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On the algebra generated by derivatives of interval functions

It is well known (See A. Bruckner [1]) that the product of two derivatives need not itself be a derivative.

This consideration led to the question of characterizing the class of functions expressible as a product of two or more derivatives and it raised the larger question of characterizing the algebra of functions generated by derivatives.

The second of these questions was solved by D. Preiss in [6]: he showed that every real Baire 1 function u of one variable can be expressed in the form $u = f' \cdot g' + h'$.

On the other hand, a function of the form $f' \cdot g' + h'$ must be Baire 1. Hence the class of one variable real functions having the representation above coincides with the first class of Baire.

In this note we generalize Preiss' result to higher dimensions, that is, we show that a real Baire 1 function u of k variables ($k>1$) also admits a representation $u = DF \cdot DG + DH$, where

F, G and H are additive ordinarily differentiable interval functions.

Let Γ be the collection of all compact intervals in R^k.

For $\alpha \in (0,1)$ let Γ_α be the collection of all compact intervals in R^k having parameter of regularity greater than α.

For $x \in R^k$, let

$\Gamma(x) = \{ I \in \Gamma : x \in I \}$, $\Gamma_\alpha(x) = \{ I \in \Gamma_\alpha : x \in I \}$.

A function $f : X \subseteq R^k \to R$ is said to be <u>ordinarily approximately continuous</u> if it is continuous with respect to the ordinary density topology in R^k (See [3]). The class of ordinarily approximately continuous real functions on R^k will be denoted by Ω^*.

We refer the reader to [8] (chapter III) for the definition of absolute continuity of an interval function.

We also refer the reader to [8] (chapter IV) and to [5] (chapter VI) for what concerns differentiation of an interval function.

Δ^* will denote the class of additive ordinarily differentiable real interval functions defined on Γ. When $F \in \Delta^*$, DF will denote the derivative of F.

B^1 will denote the class of Baire 1 real functions on R^k.

μ will denote the Lebesgue measure in R^k.

For $X \subseteq R^k$, χ_X will denote the characteristic function of X.

A point x is said to be a <u>Lebesgue point</u> of a real function f defined on R^k if

$$\lim_{\substack{I \to x \\ I \in \Gamma_\alpha(x)}} \frac{1}{\mu(I)} \int_I |f - f(x)| \, d\mu = 0$$

for every $\alpha \in (0,1)$.

If x is a Lebesgue point of f for every $x \in R^k$, f is said to be a Lebesgue function. The class of all Lebesgue functions on R^k will be denoted by Λ^*.

Every bounded function $f \in \Omega^*$ is a Lebesgue function.

A Lebesgue function f is ordinarily approximately continuous and it is the derivative (in the ordinary sense) of the interval function $\int_I f \, d\mu$ (See [5] ch. VI).

<u>Proposition</u> 1. Let $X \subseteq R^k$ be measurable, $C_1, \ldots, C_n$ pairwise disjoint closed subsets of X contained in the set of ordinary density points of X, $a_1, \ldots, a_n$ real numbers. Then there exists a Lebesgue function Φ such that $\Phi(x) = a_i$ for $x \in C_i$, $\Phi(x) = 0$ for $x \notin X$ and $\sup_{x \in R^k} |\Phi(x)| \leq \max |a_i|$.

<u>Proof</u> . The result easily follows from the k-dimensional version ([3] , lemma 4) of the analogous result by Zahorski [9] (Also see [4]).

Proposition 2. Let $u \in B^1$. There are $v \in B^1$, a sequence $\{H_n\}_{n\in N}$ of pairwise disjoint compact subsets of R^k and a sequence $\{\beta_n\}_{n\in N}$ of positive numbers such that

i) $u - v \in \Lambda^*$,

ii) v is ordinarily approximately continuous at every point of $\bigcup_{n\in N} H_n$,

iii) $v(x) = 0$ whenever $x \in H_n$ is not an ordinary density point of H_n,

iv) $|v| \leq \sum_{n\in N} \beta_n \chi_{H_n}$,

v) $\sum_{n\in N} \beta_n \chi_{H_n} \in B^1$,

vi) v is bounded provided that u is bounded.

