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A VECTOR SPACE OF MORSE FUNCTIONS

1. Let F be a continuous even function (that is, $F(x) = F(-x)$) on the closed interval $-1 \leq x \leq 1$. We say that an extended real number r is a right (left) derived number of F at a point $x \in (-1,1)$ if there exists a sequence (x_n) converging to x such that

$$\lim (F(x_n) - F(x))(x_n-x)^{-1} = r$$

and $x_n > x$ ($x_n < x$) for each n. In [2] Anthony Morse constructed such a function F with the property that F has both a finite and an infinite right derived number and both a finite and an infinite left derived number at each $x \in (-1,1)$. We say that a function F satisfying this condition is a Morse function. Thus a Morse function can have no finite or infinite unilateral derivative at any $x \in (-1,1)$. In this paper we construct another Morse function by using Cantor-like sets [1] of positive measure. Our work will require much less numerical computation than the example in [2], and will be shorter.

The continuous functions on $[-1,1]$ form a real vector space under the usual operations of pointwise addition and scalar multiplication, and form a Banach space under the sup norm. We will prove that there exists a real vector subspace of dimension $c = 2^{\aleph_0}$ of this space, all of whose nonzero vectors are Morse functions. We will also deduce that there exists such a vector space that is dense in the space of all continuous functions on $[0,1]$ under the sup norm.

There are c vectors in any basis of our vector space. One cannot obtain any one of these functions cheaply from the others.

2. In this section, we provide the notation and definitions needed for the construction of our functions. Let $[a,b]$ be an interval of length ≤ 2. By the 1-set on $[a,b]$ we mean the Cantor-like set constructed as follows. Let the open interval $I(1,1)$ be the middle 4-th part of $[a,b]$, and delete $I(1,1)$

from [a,b]. This is the first step. Delete the middle 8-th parts $I(1,2)$, $I(2,2)$ (from left to right) of the remaining two intervals. Delete the middle 16-th parts $I(1,3)$, $I(2,3)$, $I(3,3)$, $I(4,3)$ (from left to right) of the remaining 4 intervals.

At the n-th step, there are 2^{n-1} intervals remaining, all of the same length, L_n. We delete from them their middle 2^{n+1}-th parts, $I(1,n)$, $I(2,n),\dots,I(2^{n-1},n)$ (from left to right). And so forth. Finally, the set $E = [a,b]\setminus \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^{n-1}} I(j,n)$ is the 1-set on $[a,b]$. Then E is a nowhere dense perfect set, and any open interval K that meets E, necessarily meets E in a set of positive measure; this follows from the fact that the measure of E is $\prod_{n=1}^{\infty} (1-2^{-n-1})(b-a) > 0$, and the part of E in any one interval remaining at the n-th step is congruent to the part in any other such interval.

For convenience of language, if I is a concentric subinterval of an interval J, we say that I <u>bisects</u> J. In the preceding construction, $I(j,n)$ bisects an interval whose length is L_n, and the length of $I(j,n)$ is $2^{-n-1}L_n$.

Now let i be a positive integer. Partition $[a,b]$ into i closed nonoverlapping subintervals $J_1,\dots,J_i$, of equal length. Let E_j be the 1-set on J_j $(j = 1,\dots,i)$. We call $S = \bigcup_{j=1}^{i} E_j$ the <u>i-set</u> on $[a,b]$. Then S is also a nowhere dense perfect set.

We define a function g on $[a,b]$ as follows. Let E be the 1-set on $[a,b]$. On the closure of each interval $I(1,n) \cap [a,\frac{1}{2}(a+b)]$ let $g = L_n$ if n is even and $g = L_n^{1/2}$ if n is odd. If v is the left endpoint of $I(1,n)$ and u is the right endpoint of $I(1,n+1)$, let

$$g(x) = g(u) + (g(v)-g(u))\lambda((u,x) \cap E)\lambda((u,v) \cap E)^{-1}$$

for $u < x < v$ where λ denotes Lebesgue measure. Let $g(a) = 0$. Thus g is defined on $[a,\frac{1}{2}(a+b)]$. Make g symmetric about $\frac{1}{2}(a+b)$. Then g is defined and continuous on $[a,b]$ and vanishes only at a and b. For any $x \in (a,b)$, g is a Lipschitz function bounded away from 0 on some neighborhood of x because $\lambda((u,v) \cap E) \geq (v-u) \prod_{n=1}^{\infty} (1-2^{-n-1})$ in our definition of g. Moreover, g is constant on each interval $I(j,n)$ – in such a situation we write $g(I(j,n))$ for the constant value – and g obtains its maximum value on $I(1,1)$.

