Lemma 2. Consider (0.5). For $\alpha < 2$, we have

$$\sqrt{\pi/2}k_{\alpha}(\theta) = \cos(\theta - \tfrac{\pi}{2}\alpha) + \frac{1}{\Gamma(\alpha-1)}\int_{\theta}^{\infty}\xi^{\alpha-2}\sin(\xi-\theta)d\xi \quad (\theta > 0);$$

and the integral in the above is defined to be zero for integer α less than or equal to 1.

Proof. Put $K_{\alpha}(\theta) = \sqrt{\pi/2}k_{\alpha}(\theta)$. By differentiating $K_{\alpha}(\theta)$ twice, we see that

$$K''_{\alpha}(\theta) + K_{\alpha}(\theta) = \frac{1}{\Gamma(\alpha-1)}\theta^{\alpha-2}.$$

On the other hand, for $\alpha < 2$

$$\frac{1}{\Gamma(\alpha-1)}\int_{\theta}^{\infty}\xi^{\alpha-2}\sin(\xi-\theta)d\xi$$

is a special solution to the differential equation

$$y'' + y = \frac{1}{\Gamma(\alpha-1)}\theta^{\alpha-2} .$$

Thus

$$K_\alpha(\theta) = \cos(\theta - \tfrac{\pi}{2}a_\alpha) + \frac{1}{\Gamma(\alpha-1)} \int_\theta^\infty \xi^{\alpha-2} \sin(\xi-\theta)d\xi$$

for some constant a_α depending on α. Now the above and (0.5) yield

$$K_{\alpha-1}(\theta) = K'_\alpha(\theta) = \cos(\theta - \tfrac{\pi}{2}(a_\alpha - 1)) + \frac{1}{\Gamma(\alpha-2)} \int_\theta^\infty \xi^{\alpha-3} \sin(\xi-\theta)d\xi.$$

Thus we may take

(1.0) $$a_\alpha - 1 = a_{\alpha-1} .$$

By taking $\theta = 0$, we have for $0 < \alpha < 2$

$$0 = K_\alpha(0) = \cos \tfrac{\pi}{2}a_\alpha + \frac{1}{\Gamma(\alpha-1)} \int_0^\infty \xi^{\alpha-2} \sin \xi \; d\xi.$$

Since

$$\frac{1}{\Gamma(\alpha-1)} \int_0^\infty \xi^{\alpha-2} \sin \xi \; d\xi = - \cos \tfrac{\pi}{2}\alpha ,$$

we may take $a_\alpha = \alpha$ for $0 < \alpha < 2$.

This completes the proof of Lemma 2 by virtue of (1.0).

Lemma 3. For $-\frac{1}{2} < \alpha < \frac{3}{2}$, the series $k_\alpha(\theta)$ in (0.5) defines a bounded transforms on $L^2(0, \infty)$.

Proof. We assume $\alpha \neq 0, 1$. It is convenient to consider $K_\alpha(\theta) = \sqrt{\pi/2}\ k_\alpha(\theta)$. Define

$$\delta(a, b) = \begin{cases} 1, & 0 < a \le b \\ 0, & 0 < b < a \end{cases}$$

and

$$\beta(\theta) = \theta^{\alpha} \cdot \delta(\theta, 1), \qquad \gamma(\theta) = \theta^{\alpha-2} \cdot \delta(1, \theta)$$

$$\eta(\theta) = K_\alpha(\theta) \cdot \delta(1, \theta), \qquad \epsilon(\theta) = \cos(\theta - \tfrac{\pi}{2}\alpha) \cdot \delta(1, \theta).$$

Write as

$$K_\alpha(\theta) = (K_\alpha(\theta) - \eta(\theta)) + (\eta(\theta) - \epsilon(\theta)) + \epsilon(\theta).$$

Considering the power series expansion of $K_\alpha(\theta)$, we see that

$$|K_\alpha(\theta) - \eta(\theta)| = O(\beta(\theta))$$

and by Lemma 2

$$|\eta(\theta) - \epsilon(\theta)| = O(\gamma(\theta)).$$

So, it suffices to show that

$$\beta(\theta), \quad \gamma(\theta), \quad \epsilon(\theta)$$

are all the kernels of bounded transforms on $L^2(0, \infty)$.

