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A BAIRE TWO FUNCTION WITH NON-BOREL UPPER SYMMETRIC DERIVATIVE

The upper symmetric derivative $\overline{D}f$ of the function $f:\mathbb{R}\to\mathbb{R}$ is defined by

$$\overline{D}f(x) = \limsup_{h\to 0} \frac{f(x+h)-f(x-h)}{2h} \qquad (x\in\mathbb{R}).$$

In [7] E. Wajch proved that if f is Baire 1 then $\overline{D}f$ is Baire 3. Motivated by Banach's theorem [1] stating that the Dini derivatives of a Baire α function belong to the Baire $\alpha+2$ class, E. Wajch asked whether $\overline{D}f$ has the same property. In this paper we answer her question in the negative by constructing a Baire 2 function f such that $\overline{D}f$ is not Borel measurable.

We remark that if f is Lebesgue measurable then so is $\overline{D}f$ (see [2], 2.4). However, for an arbitrary f, $\overline{D}f$ need not be Lebesgue measurable [3]. This shows another difference in the behaviour of symmetric and ordinary derivatives, since the upper bilateral derivative of an arbitrary function is Baire 2 [4].

THEOREM. There exists a Baire 2 function $f:\mathbb{R}\to\mathbb{R}$ such that $\overline{D}f$ is not Borel measurable. The function f may be taken as the characteristic function of a set $A\cup B$, where A is G_δ and B is F_σ.

PROOF. If A is G_δ and B is F_σ then $H=A\cup B$ is simultaneously $G_{\delta\sigma}$ and $F_{\sigma\delta}$ and hence χ_H, the characteristic function of H, is Baire 2.

For $H\subset\mathbb{R}$ we shall denote by $D(H)$ the set of those points $x\in\mathbb{R}$ for which there exists a sequence $\{h_k\}_{k=1}^{\infty}$ of positive numbers such that $h_k\to 0$, $x+h_k\in H$, but $x-h_k\notin H$ $(k=1,2,\ldots)$. It is easy to check that $\overline{D}\chi_H(x)=\infty$ for $x\in D(H)$ and $\overline{D}\chi_H(x)\leqq 0$ for $x\notin D(H)$. We shall construct a G_δ set A and an F_σ set B such that $D(A\cup B)$ is not Borel; it then follows that $\chi_{A\cup B}$ satisfies the requirements of the theorem.

It is well-known that there exists a non-empty perfect set P such that the elements of P are linearly independent over the rational numbers. (In [6] von Neumann constructs a strictly increasing function φ defined on $(0,\infty)$ such that the elements of $R(\varphi)$, the range of φ, are linearly independent (moreover, algebraically independent) over the rationals. Since $R(\varphi)$ is uncountable and G_δ , it contains a non-empty perfect set.)

The condition that the elements of P are linearly independent over the rationals may be expressed in the following form. If $x_1,\dots,x_n \in P$, $r_1,\dots,r_n$ are rationals and $\sum_1^n r_i x_i = 0$, then for every i, $\sum_{x_j=x_i} r_j = 0$.

We shall refer to this property as property (I). We may assume that P consists of positive elements; otherwise we take $P' = \{|x| : x \in P\}$.

Let P_1 and P_2 be disjoint, non-empty, bounded and perfect subsets of P and let K be an analytic and non-Borel subset of P_1.

We shall construct A and B in such a way that

$$D(A \cup B) \cap P_1 = K \tag{1}$$

holds. This will imply that $D(A \cup B)$ is not Borel.

First we choose a G_δ set $U \subset P_1 \times P_2$ such that $K = \{ x \in P_1 :$ there is $y \in P_2$ with $(x,y) \in U \}$. (See [5], §38, IV, Remark 2 and §36, V, Corollary 2.)

For every natural number k we define

$$A_k = \{ x + \frac{1}{k} y : (x,y) \in U \}.$$

We prove that A_k is G_δ. Let $g: (P_1 \times P_2) \to \mathbb{R}$ be defined by $g(x,y) = x + \frac{1}{k} y$ $(x \in P_1,\ y \in P_2)$. Then g is continuous and one-to-one on $P_1 \times P_2$. Indeed, if $(x_i, y_i) \in P_1 \times P_2$ $(i=1,2)$ and $x_1 + \frac{1}{k} y_1 = x_2 + \frac{1}{k} y_2$ then $x_1 = x_2$ and $y_1 = y_2$ by property (I).

Now $U \subset P_1 \times P_2$ is G_δ and hence $(P_1 \times P_2) \setminus U$ is an F_σ subset of $\mathbb{R}^2$. This implies that $g((P_1 \times P_2) \setminus U) = V$ is F_σ and $A_k = g(U) = g(P_1 \times P_2) \setminus V$ is G_δ.

We define $A = \bigcup_{k=2}^{\infty} A_k$; we show that A is G_δ, too. To prove this we first remark that $A_k \cap P_1 = \emptyset$ for every k, by property (I). On the other hand, for every $x \in A_k$, $\mathrm{dist}(P_1,x) \leqq \frac{m}{k}$, where $m = \max P_2$. Therefore we have

$$A = \bigcup_{k=2}^{\infty} A_k = \bigcap_{i=2}^{\infty} \left\{ \{x:\ 0 < \mathrm{dist}(P_1,x) \leqq \tfrac{m}{i}\} \cup \bigcup_{k=2}^{i} A_k \right\} .$$

Since each term of the intersection is G_δ, so is A.