Proof. Put $D=\{(x_1,\dots,x_k)\in R^k : \exists\, i\in\{1,\dots,k\}: x_i \text{ is rational}\}$. D is a dense set of measure zero.

Let M be the measure zero set of points of R^k at which u is not ordinarily approximately continuous.

Then there is $\phi_1 \in \Lambda^*$ such that $\phi_1(x) = u(x)$ for $x \in D \cup M$ (cf. [2], th 2.5 and [4]). (In case u is bounded, ϕ_1 may be taken to be bounded).

Since $u - \phi_1 \in B^1$, we have $\log|u - \phi_1| \in B^1$ on the space $A = \{x \in R^k : (u - \phi_1)(x) \neq 0\}$.

Then there is a function $g \in B^1$ such that $g(A)$ is an isolated set and $\sup_{x \in A} | \log | u - \phi_1 |(x) - g(x)| \leq 1$ (cf. [7], I, 27, VII).

Since A is an F_σ set and a zero-dimensional space, there is a sequence $\{H_n\}_{n \in N}$ of pairwise disjoint compact sets such that $A = \bigcup_{n \in N} H_n$ and the restriction of g to H_n is constant (cf. [7], II, 26, V).

Thus, if we call ω the function assuming the values $\omega(x) = 0$ for $x \notin A$ and $\omega(x) = e^{g(x)+1}$ for $x \in A$, ω can be written as follows: $\omega(x) = \sum_{n \in N} \beta_n \chi_{H_n}$, where β_n are positive numbers for every $n \in N$. Moreover, $\omega \geq 0$ and, for every $a > 0$, the sets

$$\{ x \in R^k : \omega(x) > a \} \quad \text{and} \quad \{x \in R^k : \omega(x) < a\}$$

are F_σ sets.

That proves v).

Let now E be a set of measure zero containing all points of each H_n that are not ordinary density points of H_n and all points at which $u - \phi_1$ is not ordinarily approximately continuous; we can construct $\phi \in \Lambda^*$ such that $\phi = u - \phi_1$ on E.

Put $\phi_2(x)=\max\{\min\{\phi(x),\max\{(u-\phi_1)(x),0\}\};\min\{(u-\phi_1)(x),0\}\}$.

Using the argument of [6], Lemma 3, we can show that $\phi_2 \in \Lambda^*$ and the function $v = u - \phi_1 - \phi_2$ satisfies i), ii), iii), iv), v) and vi).

Proposition 3. Let $\{K_n\}_{n\in N}$ be a sequence of pairwise disjoint compact subsets of R^k and let $\{\gamma_n\}_{n\in N}$ be a sequence of nonnegative numbers such that $\sum_{n\in N}\gamma_n \chi_{K_n} \in B^1$. Then there is a sequence of positive numbers $\{\lambda_n\}_{n\in N}$ satisfying the following properties:

a) if, for each $n \in N$, F_n is a real additive and absolutely continuous (with respect to μ) interval function defined on Γ, satisfying:

i) $F_n \in \Delta^*$, $\sup_\Gamma |F_n| \leq \lambda_n$, $\sup_{R^k} |DF_n| \leq \gamma_n$,

ii) $(I \in \Gamma,\ I \cap K_n = \emptyset) \Longrightarrow F_n(I) = 0$,

then $\sum_{n\in N} F_n$ converges on Γ to $F \in \Delta^*$ such that $DF = \sum_{n\in N} DF_n$;

b) if, for each $n \in N$, ϕ_n is a real function on R^k such that

$\phi_n \in \Lambda^*$, $\sup_{R^k} |\phi_n| \leq \gamma_n$, $\int_{K_n} |\phi_n(x)|\,d\mu \leq \lambda_n$,

$$\{x \in R^k : \phi_n(x) \neq 0\} \subseteq K_n,$$

then $\sum_{n\in N} \phi_n$ converges to $\phi \in \Lambda^*$.

Proof. Let $\{C_n\}_{n\in N}$ be a sequence of compact subsets of R^k such that for each $t > 0$ the set $\{x \in R^k : \sum_{n\in N}\gamma_n \chi_{K_n}(x) < t\}$ can be expressed by a union of a subsequence of $\{C_n\}_{n\in N}$.