Given any $f \in L^2(0, \infty)$, we have first for $\beta(\theta)$

$$F(u) = \int_0^\infty f(t)\beta(tu)dt = \int_0^{1/u} f(t)(tu)^\alpha dt$$

$$= u^{-1}A_{\alpha+1}f(u^{-1}).$$

Since $\alpha+1 > \frac{1}{2}$, and

$$\left(\int_0^\infty |F(u)|^2 du\right)^{1/2} = \left(\int_0^\infty |A_{\alpha+1}f(u)|^2 du\right)^{1/2} \le (\alpha + \tfrac{1}{2})^{-1}\|f\|_2$$

by Lemma 1, which shows that $\beta(\theta)$ is the kernel of a bounded transform on $L^2(0, \infty)$. As for $\gamma(\theta)$, we have

$$F(u) = \int_0^\infty f(t)\gamma(tu)dt = \int_{u^{-1}}^\infty f(t)(tu)^{\alpha-2}dt$$

$$= u^{-1}B_{\alpha-1}f(u^{-1}).$$

Since $\alpha-1 < \frac{1}{2}$, and

$$\left(\int_0^\infty |F(u)|^2 du\right)^{1/2} = \left(\int_0^\infty |B_{\alpha-1}f(u)|^2 du\right)^{1/2} \le \left(\tfrac{1}{2} - (\alpha-1)\right)^{-1}\|f\|_2$$

by Lemma 1, which shows that $\gamma(\theta)$ is the kernel of a bounded transform on $L^2(0, \infty)$. Finally, to show $\epsilon(\theta)$ is the kernel of a bounded transform on $L^2(0, \infty)$, it is enough to work on

$$\bar{\epsilon}(\theta) = \cos(\theta - \tfrac{\pi}{2}\alpha)\cdot\delta(\theta, 1),$$

since $\cos(\theta - \frac{\pi}{2}\alpha) = \epsilon(\theta) + \bar{\epsilon}(\theta)$. Obviously

$$\bar{\epsilon}(\theta) = O(\beta(\theta))$$

for any fixed $\alpha < 0$. And we have shown that $\beta(\theta)$ is the kernel of a bounded transform for $-\frac{1}{2} < \alpha < 0$, which implies that $\bar{\epsilon}(\theta)$ is the kernel of a bounded transform on $L^2(0, \infty)$.

This completes the proof of Lemma 3.

§2. Proof of Theorem 1. Note first that

$$k_\alpha(\theta) = \sqrt{2/\pi}\ I_\alpha \cos\theta \qquad (\alpha \geq 0). \tag{2.0}$$

Denote

$$D_\alpha f(u) = \underset{T\to\infty}{\mathrm{l.i.m.}} \int_0^T f(t)k_\alpha(tu)dt \qquad (-\tfrac{1}{2} < \alpha < \tfrac{3}{2}) \tag{2.1}$$

for $f \in L^2(0, \infty)$. Lemma 3 shows that D_α is a bounded transform on $L^2(0, \infty)$.

Theorem 1. Consider the transform C_α in (0.1). We have

$$C_\alpha D_\alpha f = D_\alpha C_\alpha f = f \qquad (-\tfrac{1}{2} < \alpha < \tfrac{3}{2}) \tag{2.2}$$

for $f \in L^2(0, \infty)$.

Proof. By the Plancherel's Theorem, (2.2) holds good for $\alpha = 0, 1$.

Step 1. We show first that (2.2) holds good for $0 < \alpha < 1$.