More generally, let i be any positive integer. Partition $[a,b]$ into i nonoverlapping subintervals $J_1,\ldots,J_i$ of equal length. Let G_j be the function defined on J_j $(j = 1,\ldots,i)$ the same way g was defined on $[a,b]$. Let g_i coincide with G_j on J_j. Then g_i is defined and continuous on $[a,b]$.

Definition 1. Let g_i be the function defined in the preceding paragraph. By an i-<u>function</u> on $[a,b]$ we mean the function tg_i for some constant $t \geq 1$.

We turn now to our generator functions $F_1,F_2,\ldots$ on $[-1,1]$. Let E_1 be the 4-set on $[-1,1]$ and let F_1 be a 4-function on $[-1,1]$. Let F_2 vanish on E_1. To complete the definition of F_2, let I be a complementary interval of E_1. Then there is an interval $J \supset I$ for which $J \cap E_1$ is the 1-set on J. Say $I = I(j,n)$ in our notation. Let M = minimum of $F_1(I(j,n))$ and $F_1(I(1,n))$. Let i be the smallest integer for which the maximum of some i-function on I equals $\frac{1}{4}M$. (In other words, i is the smallest integer for which $(\lambda(I)/i)^{\frac{1}{2}} \leq \frac{1}{4}M$.) Let F_2 coincide with that i-function on I, and let $A(j,n)$ be the i-set on I. Then F_2 is continuous on J. (Note that $F_1(I(1,n)) \to 0$ and hence $\max F_2(I(j,n)) \to 0$ as $n \to \infty$.) It follows that F_1 and F_2 are defined and continuous on $[-1,1]$, and $F_2 \leq \frac{1}{4}F_1$. Form E_2 by adjoining to E_1 all the sets $A(j,n)$ constructed in this way on the complementary intervals of E_1. Then E_2 is also a nowhere dense perfect set.

We construct E_3 and F_3 from E_2 and F_2 the same way E_2 and F_2 were constructed from E_1 and F_1. In general, we construct E_{n+1} and F_{n+1} from E_n and F_n the same way E_2 and F_2 were constructed from E_1 and F_1. For each n, E_n is a nowhere dense perfect set, F_n is continuous on $[-1,1]$, $F_{n+1} \leq \frac{1}{4}F_n$, and equality holds on certain subintervals of $[-1,1]$. Moreover, F_n vanishes on E_j if $n > j$.

Let $[a,b]$ be an interval such that F_n coincides with a 1-function on $[a,b]$, and let $F_n(I(j,m)) \geq F_n(I(1,m))$ for some j and m. Then there is a subinterval K of $I(j,m)$ with endpoints in E_{n+1} on which F_{n+1} is constant and $F_{n+1} = \frac{1}{4}F_n(I(1,m))$. Let k be any positive integer. Repeated application of this principle (with $m = 1$) shows there is a subinterval of $I(j,m)$ with endpoints in E_{n+k} on which F_{n+k} is constant and $F_{n+k} = (\frac{1}{4})^k F_n(I(1,m))$. If x is one of the endpoints, then $x \in E_{n+k}$ and $F_{n+k}(x) = (\frac{1}{4})^k F_n(I(1,m))$. In particular, if $m = 1$, then $F_{n+k}(x) = (\frac{1}{4})^k F_n(I(1,1))$.

Observe that if (c_n) is a bounded sequence of real numbers, then $\Sigma_n\, c_n F_n$ sums to a continuous function on $[-1,1]$. In Section 3 we will prove:

Theorem 1. Let (c_n) be a bounded sequence of real numbers such that $\limsup c_n > 0$ and $\limsup c_n + \liminf c_n = 0$. Then $F = \Sigma_n\, c_n F_n$ is a Morse function on $[-1,1]$.

In particular, $\Sigma(-1)^n F_n$ is a Morse function. (Compare with the function F in [2].) But there are other possibilities. For example, if $c_n = -1$ when n is a prime integer, and $c_n = 1$ otherwise, then (c_n) satisfies the hypothesis of Theorem 1. In Section 4 we will construct a vector space all of whose nonzero vectors are some of the functions F described in Theorem 1.