Put, for every $n \in N$,

$$K'_n = \bigcup_{m<n} K_m \cup \bigcup_{m<n} \{C_m : C_m \cap K_n = \emptyset\},\quad \lambda_n = \frac{1}{2^n}\min\{d^{k+1}(K_n, K'_n), 1\}$$

To prove a), let $\{F_n\}_{n\in N}$ be a sequence of real functions with the properties described there. $\sum_{n\in N} F_n$ and $\sum_{n\in N} DF_n$ are well-defined (on Γ and R^k respectively).

It will be shown that $D \sum_{n\in N} F_n = \sum_{n\in N} DF_n$.

For this purpose, let $x_o \in \bigcup_{n\in N} K_n$ and let $m \in N$ be such that $x_o \in K_m$.

If n is a positive integer such that $n > m$, then $x_o \notin K_n$ and $x_o \in K'_n$.

Let $\alpha \in (0,1)$ and let $I \in \Gamma_\alpha(x_o)$. There are two possibilities: either $I \cap K_n$ is empty or it is not.

If $I \cap K_n = \emptyset$, from the hypothesis we have $F_n(I) = 0$.

If $I \cap K_n \neq \emptyset$, we have for every $x \in K_n \cap I$ $d(K'_n, K_n) \leq d(x_o, x)$.

Thus

$$\frac{|F_n(I)|}{\mu(I)} \leq \frac{d^{k+1}(K'_n,K_n)}{2^n \mu(I)} \leq \frac{d^{k+1}(x_o,x)}{2^n \mu(I)} \leq \frac{\mathrm{diam}^{k+1}(I)}{2^n \mu(I)} \leq M \frac{\mathrm{diam}(I)}{2^n},$$

where M is a constant, depending only on k and α, such that $\mathrm{diam}^k(I) \leq M\,\mu(I)$ for every $I \in \Gamma_\alpha$.

Consequently, $D\left(\sum_{n>m} F_n\right)(x_o) = 0$.

On the other hand, from the hypothesis we have $DF_n(x_o) = 0$ if $n > m$.

Hence $D\left(\sum_{n>m} F_n\right)(x_o) = \sum_{n>m} DF_n(x_o)$.

Let now $x_o \varepsilon R^k - \bigcup_{n\varepsilon N} K_n$.

For each $t > 0$, let's call p a positive integer such that $x_o \varepsilon C_p$ and $\sum_{n\varepsilon N} \gamma_n \chi_{K_n}(x) < t$ for every $x \varepsilon C_p$.

For $n \leq p$ we have

$F_n(I) = 0$ for every $I \varepsilon \Gamma(x_o)$ with $I \subseteq R^k - \bigcup_{n\leq p} K_n$.

For $n > p$ we assume first that $K_n \cap C_p = \emptyset$.

In this case $C_p \subseteq K'_n$, so that $x_o \varepsilon K'_n$. Then let $I\varepsilon \Gamma(x_o)$.

If $I \cap K_n = \emptyset$, from the hypothesis we have $F_n(I) = 0$.

If $I \cap K_n \neq \emptyset$, let $x \varepsilon I \cap K_n$. Then we have

$$d(K'_n,K_n) \leq d(x_o,x) \leq \mathrm{diam}(I)$$

and, consequently,

$$|F_n(I)| \leq \frac{d^{k+1}(K'_n,K_n)}{2^n} \leq \frac{\mathrm{diam}^{k+1}(I)}{2^n} .$$

Now assume $K_n \cap C_p \neq \emptyset$ and let $x \varepsilon K_n \cap C_p$. Then we have

$$\gamma_n = \sum_{n\varepsilon N} \gamma_n \chi_{K_n}(x) < t.$$

From this and since $|DF_n| \leq \gamma_n \chi_{K_n}$, we have (cf.[8],th.(7.8) page 121)

$$|F_n(I)| \leq t\, \mu(I\cap K_n).$$

To sum up, we may assert that for $I \varepsilon \Gamma_\alpha(x_o)$ with sufficiently small diameter we have

$$\frac{|F_n(I)|}{\mu(I)} \leq \frac{M}{2^n}\,\mathrm{diam}(I) + t\,\frac{\mu(I \cap K_n)}{\mu(I)},$$

where M is the constant mentioned above.