Let $f \in C^\infty(0, \infty)$ with support a compact subset of $(0, \infty)$. Note that $\mathrm{supp}\ J_\alpha f(v) \subset [0, A]$ for $\forall$ large A. By the definition of $I_\alpha \cos\theta$, we have

$$(2.3) \qquad F(u) = D_\alpha f(u) = \sqrt{2/\pi}\int_0^\infty f(t)I_\alpha \cos(tu)dt$$

$$= \sqrt{2/\pi}\,\frac{1}{\Gamma(\alpha)}\int_0^\infty f(t)\int_0^{tu}(tu-v)^{\alpha-1}\cos\ v\ dvdt$$

$$= \sqrt{2/\pi}\,\frac{u^\alpha}{\Gamma(\alpha)}\int_0^\infty \cos\ uv \int_v^\infty (t-v)^{\alpha-1}f(t)dtdv.$$

This gives

$$(2.4) \qquad u^{-\alpha}F(u) = \sqrt{2/\pi}\int_0^\infty J_\alpha f(v)\cos\ uv\ dv$$

and

$$(2.5) \qquad F(u) = O(u^\alpha) \qquad (u \to 0^+)$$

$$(2.6) \qquad F(u) = O(u^{\alpha-2}) \qquad (u \to +\infty),$$

by taking integration by parts twice in (2.4).

By Plancherel's Theorem, (2.4) yields

$$(2.7) \qquad J_\alpha f(v) = \sqrt{2/\pi}\int_0^\infty u^{-\alpha}F(u)\cos uvdu.$$

By using (0.2), equation (2.7) yields

$$(2.8) \qquad \sqrt{\pi/2}\,\Gamma(1-\alpha)J_1f(v) = \sqrt{\pi/2}\int_v^\infty (t-v)^{-\alpha}J_\alpha f(t)dt$$

$$= \int_v^\infty (t-v)^{-\alpha}\int_0^\infty u^{-\alpha}F(u)\cos tududt$$

$$= \int_v^\infty (t-v)^{-\alpha}\{-\frac{1}{t}\int_0^\infty [u^{-\alpha}F(u)]'\sin tudu\}dt,$$

since $u^{-\alpha}F(u)\sin tu\Big|_{u=0}^{u=\infty} = 0$, by (2.5) and (2.6),

$$= \int_0^\infty [u^{-\alpha}F(u)]'\int_v^\infty (t-v)^{-\alpha}(-\frac{1}{t})\sin tudtdu,$$

by the fact $[u^{-\alpha}F(u)]' = \min(O(1), O(u^{-2}))$ following from (2.4) and hence

$$[u^{-\alpha}F(u)]'(t-v)^{-\alpha}t^{-1} \in L^1_{(0,\infty)}(u) \times L^1_{(v,\infty)}(t),$$

$$= [u^{-\alpha}F(u)]\int_v^\infty (t-v)^{-\alpha}(-\frac{1}{t})\sin tudu\Big|_{u=0}^{u=\infty}$$

$$- \int_0^\infty u^{-\alpha}F(u)\int_v^\infty (t-v)^{-\alpha}(-\tfrac{1}{t})(\cos\ tu)tdtdu$$

$$= \int_0^\infty u^{-\alpha}F(u)\int_v^\infty (t-v)^{-\alpha}\cos\ tudtdu$$

$$= \Gamma(1-\alpha)\int_0^\infty u^{-1}F(u)J_{1-\alpha}\cos(uv)du.$$

Differentiating both sides of (2.8) gives rise to

$$f(v) = \sqrt{2/\pi}\int_0^\infty F(u)J_{1-\alpha}\sin(uv)du \tag{2.9}$$

$$= \sqrt{2/\pi}\int_0^\infty F(u)\cos(uv - \tfrac{\pi}{2}\alpha)du, \qquad \text{by (0.3)},$$

$$= C_\alpha^{\cdot}F(v) = C_\alpha D_\alpha f(v).$$

Conversely, consider again $f \in C^\infty(0, \infty)$ with support a compact subset of $(0, \infty)$. By (0.3), we have

$$F(v) = C_\alpha f(v) = \sqrt{2/\pi}\int_0^\infty f(u)J_{1-\alpha}\sin(uv)du \tag{2.10}$$

$$= \sqrt{2/\pi}\,\frac{1}{\Gamma(1-\alpha)}\int_0^\infty f(u)\int_{uv}^\infty (t-uv)^{-\alpha}\sin\ t\ dtdu$$

$$= \sqrt{2/\pi}\ \frac{1}{\Gamma(1-\alpha)} \int_v^\infty (t-v)^{-\alpha} \int_0^\infty u^{1-\alpha} f(u) \sin tu du dt.$$