For convenience we let $\Delta f(x,u)$ denote $(f(x)-f(u))(x-u)^{-1}$ for any function f. Let E_0 be the void set.

3. We begin with a nuts and bolts lemma that is essentially all we need to prove Theorem 1.

Lemma 1. Let $[a,b]$ be an interval of length ≤ 2, let E be a 1-set on $[a,b]$, and let F be a 1-function on $[a,b]$. Let x be any point with $a \leq x < b$.

(i) There exist $z \in E$, $w \in E$ $(z > x,\ w > x)$ such that for every real number y,

$$|(F(w)-y)(w-x)^{-1}| + |(F(z)-y)(z-x)^{-1}| \geq \tfrac{1}{4}(z-x)^{-\frac{1}{2}} - 6.$$

(ii) If $x \in E$ is a right accumulation point of E, then there are points a_i, b_i and indices j_i, n_i $(i = 1,2,\ldots)$ such that $(a_i,b_i) = I(j_i,n_i)$, $F(I(j_i,n_i)) \geq F(I(1,n_i)) \geq \frac{1}{2}(b_i-x)^{\frac{1}{2}}$, $\lim a_i = \lim b_i = x$, and $a_i > x$ for all i.

(iii) If $x \in E$ is a right accumulation point of E, there are points $w_i \in E$ $(i = 1,2,\ldots)$ such that $\lim w_i = x$, $w_i > x$ for all i, and the sequence $(\Delta F(x,w_i))_i$ is bounded.

Proof. In the notation of Section 2, $F = tg$ for some constant $t \geq 1$. Without loss of generality, we let $t = 1$.

(i) Let n be the smallest index for which some interval $I(\cdot,n)$ has an endpoint to the right of x. Indeed there is at most one such interval $I(j,n)$ for this index n; if there were two, a proscribed interval would lie between them. Say $(w',w) = I(j,n)$ and (w',w) bisects the interval $[s,b]$. Let $I(k,n+1)$ be the (unique) interval bisecting the interval $[w,b]$. Say $(z',z) = I(k,n+1)$. Observe that $s < w' < w < z' < z < b$ and $s \leq x < w$. Moreover $b-w = w'-s > w-w'$ because (w',w) bisects $[s,b]$, and $b-w < 2(z-w)$ because (z',z) bisects $[w,b]$. Thus $z-x < b-s = (b-w) + (w-w') + (w'-s) < 4(b-w)$ and

$$(z-x)^{\frac{1}{2}} < 2(b-w)^{\frac{1}{2}}. \tag{1}$$

Likewise $b-s = (b-w) + (w-w') + (w'-s) < 3(b-w) < 6(z-w)$ and

$$b-s < 6(z-x). \tag{2}$$

If n is even, then $F(w) = L_n = b-s$ and $F(z) = L_{n+1}^{\frac{1}{2}} = (b-w)^{\frac{1}{2}}$; if n is odd, then $F(w) = L_n^{\frac{1}{2}} = (b-s)^{\frac{1}{2}}$ and $F(z) = L_{n+1} = b-w$. Because $b-s > b-w$ we have in either case

$$|F(w) - F(z)| \geq (b-w)^{\frac{1}{2}} - (b-s). \tag{3}$$

From (1), (2) and (3) we obtain

$$|F(w) - F(z)| \geq \tfrac{1}{2}(z-x)^{\frac{1}{2}} - 6(z-x). \tag{4}$$

Finally, $z-x > w-x$, so

$$|(F(w)-y)(w-x)^{-1}| + |(F(z)-y)(z-x)^{-1}| \geq (|F(w)-y| + |F(z)-y|)(z-x)^{-1}, \text{ and}$$

(5)

$$|(F(w)-y)(w-x)^{-1}| + |(F(z)-y)(z-x)^{-1}| \geq |F(w) - F(z)|(z-x)^{-1}.$$

The conclusion for (i) follows from (4) and (5).