Thus

$$\limsup_{\substack{I \to x_o \\ I \in \Gamma_\alpha(x_o)}} \frac{\left|\sum_{n\in N} F_n(I)\right|}{\mu(I)} \leq \limsup_{\substack{I \to x_o \\ I \in \Gamma_\alpha(x_o)}} (\,M\mathrm{diam}(I) + t) = t\ .$$

This shows that $D\left(\sum_{n\in N} F_n\right)(x_o) = 0$, which togheter with $\sum_{n\in N} DF_n(x_o) = 0$ proves (1).

In order to prove b), first observe that $\phi = \sum_{n\in N} \phi_n$ is well-defined; then, let $x \in R^k$. Putting

$$\psi(x) = \sum_{n:x\in K_n} \phi_n(x) \quad , \qquad \tilde{\psi}(x) = \sum_{n:x\notin K_n} \phi_n(x) \ ,$$

we have $\phi(x) = \psi(x) + \tilde{\psi}(x)$.

Since ψ is bounded and ordinarily approximately continuous, x is a Lebesgue point of ψ .

On the other hand, a) implies that $|\tilde{\psi}|$ is a derivative, which togheter with $\tilde{\psi}(x) = 0$ implies that x is a Lebesgue point of $\tilde{\psi}$. Hence x is a Lebesgue point of ϕ .

Proposition 4. Let X be a bounded measurable subset of R^k, c a positive number, v a measurable function such that $\sup_X |v| \leq c$.

Then for every $\varepsilon > 0$ there are two real additive and absolutely continuous (with respect to μ) interval functions F and G defined on Γ such that

i) $(I \in \Gamma\ ,\ I \cap X = \emptyset) \Longrightarrow F(I) = G(I) = 0$,

ii) $\sup_{I\in\Gamma} |F(I)| < \varepsilon$, $\sup_{I\in\Gamma} |G(I)| < \varepsilon$,

iii) $F \in \Delta^*$, $G \in \Delta^*$, $DF \in \Lambda^*$, $DG \in \Lambda^*$,

iv) $\sup_{R^k} |DF| \leq \max\{c,\sqrt{c}\}$, $\sup_{R^k} |DG| \leq \min\{\sqrt{c},1\}$,

v) $\int_X |v - DF\cdot DG|\,d\mu < \varepsilon$.

Proof . Let J be an interval containing X. Let $\varepsilon > 0$. We can construct a finite family Θ covering X of nonoverlapping intervals with the following property: whenever I is an interval contained in J, the measure of the union of the intervals from Θ which intersect the boundary of I is less than $\varepsilon / 3\max\{1,c\}$.

It follows that X can be written as a union of nonempty, disjoint, measurable sets $X_1,\dots,X_n$ such that for each $i\in\{1,\dots,n\}$

$$\sup_{x,y\in X_i} |v(x) - v(y)| < \varepsilon / 3(\mu(X)+1)$$

and X_i is contained in only one element of Θ.

If I is an interval contained in J, put

$N_I = \{i \in \{1,\dots,n\} : X_i$ intersect the boundary of $I\}$.

We have $\mu\left(\bigcup_{i\in N_I} X_i\right) < \varepsilon / 3\max\{1,c\}$.

Then let P_i and Q_i be closed, disjoint subsets of X_i such that $d_o(x,X_i) = 1$ for $x \in P_i \cup Q_i$, $\mu(P_i) = \mu(Q_i)$ and $\mu(X_i - (P_i \cup Q_i)) < \varepsilon / 3n\max\{1,c\}$.

If $x_i \in X_i$, put

$a_i = \max\{|v(x_i)|, \sqrt{|v(x_i)|}\}$, $b_i = \min\{\sqrt{|v(x_i)|}, 1\} \operatorname{sgn} v(x_i)$.