Taking integration by parts, we see that

(2.11) $$\int_0^\infty u^{1-\alpha} f(u) \sin tu du = O(t^{-k}) \qquad (t \to +\infty)$$

for any integer $k > 0$. So

(2.12) $$F(v) = O(v^{-k}) \qquad (v \to +\infty)$$

for any integer $k > 0$. Now by (0.2), (2.10) gives rise to

$$J_\alpha F(v) = \sqrt{2/\pi} \int_v^\infty \int_0^\infty u^{1-\alpha} f(u) \sin tu du dt$$

and hence

$$(J_\alpha F)'(v) = -\sqrt{2/\pi} \int_0^\infty u^{1-\alpha} f(u) \sin uv du.$$

Consequently, we have by using (2.12)

(2.13) $$u^{1-\alpha} f(u) = -\sqrt{2/\pi} \int_0^\infty (J_\alpha F)'(v) \sin uv dv$$

$$= u\sqrt{2/\pi}\int_0^\infty (J_\alpha F)(v)\cos uvdv \ ,$$

and

$$u^{-\alpha}f(u) = \sqrt{2/\pi}\,\frac{1}{\Gamma(\alpha)}\lim_{A\to\infty}\int_0^A \cos uv \int_v^\infty (t-v)^{\alpha-1}F(t)dtdv$$

$$= \sqrt{2/\pi}\,\frac{1}{\Gamma(\alpha)}\lim_{A\to\infty}\{\int_0^A F(t)\int_0^t (t-v)^{\alpha-1}\cos uvdvdt + \int_A^\infty F(t)\int_0^A (t-v)^{\alpha-1}\cos uvdvdt\}$$

$$= \sqrt{2/\pi}\,u^{-\alpha}\int_0^\infty F(t)I_\alpha \cos(tu)dt \ .$$

So

$$f(u) = \int_0^\infty F(t)k_\alpha(tu)dt = D_\alpha F(u) = D_\alpha C_\alpha f(u).$$

By a standard continuity argument, we see that (2.2) holds good for $0 < \alpha < 1$.

<u>Step 2</u>. We now show that (2.2) holds good for $1 < \alpha < \frac{3}{2}$. Note first that $I_\alpha \cos\theta = I_{\alpha-1}\sin\theta$, and $0 < \alpha-1 < \frac{1}{2}$. We proceed the same argument as in Step 1 with $I_{\alpha-1}\sin\theta$ in the place of $I_{\alpha-1}\cos\theta$. Put $\beta = \alpha-1$.

Corresponding to (2.3), we have

$$F(u) = D_\beta f(u) = \sqrt{2/\pi}\,\frac{u^\beta}{\Gamma(\beta)}\int_0^\infty \sin\,uv \int_v^\infty (t-v)^{\beta-1} f(t)\,dt\,dv. \tag{2.14}$$

This gives

$$u^{-\beta}F(u) = \sqrt{2/\pi}\int_0^\infty J_\beta f(v)\sin\,uv\;dv \tag{2.15}$$

and

$$F(u) = O(u^{\beta+1}) \qquad (u \longrightarrow 0^+) \tag{2.16}$$

and

$$u^{-\beta}F(u) = \sqrt{2/\pi}\;J_\beta f(0)u^{-1} + O(u^{-2}) \qquad (u \longrightarrow +\infty), \tag{2.17}$$

by taking integration by parts in (2.15).

By Plancherel's Theorem and (2.17), we see that

$$J_\beta f(v) = \sqrt{2/\pi}\int_0^\infty u^{-\beta}F(u)\sin\,uv\,du. \tag{2.18}$$