(ii) We assume without loss of generality that $a < x < b$; for if $x = a$, then $(a_i,b_i) = I(1,2i+1)$ suffices. So $F(x) > 0 = F(a)$. Let $\varepsilon > 0$ be such that $\inf F(x,x+\varepsilon) > \sup F(a,a+2\varepsilon)$. Let m' be the smallest index for which some interval $I(\cdot,m')$ meets $(x,x + \varepsilon)$. Indeed there is at most one such interval $I(j,m')$ for this index m'; if there were two, a proscribed interval would lie between them. Say $(u_3',u_3) = I(j,m')$. Let m be the smallest index $> m'$ for which some interval $I(\cdot,m)$ meets (x,u_3'). Again there is only one such $I(k,m)$ for this index m. Say $(u_1',u_1) = I(k,m)$ and (u_1',u_1) bisects the interval $[q,u_3']$. Let $(u_2',u_2) = I(\cdot,m+1)$ be the (unique) interval bisecting $[u_1,u_3']$. Observe that $q \leq x < u_1' < u_1 < u_2' < u_2 < u_3'$, and $u_3'-u_1 = u_1'-q \geq \max(u_1'-x,u_1-u_1')$ because (u_1',u_1) bisects $[q,u_3']$. Hence $L_m = u_3'-q \geq u_1-x$ and $u_3' - x = (u_3'-u_1) + (u_1-u_1') + (u_1'-x) \leq 3(u_3'-u_1) = 3L_{m+1}$. It follows that $u_2 - x \leq u_3'-x \leq 3L_{m+1}$ and

(6)
$$L_{m+1}^{1/2} \geq 1/2(u_2-x)^{1/2} ,$$

(7)
$$L_m^{1/2} \geq (u_1-x)^{1/2}.$$

The distance between (u_1',u_1) and (u_3',u_3) is $< \varepsilon$, so the distance from a to $I(1,m)$ is $< \varepsilon$. Moreover $u_1 - u_1' < \varepsilon$, so $I(1,m) \subset (a,a + 2\varepsilon)$. Hence $I(1,m+1) \subset (a,a + 2\varepsilon)$ also. It follows that $F(u_1',u_1) \geq F(I(1,m))$, and $F(u_2',u_2) \geq F(I(1,m+1))$. If m is odd, then by (7), we have $F(I(1,m) \geq 1/2(u_1-x)^{1/2}$; if m is even, then by (6), we have $F(I(1,m+1)) \geq 1/2(u_2-x)^{1/2}$. Because ε is arbitrarily small, the conclusion of (ii) is clear.

(iii) The conclusion is obvious when $a < x < b$; for in the case, $F(x) > 0$ and F is a Lipschitz function on some neighborhood of x. But if $x = a$, we need only make w_i the right endpoint of $I(1,2i)$. □

We now turn immediately to Theorem 1.

Proof of Theorem 1. Each F_n is an even function by construction, and hence F must be even. It remains only to prove that F has finite and infinite right (left) derived numbers at each $x \in (-1,1)$. We will show this only for right derived numbers; the result for left derived numbers follows from the fact that F is an even function. Select any $x \in (-1,1)$. We divide our proof into three cases.

Case 1. $x \in (-1,1) \setminus \bigcup_{n=1}^{\infty} E_n$.

For each $n \geq 1$, x lies in a complementary interval J_n of E_{n-1} where F_n coincides with some i-function on J_n. It follows from Lemma 1(i) that for each $n > 1$ there are points x_n and $x_n' \in E_n \cap J_n$ such that $x_n > x$, $x_n' > x$ and

$$(1) \qquad |(y_n-F_n(x_n))(x-x_n)^{-1}| + |(y_n-F_n(x_n'))(x-x_n')^{-1}| \geq \tfrac{1}{2}(x_n-x)^{-\frac{1}{2}} - 6$$

for every real number y_n. Moreover $F_1,\ldots,F_{n-1}$ are each constant on an interval containing x, x_n, x_n', and $F_j(x) = F_j(x_n) = F_j(x_n')$ for $1 \leq j \leq n-1$. Also $F_j(x_n) = F_j(x_n') = 0$ for $j > n$. For those n satisfying $c_n \neq 0$, put $y_n = \sum_{j=n}^{\infty} c_jF_j(x)/c_n$. It follows that for these n

$$|\Delta F(x,x_n)| = |c_n(y_n-F_n(x_n))(x-x_n)^{-1}|,$$

$$|\Delta F(x,x_n')| = |c_n(y_n-F_n(x_n'))(x-x_n')^{-1}|.$$

From (1) we obtain

$$(2) \qquad |c_n|^{-1}(|\Delta F(x,x_n)| + |\Delta F(x,x_n')|) \geq \tfrac{1}{2}(x_n-x)^{-\frac{1}{2}} - 6.$$

But the length of the interval J_n containing x, x_n and x_n' tends to 0 as $n \to \infty$. It follows from this and the fact that $\limsup c_n > 0$, that F has an infinite right derived number at x.

