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AN EXISTENCE THEOREM FOR HIGHER PEANO DERIVATIVES IN $\mathbb{R}^m$

INTRODUCTION. It follows from the Denjoy-Young-Saks theorem that if $f(x + h) = f(x) + O(|h|)$ for every x, then f is almost everywhere differentiable. For higher Peano derivatives Denjoy generalized this theorem [D], but he had to suppose the continuity of f. A year later Marcinkiewicz and Zygmund proved it for measurable functions [MZ]. At the same time Marcinkiewicz [M1] found a third proof. We quote that Zygmund wrote in [M2,p.6] about the difference between these proofs: In paper [MZ] this theorem " is proved in a very complicated way; in particular, the proof uses rather deep results about the behaviour of an analytic function near the boundary of its domain of existence. A simplified proof, using exclusively real variable is given in [M1]". In a footnote Marcinkiewicz remarked that his proof works for arbitrary functions as well.

The first generalization for higher dimensions was done by W.H. Oliver. His method uses the one dimensional results and Fubini's theorem and he had to suppose the measurability of f. In [St,Ch.VIII.§3.] E.M. Stein gives a proof of the multidimensional version of the first order case for arbitrary functions. This proof is based on ideas similar to the proof of Marcinkiewicz and Zygmund, that is based on non-tangential boundedness and convergence of certain Poisson integrals. In [St,Ch.VIII.6.1] Stein remarks that this method is also applicable for higher Peano derivatives.

In Theorem 1 we present a "real variable" proof of this most general version of the theorem (for arbitrary function, dimension and order). As far as we know this is the first published proof of this theorem. The difference between the methods in [St] and ours is similar to that between those in [MZ] and [M1]. It is interesting that our proof is closest to Denjoy's [D], the first one of the five proofs mentioned above. In both proofs Lagrange interpolation polynomials are used to establish the Lipschitz properties of the Peano derivatives of the original function. Then the original Denjoy-Young-Saks theorem (Theorem B) is applied for the (k-1)st derivative and finally this result is integrated in order to get an estimation for the original function. (In the other "real variable" proof in [M1], Marcinkiewicz uses Theorem B as the initial step of an inductive argument and he splits the original function into the sum of two functions. One of these functions is the "good" part of the original function; that is, the previous step of the induction is applicable for its derivative. During the proof it turns out that the other, "bad", part of the splitting is not too bad. That is, the original function, as the sum of the good and bad parts still has good differentiability properties.)

We also remark that during the investigation of the history of this theorem, we found first the results in [St] and there a reference to [CZ]; in [CZ] we found further references to [01] and [02]; and finally, [01] referred to [MZ], [M1] and [D].

PRELIMINARIES. We denote by A^{cl} the closure of the set A. We denote by $mes_m(A)$ the m dimensional Lebesgue measure of A. For a multiindex $j = (j_1, j_2, \ldots, j_m)$ we put $|j| := j_1 + j_2 + \ldots + j_m$, $j! := j_1! \, j_2! \ldots j_m!$ and $x^j := x_1^{j_1} x_2^{j_2} \ldots x_m^{j_m}$ where $x \in \mathbb{R}^m$. For an $x = (x_1, x_2, \ldots, x_m)$ we put $\|x\| := \sqrt{x_1^2 + x_2^2 + \ldots + x_m^2}$. For an $x \in \mathbb{R}^m$ and an $r > 0$ we let $S(x,r) := \{x' \in \mathbb{R}^m : \|x' - x\| < r\}$, $W(x,r) := \{x' \in \mathbb{R}^m : x' - x \in [0,r]^m\}$.

DEFINITION 1. Let k be a positive integer. A function $f : \mathbb{R}^m \to \mathbb{R}$ is said to have k^{th} Peano derivative at x with respect to the closed set F if there exist numbers $f_{(j)}(x)$ for each multiindex j with $|j| \le k$ such that if we put $P_k(f,x,h) := \sum_{|j| \le k} f_{(j)}(x) h^j / j!$, then $f(x + h) = P_k(f,x,h) + o(\|h\|^k)$ when $h \to 0$ and $x + h \in F$.