By proposition 1 there are $f \in \Lambda^*$ and $g \in \Lambda^*$ such that

$f(x)=a_i$ if $x \in P_i$, $f(x)=-a_i$ if $x \in Q_i$, $f(x)=0$ if $x \notin X$,

$g(x)=b_i$ if $x \in P_i$, $g(x)=-b_i$ if $x \in P_i$, $g(x)=0$ if $x \notin X$

and $|f| \le \max_i a_i$, $|g| \le \max_i b_i$.

Then define F and G by

$F(I) = \int_I f\, d\mu$, $G(I) = \int_I g\, d\mu$

for every $I \in \Gamma$.

Since

$$\left|\int_{X_i} f\, d\mu\right| < \frac{\varepsilon}{3n},$$

for $I \in \Gamma$, $I \subseteq J$, we have

$$\left|\int_I f \, d\mu\right| \leq \sum_{i \in N_I} \left|\int_{I \cap X_i} f \, d\mu\right| + \sum_{i \notin N_I} \left|\int_{X_i} f \, d\mu\right| \leq$$

$$\max\{c, \sqrt{c}\} \, \mu\left(\bigcup_{i \in N_I} X_i \cap I\right) + \frac{\varepsilon}{3} < \varepsilon .$$

Applying the same argument to g, we obtain for $I \in \Gamma$, $I \subseteq J$

$$\left|\int_I g \, d\mu\right| < \varepsilon .$$

Hence F and G satisfy i) and ii). In order to prove iii) and iv), it's enough to observe that for $x \in R^k$ we have $DF(x) = f(x)$ and $DG(x) = g(x)$ (cf. [5]).

In order to prove v), we observe that

$DF(x) \cdot DG(x) = a_i \cdot b_i = v(x_i)$ for $x \in P_i \cup Q_i$, so

$$\int_{P_i \cup Q_i} |v(x_i) - DF \cdot DG| \, d\mu = 0.$$

Hence

$$\int_X |v - DF \cdot DG| \, d\mu \leq \sum_{i=1}^n \int_{X_i} |v - v(x_i)| \, d\mu +$$

$$\sum_{i=1}^n \int_{X_i - (P_i \cup Q_i)} |v(x_i) - DF \cdot DG| \, d\mu < \varepsilon$$

which completes the proof.

Proposition 5 . Let $u \in B^1$. Then there are F, G and H belonging to Δ^* such that $u = DF \cdot DG + DH$. Moreover, one can find this representation such that DG is bounded, $DH \in \Lambda^*$ and, if u is bounded, such that DF and DH are also bounded.

Proof. Let $v \in B^1$, let $\{H_n\}_{n \in N}$ be a sequence of pairwise disjoint compact subsets of R^k and let $\{\beta_n\}_{n \in N}$ be a sequence of positive numbers as in proposition 2.

For $n \in N$, put $\gamma_n = 2 \max\{\beta_n, \sqrt{\beta_n}\}$. For the sequences $\{H_n\}_{n \in N}$ and $\{\gamma_n\}_{n \in N}$ we find a sequence of positive numbers $\{\lambda_n\}_{n \in N}$ according to proposition 3.

Since, for each $n \in N$, $\sup_{x \in H_n} |v(x)| \leq \beta_n$, we use proposition 4 with $X = H_n$, $c = \beta_n$ and $\varepsilon = \lambda_n$ to construct functions F_n and G_n with the properties described there.

Proposition 3 implies that $\sum_{n \in N} F_n$ and $\sum_{n \in N} G_n$ converge to $F \in \Delta^*$ and $G \in \Delta^*$ respectively and $DF = \sum_{n \in N} DF_n$, $DG = \sum_{n \in N} DG_n$.

Using ii) from proposition 2, the properties of F_n and G_n described in proposition 4 and b) from proposition 3, we obtain

$$v - DF \cdot DG = \sum_{n \in N} (v - DF_n \cdot DG_n)\chi_{H_n} \in \Lambda^*.$$

Since u- v also belong to Λ^* , $(u - v) + (v - DF \cdot DG) = DH$, where $H \in \Delta^*$.

The proof is complete when we observe that

$u = (u - v) + (v - DF \cdot DG) + DF \cdot DG.$

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