Now let w_n be the right endpoint of the interval J_n. Then $F_j(x) = F_j(w_n)$ for $1 \leq j \leq n-1$, $F_j(w_n) = 0$ for $j \geq n$ and $F_n(x) > 0$. Moreover, $F_j \leq (\tfrac{1}{4})^{j-n}F_n$ for $j > n$ by construction. It follows that

$$(3) \qquad \Delta F(x,w_n) = -c_nF_n(x)(x_n-x)^{-1} - \sum_{j=n+1}^{\infty} -c_jF_j(x)(w_n-x)^{-1}.$$

For $c_n > \tfrac{1}{2} \limsup_j |c_j|$ it follows from (3) that

$$(4) \qquad \Delta F(x,w_n) < -\tfrac{1}{2}\, c_nF_n(x)(w_n-x)^{-1} + \sum_{j=n+1}^{\infty} |c_j|F_j(x)(w_n-x)^{-1} < 0.$$

Thus F has nonpositive right derived number at x. Similarly, F has a nonnegative right derived number at x (use an analogous argument with $c_n < -\tfrac{1}{2} \limsup_j |c_j|$). Because F is continuous, it follows that 0 is a right derived number of F at x. This completes case 1.

Case 2. $x \in E_n \setminus E_{n-1}$ for some $n \geq 1$ and x is a right accumulation point of E_n.

Let k be the positive integer such that $n < j < n+k$ implies $c_j = 0$ and $c_{n+k} \neq 0$. Let (a',b') be the complementary interval of E_{n-1} containing x. Then $E_n \cap [a',b']$ is an i-set and F_n coincides with an i-function on $[a',b']$. There is a subinterval $[a,b]$ of $[a',b']$ such that $E_n \cap [a,b]$ is a 1-set, $x \in [a,b)$ and F_n coincides with a 1-function on $[a,b]$. We use the notation of section 2 for this 1-set.

By Lemma 1(ii) there are numbers a_i, b_i and indices j_i, n_i $(i = 1,2,\ldots)$ such that $(a_i,b_i) = I(j_i,n_i)$, $F_n(I(j_i,n_i)) \geq F_n(I(1,n_i)) \geq \tfrac{1}{2}(b_i-x)^{\frac{1}{2}}$, $\lim a_i = \lim b_i = x$ and $a_i > x$ for each i. In section 2 we saw that there is a $z_i \in E_{n+k} \cap (a_i,b_i)$ such that $F_{n+k}(z_i) = (\tfrac{1}{4})^kF_n(I(1,n_i))$ and hence

$$(5) \qquad F_{n+k}(z_i) \geq \tfrac{1}{2}(\tfrac{1}{4})^k(b_i-x)^{\frac{1}{2}} .$$

We observe that $F_j(x) = F_j(a_i) = F_j(b_i) = F_j(z_i)$ for $1 \leq j \leq n-1$, $F_n(a_i) = F_n(b_i) = F_n(z_i)$, $F_j(x) = F_j(a_i) = F_j(b_i) = 0$ for $j > n$, and $F_j(z_i) = 0$ for $j > n+k$. It follows that

$$\Delta F(x,z_i) = c_n \Delta F_n(x,z_i) + c_{n+k} F_{n+k}(z_i)(z_i-x)^{-1}, \tag{6}$$

$$\Delta F(x,a_i) = c_n \Delta F_n(x,a_i) \; . \tag{7}$$

From (5) we obtain

$$F_{n+k}(z_i)(z_i-x)^{-1} \geq \tfrac{1}{2}(\tfrac{3}{4})^k (b_i-x)^{-\frac{1}{2}} \; . \tag{8}$$

Also $|\Delta F_n(x,z_i)| = |(F_n(z_i)-F_n(x))(z_i-x)^{-1}| \leq |(F_n(a_i)-F_n(x))(a_i-x)^{-1}| = |\Delta F_n(x,a_i)|$ and by (7),

$$|c_n \Delta F_n(x,z_i)| \leq |\Delta F(x,a_i)| . \tag{9}$$

Finally, if the sequence $(c_n \Delta F_n(x,z_i))_i$ is bounded, then it follows from (6) and (8) that F has an infinite right derived number at x; if $(c_n \Delta F_n(x,z_i))_i$ is unbounded, it follows from (9) that F has an infinite right derived number at x. In either case, F has an infinite right derived number at x.