DEFINITION 2. Let $f : F \to \mathbb{R}$ where $F \subseteq \mathbb{R}^m$ and F is closed. We say that f belongs to Lip(k+1,F) if there exist functions $f_{(j)}$, $0 \le |j| \le k$ defined on F, with $f_{(0)} = f$ and $M > 0$ so that if for every $x, x + h \in F$, $|j| \le k$, $f_{(j)}(x + h) = \sum_{|j+t| \le k} f_{(j+t)}(x) h^t / t! + R_j(x,h)$, then

$$(1) \qquad |f_{(j)}(x)| \le M \quad \text{and} \quad |R_j(x,h)| \le M\|h\|^{k+1-|j|} \quad .$$

The norm of an element in Lip(k+1,F) is the smallest M for which the inequality (1) holds.

We remark that if $f \in$ Lip(k+1,F), then obviously f is k times Peano differentiable with respect to F.

THEOREM A. Suppose k is a non-negative integer and F is a closed set in $\mathbb{R}^m$. Then for any $f \in$ Lip(k+1,F) there exists $\bar{f} \in$ Lip(k+1,$\mathbb{R}^m$) such that $\bar{f}|_F = f$ and the norm of $\bar{f}$ is at most a constant times the norm of f with a constant independent of f.

The proof of this Whitney-type extension theorem can be found in [St,Ch.VI.Th.4.]. We also remark that from the proof of this theorem it follows that there exists $\bar{f}$ fulfilling Theorem A such that if $x \in \mathbb{R}^m\setminus F$, then $\bar{f} \in C^\infty(S(x,r))$ for an $r > 0$.

THEOREM B. Let $f : \mathbb{R}^m \to \mathbb{R}$. Then f is differentiable almost everywhere if and only if $f(x + h) = f(x) + O(\|h\|)$ as $h \to 0$ for almost every x.

Theorem B for $m = 1,2$ can be found in [S Ch.IX.]. For higher dimensions see [R] and [St Ch.VIII.Th.3.].

THEOREM 1. Let $f : F \to \mathbb{R}$ where $F \subseteq \mathbb{R}^m$, F is closed and

$$(2) \qquad f(x + h) = P_{k-1}(f,x,h) + O(\|h\|^k) \quad .$$

Then for almost every $x \in F$ there exist numbers $f_{(j)}(x)$ for $|j| = k$ so that

$$f(x + h) = P_k(f,x,h) + o(\|h\|^k) \qquad (h \to 0, \quad x + h \in F) \quad .$$

LEMMA 1. Suppose that $p(x) = a_0 + a_1x^1 + \dots + a_kx^k$, $(x \in \mathbb{R})$ and there exist x_i such that

$$(3) \qquad |x_i - (i/(k + 1))| \le 1/(3(k + 1))$$

and

(4) $$|p(x_i)| < \varepsilon \quad \text{for} \quad i = 1,2,\dots,k+1 \ .$$

Then $|a_i| < C_k \cdot \varepsilon$ holds for every $i = 0,1,\dots,$ with a constant C_k depending only on k.

PROOF of Lemma 1. We put for $i = 1,2,\dots,k+1$

$$L_i(x,x_1,\dots,x_{k+1}) := \frac{\prod_{j=1}^{k+1} (x_j - x)}{(x_i - x) \prod_{\substack{j=1 \\ j \neq i}}^{k+1} (x_j - x_i)} ;$$

that is, L_i are the polynomials used in Lagrange interpolation. Plainly

$$p(x) = \sum_{i=1}^{k+1} p(x_i) L_i(x,x_1,\dots,x_{k+1}) \ .$$

Since the coefficients of L_i are continuous functions of the variables x_j, (3) implies that the coefficients of L_i are bounded by constants depending only on k. Thus using (4) Lemma 1 is proved.