Corresponding to (2.8), we have by (2.18)

$$J_1 f(v)$$

$$= \frac{1}{\Gamma(1-\beta)} \int_v^A (t-v)^{-\beta} J_\beta f(t) dt$$

$$= \sqrt{2/\pi}\ \frac{1}{\Gamma(1-\beta)} \int_v^A (t-v)^{-\beta} \int_0^\infty u^{-\beta} F(u) \sin tu du dt$$

$$= \sqrt{2/\pi}\ \frac{1}{\Gamma(1-\beta)} \lim_{B \to \infty} \{ \int_0^B u^{-\beta} F(u) \int_v^A (t-v)^{-\beta} \sin tu dt du$$

$$+ \int_v^A (t-v)^{-\beta} \int_B^\infty u^{-\beta} F(u) \sin tu du dt \}$$

$$= \sqrt{2/\pi}\ \frac{1}{\Gamma(1-\beta)} \int_0^\infty u^{-\beta} F(u) \int_v^A (t-v)^{-\beta} \sin tu dt du, \text{ by (2.17)}$$

$$= \sqrt{2/\pi}\ \frac{1}{\Gamma(1-\beta)} \int_0^\infty u^{-1} F(u) \int_{uv}^\infty (t-uv)^{-\beta} \sin t dt du, \text{ by making } A \to \infty,$$

since by writing

$$\int_v^A (t-v)^{-\beta} \sin tu dt = \int_v^{v+1} (t-v)^{-\beta} \sin tu dt + \int_{v+1}^A (t-v)^{-\beta} \sin tu dt$$

and by taking integration by parts

$$\int_{v+1}^{A}(t-v)^{-\beta}\sin\ tudt = O(u^{-1})$$

uniformly with respect to $A \to +\infty$. So the above gives

$$J_1f(v) = \sqrt{2/\pi}\int_0^{\infty}u^{-1}F(u)J_{1-\beta}\sin\ (uv)du\ . \tag{2.19}$$

Now by using (2.17), we can differentiate both sides of (2.19) and get

$$f(v) = -\sqrt{2/\pi}\int_0^{\infty}F(u)J_{1-\beta}\cos\ uvdu$$

$$= \sqrt{2/\pi}\int_0^{\infty}F(u)\cos\ (uv - \tfrac{\pi}{2}\alpha)du.$$

$$= C_\alpha F(v) = C_\alpha D_\alpha f(v),\quad (1 < \alpha < \tfrac{3}{2}).$$

To justify the above equation, it is enough to show

$$\frac{d}{dv}\int_1^{\infty}u^{-1}F(u)J_{1-\beta}\sin\ uvdu = \int_1^{\infty}u^{-1}F(u)\frac{d}{dv}(J_{1-\beta}\sin\ uv)du.$$

Now (2.17) gives

$$u^{-1}F(u) = \lambda u^{\beta-2} + O(u^{\beta-3})$$

for some constant λ. By appealing to (0.3), we see that

$$\frac{d}{dv}\int_1^\infty u^{\beta-2}J_{1-\beta}\sin\, uvdu = \int_1^\infty u^{\beta-2}\, \frac{d}{dv}(J_{1-\beta}\sin\, uv)du$$

and since $F(u) - \lambda u^{\beta-1} = O(u^{\beta-2}) \in L^1(0, \infty)$, which combined together prove what we want.

Conversely, corresponding to (2.10), we have

$$F(v) = C_\alpha f(v) = -\sqrt{2/\pi}\int_0^\infty f(u)J_{1-\beta}\cos(uv)du \tag{2.20}$$

$$= -\sqrt{2/\pi}\,\frac{1}{\Gamma(1-\beta)}\int_v^\infty (t-v)^{-\beta}\int_0^\infty u^{1-\beta}f(u)\cos\, tududt$$

and to (2.11), (2.12) we have

$$\int_0^\infty u^{1-\beta}f(u)\cos\, tudu = O(t^{-k}) \quad (t \to +\infty) \tag{2.21}$$

$$F(v) = O(v^{-k}) \quad (v \to +\infty) \tag{2.22}$$

for any integer $k > 0$.