Now we turn to the construction of the set B. We put

$$B_1 = \{\ 2x_1 - x_2 - \tfrac{1}{k}\, y\ :\ x_1,x_2 \in P_1,\ y \in P_2,\ x_1 \neq x_2,\ x_1 < x_2 + \tfrac{1}{k}\, y\ ,\ k \in \mathbb{N},\ k \geqq 2\ \}.$$

Next we define the sets B_n inductively by

$$B_n = \{\ x-h\ :\ x \in P_1,\ h > 0,\ x+h \in B_{n-1}\ \} \qquad (n \geqq 2).$$

We show by induction on n that each B_n is F_σ. Let

$$F_k = \{\ (x,y,z)\ :\ x \neq y,\ x < y + \tfrac{1}{k} z\ \} \cap (P_1 \times P_1 \times P_2)$$

for every $k=1,2,\ldots$. It is easy to see that each F_k is an F_σ subset of $\mathbb{R}^3$. Since the map $g_k(x,y,z) = 2x - y - \frac{1}{k} z$ is continuous, this implies that $g_k(F_k)$ is F_σ for every k. Therefore $B_1 = \bigcup_{k=2}^{\infty} g_k(F_k)$ is F_σ. Let $n > 1$ and suppose that B_{n-1} is F_σ. Then B_n is the image of the F_σ set

$$\{\ (x,y)\ :\ y > x\ \} \cap (P_1 \times B_{n-1})$$

under the continuous map $G(x,y) = 2x - y$. Consequently, B_n is F_σ for every n.

We define $B = \bigcup_{n=1}^{\infty} B_n$, then it follows that B is also F_σ.

We are going to verify (1). We have to show that for any $x \in P_1$, $x \in K$ if and only if $x \in D(A \cup B)$.

Let $x \in P_1 \setminus K$. In order to show $x \notin D(A \cup B)$ it is enough to prove that for every $h > 0$, $x+h \in A \cup B$ implies $x-h \in A \cup B$.

Let $h > 0$ be given and suppose first $x+h \in A$. Then $x+h = x_1 + \frac{1}{k} y$, where $(x_1, y) \in U$ and $k \geqq 2$. Since $x \notin K$, we have $x \neq x_1$ by the definition of U. Hence $x-h = 2x - (x+h) = 2x - x_1 - \frac{1}{k} y$, where $x, x_1 \in P_1$, $y \in P_2$, and $x \neq x_1$. Also, because $h > 0$, it follows that $x < x_1 + \frac{1}{k} y$. Therefore $x-h \in B_1$ and hence $x-h \in B$.

If $x+h \in B$, then $x+h \in B_n$ for some $n \geqq 1$ and thus $x-h \in B_{n+1}$ by the definition of the sets B_n. Hence $x+h \in A \cup B$ implies $x-h \in A \cup B$ as we stated.

Next let $x \in K$ be fixed. Our aim is to show that $x \in D(A \cup B)$. Since $x \in K$, there is a $y \in P_2$ such that $(x,y) \in U$. Let $h_k = \frac{1}{k} y$ $(k=2,3,\dots)$, then $h_k > 0$, $h_k \to 0$ and $x+h_k = x + \frac{1}{k} y \in A_k \subset A \cup B$ holds for every $k \geqq 2$. In order to complete the proof of $x \in D(A \cup B)$ it is enough to show that $x - \frac{1}{k} y \notin A \cup B$ for every $k \geqq 2$.

Suppose that $x - \frac{1}{k} y \in A$. Then $x - \frac{1}{k} y = x_1 + \frac{1}{j} y_1$, where $x_1 \in P_1$, $y_1 \in P_2$ and $j \geqq 2$. However, this clearly contradicts property (I).

Finally, suppose that $x - \frac{1}{k} y \in B_n$ with $n \geqq 1$. We show that every $z \in B_n$ can be written in the form

(2) $\quad z = 2(x_n - x_{n-1} + \ldots + (-1)^{n-1} x_1) + (-1)^n (x_o + \frac{1}{j} u)$,

where $x_o, x_1, \ldots, x_n \in P_1$, $u \in P_2$, $j \geqq 2$ and

$x_1 > x_2 > \ldots > x_n > z$.

This statement will be proved by induction on n. For $n=1$ the assertion follows from the definition of B_1. Suppose the assertion is valid for $n-1 \geqq 1$ and let $z \in B_n$. Then there exist $x_n \in P_1$ and $h > 0$ such that $z = x_n - h$ and $x_n + h \in B_{n-1}$. By the induction hypothesis we have

(3) $$x_n + h = 2(x_{n-1} - x_{n-2} + \ldots + (-1)^{n-2} x_1) + \\ + (-1)^{n-1}(x_o + \frac{1}{j} u),$$

where $x_o, \ldots, x_n \in P_1$, $u \in P_2$, $j \geqq 2$ and $x_1 > \ldots > x_{n-1} > x_n + h$. Then $z = x_n - h < x_n$. Also, $z = 2x_n - (x_n + h)$, and, by substituting the value of $x_n + h$ given by (3), we obtain (2).

Now, if $x - \frac{1}{k} y \in B_n$ then $x - \frac{1}{k} y$ equals the right hand side of (2). Using property (I) it follows that $j = k$, n is odd, $y = u$ and hence

$$x = 2(x_n - x_{n-1} + \ldots + x_1) - x_o.$$

For $n=1$, x_1 and x_o are distinct, and thus the last equation contradicts property (I). For $n > 1$, the elements $x_1, \ldots, x_n$ are distinct, so the last equation again contradicts property (I). This contradiction completes the proof of the theorem.

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