LEMMA 2. Suppose that $P(x) = \sum_{|j| \leq k} a_j x^j$ where $x \in \mathbb{R}^m$ and we use the notation introduced in the Preliminary section. We also suppose that there exists a set Q such that

(5) $$\text{mes}_m([0,1]^m \cap Q) > 1 - (1/(3(k+1)))^m$$

and

(6) $$|P(x)| < \varepsilon \quad \text{for any} \quad x \in Q \ .$$

Then for any $|j| \leq k$, $|a_j| \leq C_k^m \cdot \varepsilon$ where C_k is as in Lemma 1.

PROOF of Lemma 2. We deduce this lemma from Lemma 1 by induction on m. For $m = 1$ it follows from (5) that we can find x_i - s fulfilling (3).

Suppose that Lemma 2 is proved when $m = M - 1$. We put $f(t) := \text{mes}_{M-1}(Q \cap [0,1]^{M-1} \times \{t\})$. From (5) and Fubini's theorem it follows that

$$\text{mes}_1(\{t \in [0,1] : f(t) > 1 - (1/3(k+1))^{M-1}\}) > 1 - 1/3(k+1) \ .$$

And hence we can choose t_i such that $f(t_i) > 1 - (1/3(k+1))^{M-1}$ and t_i fulfils (3) for $i = 1,2,\dots,k+1$. In the hyperplanes $[0,1]^{M-1} \times \{t_i\}$ we can apply Lemma 2 with $m = M - 1$ and hence the coefficients of the polynomial $P(x_1, \ldots, x_{M-1}, t_i)$ are smaller than $C_k^{M-1} \cdot \varepsilon$. But for each t_i the above coefficients are polynomials of t_i. Applying Lemma 1 for these polynomials we can prove their coefficients are smaller than $C_k^{M} \cdot \varepsilon$. But they are the coefficients of $P(x)$. Therefore, Lemma 2 is proved for $m = M$.

PROOF of Theorem 1. Since any closed set can be decomposed into countably many bounded closed sets, we can suppose that F is bounded. We let

$$F_n' := \{x \in F : \text{ for } 0 < r < 1/n,\ \mathrm{mes}_m(F \cap W(x,r))/r^m > 1 - (1/3(k + 1))^m\} .$$

We also choose a closed set $F_n \subset F_n'$ such that $\mathrm{mes}_m(F_n) > ((n-1)/n)\mathrm{mes}_m(F_n')$. We put $Q_n := \{x \in F_n :$ for all $x' \in F$,

$$|f(x') - P_{k-1}(f,x,x' - x)| \leq n\|x' - x\|^k\} .$$

We choose $x, x' \in Q_n$ and we let $h := x' - x$. It follows from the definition of Q_n that for any $x'' \in W(x',\|h\|) \cap F$ we have

$$|f(x'') - P_{k-1}(f,x',x'' - x')| \leq n\|x'' - x'\|^k \leq nm^{k/2}\|h\|^k$$

and

$$|f(x'') - P_{k-1}(f,x,x'' - x)| \leq n\|x'' - x\|^k \leq n(1 + m^{1/2})^k\|h\|^k .$$

It follows that if we put $P(y) := P_{k-1}(f,x',y) - P_{k-1}(f,x,y + x' - x)$, then we have

(7) $$|P(y)| < K\|h\|^k$$

if $y + x' \in W(x',\|h\|) \cap F$ with $K = n \cdot m^{k/2} + n(1 + m^{1/2})^k$.