In view of (0.2), (2.20) gives rise to

$$J_\beta F(v) = -\sqrt{2/\pi}\int_v^\infty \int_0^\infty u^{1-\beta} f(u)\cos tududt,$$

and hence

$$(J_\beta F)'(v) = \sqrt{2/\pi}\int_0^\infty u^{1-\beta} f(u)\cos uv\ du.$$

Consequently, we have

$$(2.23)\qquad u^{1-\beta} f(u) = \sqrt{2/\pi}\int_0^\infty (J_\beta F)'(v)\cos uv\ dv$$

$$= c + \sqrt{2/\pi}\ u\int_0^\infty (J_\beta F)(v)\sin uv\ dv$$

for some constant c; by virtue of (2.22) and by making $u \longrightarrow 0^+$ in (2.23), we see that $c = 0$. So, we have, by using (2.22),

$$u^{-\beta} f(u) = \sqrt{2/\pi}\ \frac{1}{\Gamma(\beta)} \lim_{A\to\infty} \int_0^A \sin uv \int_v^\infty (t-v)^{\beta-1} F(t)dtdv$$

$$= \sqrt{2/\pi}\ \frac{1}{\Gamma(\beta)} \lim_{A\to\infty}\{\int_0^A F(t)\int_0^t (t-v)^{\beta-1}\sin uvdvdt$$

$$+\int_A^\infty F(t)\int_0^A (t-v)^{\beta-1}\sin\ uv dv dt\}$$

$$= u^{-\beta}\sqrt{2/\pi}\int_0^\infty F(t) I_\beta \sin(tu)dt$$

and

$$f(u) = \sqrt{2/\pi}\int_0^\infty F(t) I_\alpha \cos(tu)dt$$

$$= \int_0^\infty F(t)k_\alpha(tu)dt = D_\alpha F(u) = D_\alpha C_\alpha f(u).$$

By a standard continuity argument, we see that (2.2) holds good for $1 < \alpha < \frac{3}{2}$.

Step 3. Finally we show that (2.2) holds good for $-\frac{1}{2} < \alpha < 0$. Suppose $f \in C^\infty(0, \infty)$ with support a compact subset of $(0, \infty)$.

Consider first

$$D_\alpha f(u) = \int_0^\infty f(t)k_\alpha(tu)dt$$

$$= u^{-1}f(t)k_{\alpha+1}(tu)\Big|_{t=0}^{t=\infty} - u^{-1}\int_0^\infty f'(t)k_{\alpha+1}(tu)dt.$$

By Lemma 2, we have the estimate

$$D_\alpha f(u) = O(u^{\alpha-2}) \qquad (u \longrightarrow +\infty)$$

and hence

$$uD_\alpha f(u) \in L^1(0, \infty).$$

Since $\frac{1}{2} < \alpha+1 < 1$, (2.2) gives rise to

$$-f'(t) = \int_0^\infty uD_\alpha f(u)\cos(tu - \tfrac{\pi}{2}(\alpha+1))du$$

from which it follows

$$f(t) = \int_0^\infty D_\alpha f(u)\cos(tu - \tfrac{\pi}{2}\alpha)du = C_\alpha D_\alpha f(t)$$

Consider next

$$C_\alpha f(u) = \int_0^\infty f(t)\cos(tu - \tfrac{\pi}{2}\alpha))dt.$$

Note that

$$(C_\alpha f)^{(\ell)}(u) = O(u^{-k}) \quad (u \longrightarrow +\infty)$$

for any integers $k > 0$, $\ell \geq 0$, and

$$(C_\alpha f)'(u) = -\int_0^\infty tf(t)\sin(tu - \tfrac{\pi}{2}\alpha)dt$$

$$= -\int_0^\infty tf(t)\cos(tu - \tfrac{\pi}{2}(\alpha+1))dt.$$

Since $\frac{1}{2} < \alpha+1 < 1$, (2.2) gives rise to

$$tf(t) = -\int_0^\infty (C_\alpha f)'(u)k_{\alpha+1}(tu)du$$

$$= -C_\alpha f(u)k_{\alpha+1}(tu)\Big|_{u=0}^{u=\infty} + t\int_0^\infty C_\alpha f(u)k_\alpha(tu)du.$$

Thus we see that

$$f(t) = \int_0^\infty C_\alpha f(u)k_\alpha(tu)du = D_\alpha C_\alpha f(t).$$

Similarly as before, (2.2) holds good for $f \in L^2(0, \infty)$ and $-\frac{1}{2} < \alpha < 0$.

This completes the proof of Theorem 1.

References

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