By Lemma 1(iii), there is a sequence $(w_i) \subset E_n$ such that $w_i \searrow x$ and $\Delta F_n(x,w_i)$ is bounded as $i \to \infty$. Clearly

$$\Delta F(x,w_i) = \Delta F_n(x,w_i) c_n \; , \tag{10}$$

so F has finite right derived number at x. This concludes case 2.

Case 3. x is the left endpoint of a complementary interval J of E_n for an $n > 0$.

By construction F_{n+1} coincides with an i-function on J. The proof is complete as in case 2 using F_{n+1} in place of F_n and E_{n+1} in place of E_n. So we leave it. □

It can be shown that F has finite and infinite right derived numbers at $x = -1$ and F has finite and infinite left derived numbers at $x = 1$. The proof is similar to case 2, but we will not do it here.

4. Let V denote the vector space of bounded sequences of rational numbers over the rational field Q and let W denote the vector space of bounded sequences of real numbers over the real field R. Let $v_1,\dots,v_n$ be vectors in V that are linearly dependent in W. Let M denote the matrix with infinitely many columns whose rows are the vectors (sequences) $v_1,\dots,v_n$. The determinant of any n by n matrix whose columns are n columns of M is zero. It follows that $v_1,\dots,v_n$ are also linearly dependent in V. Thus any basis of V can be extended to a basis of W. But the power of V is c and Q is countable. Thus the dimensions of V and W are both c.

Theorem 2. There is a real vector space of dimension c of continuous functions on $[-1,1]$, under the usual operations of addition and scalar multiplication, such that every nonzero vector in the space is a Morse function.

Proof. Let $s(1), s(2), s(3), \dots$ be the sequence of numbers $1,1,2,1,2,3,1,2,3,4,1,2,3,4,5,\dots$. Let W be the vector space of bounded sequences of real numbers over the real field R. Then W has dimension c. We construct an isomorphism of W into the vector space of continuous functions on $[-1,1]$ as follows. For any sequence $d = d_1,d_2,d_3,\dots$ in W, let $q(d)$ be the function $\sum_{j=1}^{\infty} (-1)^j d_{s(j)} F_j$. That q is a vector space homomorphism is clear. Moreover, if d is not the zero sequence, the coefficients $(-1)^j d_{s(j)}$ satisfy the hypothesis of Theorem 1, and $q(d)$ is a Morse function. Note that if $d_i \neq 0$, then $(-1)^j d_{s(j)} = d_i$ for infinitely many j, and $(-1)^j d_{s(j)} = -d_i$ for infinitely many j. So q is an isomorphism and $q(W)$ is the desired vector space.

Let S denote the vector space $q(W)$ constructed in the proof of Theorem 2. Since W is a Banach space under the sup norm, q induces a norm on S that makes S a Banach space. The topology of this norm on S, however, is finer than the topology of the sup norm on S.

Theorem 3. There exists a real vector space Sol of functions satisfying the condition of Theorem 2 such that the restrictions of the functions in Sol to $[0,1]$ form a dense subset of the family of all continuous functions on $[0,1]$ under the sup norm.

Proof. Let $p_1, p_2, \ldots, p_n, \ldots$ be a countable set of polynomials in x^2 with rational coefficients that is dense in the family of all continuous functions on $[0,1]$. Let T be a basis of any space S satisfying the conditions of Theorem 2. Let $\{f_1, f_2 \ldots, f_n, \ldots\}$ be a countably infinite subset of T. Put $g_n = p_n + (n \sup|f_n|)^{-1} f_n$. Let G denote the family of all the functions g_n together with all the functions in $T \setminus \{f_1, f_2, \ldots, f_n, \ldots\}$. Let S_0 be the vector space generated by the functions in G. Then the restrictions of the functions in S_0 to $[0,1]$ are dense in the family of continuous functions on $[0,1]$ under the sup norm because the p_n are dense. Note that if $\Sigma_i a_i h_i$ is a linear combination of functions in G, not all $a_i = 0$, then $\Sigma_i a_i h_i = f + p$ where f is a Morse function and p is a polynomial in x^2. Then $f + p$ is a Morse function on $[-1,1]$.

References

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