We define $T(y)$ on $[0,1]^m$ by $T(y) := P(y \cdot \|h\|)$. Since $x' \in F_n$ and since we have (7), we can apply Lemma 2 with $\varepsilon = K\|h\|^k$, and with $K' = C_k^m \cdot K$ we get $|a_j'| < K'\|h\|^k$, where a_j' is any coefficient of $T(y)$. It follows that $|a_j| < K'\|h\|^{k-|j|}$ where a_j is any coefficient of $P(y)$. And hence for $0 \leq |j| \leq k - 1$

(8) $$|a_j j!| = |f_{(j)}(x') - \sum_{|j+t|<k} f_{(j+t)}(x)h^t/t!| <$$

$$< K' j! \|x' - x\|^{k-|j|} .$$

In particular, note that since x was an arbitrary element of Q_n, (8) assures the uniform continuity of $f_{(j)}$ on Q_n for $0 \leq |j| \leq k-1$. Hence, $f_{(j)}$ may be extended continuously to Q_n^{cl} and these $f_{(j)}(x)$'s are the coefficients of P_{k-1} in (2). It also follows that $Q_n^{cl} \subseteq Q_n$; that is, Q_n is also closed. From (8) it follows that $f \in \mathrm{Lip}(k,Q_n)$. By Theorem A we can define a Whitney extension $\bar{f}$ of $f|_{Q_n}$ such that $\bar{f} \in \mathrm{Lip}(k,\mathbb{R}^m)$ and if $x \in \mathbb{R}^m \setminus Q_n$, then there exists a neighborhood U of x such that $f \in C^\infty(U)$. We show that Theorem 1 holds for $\bar{f}$. From $\bar{f} \in \mathrm{Lip}(k,\mathbb{R}^m)$ it follows that $\bar{f}_{(j)} \in \mathrm{Lip}(k - |j|,\mathbb{R}^m)$. Thus when $|j| = k-1$ by Theorem B $\bar{f}_{(j)}$ is totally differentiable a.e. Thus if we denote the partial derivative $\partial\bar{f}_{(j)}/\partial x_s$ by $\bar{f}_{(j+e(s))}$, then we have for $|j| = k-1$

$$(9) \qquad \bar{f}_{(j)}(x+h) = \bar{f}_{(j)}(x) + \sum_{s=1}^{m} \bar{f}_{(j+e(s))}(x)h_s + o(\|h\|)$$

where $h = (h_1,h_2,\dots,h_m)$ and plainly $o(\|h\|)$ does not depend on the direction of h.

Obviously there exists a set of full measure D so that $\bar{f}_{(j+e(s))}$ exists on D for each $s = 1,\dots,m$. From Theorem A and (8) it also follows that for $0 \leq |j| \leq k-1$ the functions $\bar{f}_{(j)}(x+wh)$ are absolutely continuous (Lipschitz) functions of w for $w \in \mathbb{R}$. That is, they are the integrals of their derivatives. Suppose that $|j| = k-2$ and the segment x, $x+h$ intersects D in a set of linear measure $\|h\|$. Then

$$(10) \qquad \bar{f}_{(j)}(x+h) = \bar{f}_{(j)}(x) + \\ + \sum_{s=1}^{m} \int_0^1 (\bar{f}_{(j+e(s))}(x)h_s + \\ + \sum_{u=1}^{m} \bar{f}_{(j+e(s)+e(u))}(x)h_u h_s w + o(h_s^2 w))dw = \\ = \sum_{|j+q| \leq k} \bar{f}_{(j+q)}(x)h^q/q! + o(\|h\|^2) \quad .$$

Since D is of full measure, there exists a set $D'(x)$ of directions of full measure such that if $h/\|h\| \in D'(x)$, then (10) is true. Plainly $o(\|h\|^2)$ is again uniform in the above directions because $o(\|h\|)$ above was also uniform. By the continuity of the functions $f_{(j)}$ for $|j| \leq k-1$ (10) is true for all h. Iterating this integration process we can prove that

$$(11)\qquad \bar f_{(j)}(x+h) = \sum_{|j+q|\le k} \bar f_{(j+q)}(x)h^q/q! + o(\|h\|^{k-|j|}) :=$$

$$:= P_{k,j}(\bar f,x,h) + o(\|h\|^{k-|j|}) \quad \text{for} \quad 0 \le |j| \le k-1 \quad .$$

When $|j| = 0$ we get Theorem 1 for $\bar f$.

Since $\bar f$ is an extension of $f|_{Q_n}$, we proved that (11) is true for almost every $x \in Q_n$ when $x + h \in Q_n$. We now prove that if x is a point of density of Q_n, then the last assumption can be replaced by $x + h \in F$. If x is a point of density of Q_n, then there exists a function $w(h)$ such that $w(h)$ is $o(\|h\|)$, and for every $x' \in S(x,\|h\|)$ there exists a point $R(x') \in Q_n$ so that $\|R(x') - x'\| < w(h)$. Suppose that $x \in Q_n$ and $x + h \in F$. Since $R(x + h) \in Q_n$, we have $|f(x + h) - P_{k-1}(f,R(x + h),x + h - R(x + h))| \le \le n\|x + h - R(x + h)\|^k$. Using (11) we obtain

$$(12)\quad P_{k-1}(f,R(x+h),x+h-R(x+h)) =$$

$$= \sum_{|j|<k} (P_{k,j}(f,x,R(x+h)-x) + o(\|h\|^{k-|j|}))(x+h-R(x+h))^j/j! =$$

$$= \sum_{|j|<k} P_{k,j}(f,x,R(x+h)-x)(x+h-R(x+h))^j/j! + o(\|h\|^k) \quad ,$$

where we used that $|x + h - R(x + h)| < w(h) = o(\|h\|)$. Using (2) at $R(x + h)$ and (12) at x with $|j| = 0$ we get

$$f(x+h) - P_k(f,x,h) = P_{k-1}(f,R(x+h),x+h-R(x+h)) +$$

$$+ O(w(h)^k) - P_k(f,x,h) =$$

$$= \sum_{|j|<k} P_{k,j}(f,x,R(x+h)-x)(x+h-R(x+h))^j/j! - P_k(f,x,h) +$$

$$+ o(\|h\|^k) = \sum_{|j|<k} P_{k,j}(f,x,R(x+h)-x)(x+h-R(x+h))^j/j! +$$

$$+ \sum_{|j|=k} f_{(j)}(x)(x+h-R(x+h))^j/j! - O(w(h)^k) - P_k(f,x,h) + o(\|h\|^k) =$$

$$= \sum_{|j|\le k} P_{k,j}(f,x,R(x+h)-x)(x+h-R(x+h))^j/j! -$$

$$- P_k(f,x,h) + o(\|h\|^k) = o(\|h\|^k) \quad ,$$

where the last equality holds because the first sum on the left hand of this equality is at $R(x + h)$ the k^{th} Taylor polynomial of the polynomial $P_k(f,x,h)$ and hence they are equal.

Therefore we proved that Theorem 1 is true for almost every point of Q_n for every n. It follows from the definition of Q_n and from the Lebesgue density theorem that Theorem 1 is true for F.

REFERENCES

[CZ] A.P. Calderon and A. Zygmund, Local properties of solutions of elliptic partial differential equations, Studia Math. 20, (1961), 171-225.

[D] A. Denjoy, Sur l'integration des coefficients differentieles d'ordre superieur, Fund. Math. (25), (1935), 273-326.

[M1] J. Marcinkiewicz, Sur les series de Fourier, Fund. Math. (27), (1936), 38-69.

[M2] J. Marcinkiewicz, Collected Papers, Warszawa, (1964).

[MZ] J. Marcinkiewicz-A. Zygmund, On the differentiability and summability of trigonometrical series, Fund. Math. (26), (1936), 1-43.

[O1] W.H. Oliver, Differential properties of real functions, Ph.D. Dissertation, Univ. of Chicago, (1951).

[O2] W.H. Oliver, An existence theorem for the n-th Peano differential Abstract, Bull. Amer. Math. Soc., 57, (1951).

[R] F. Roger, Sur la relation entre les proprietes tangentielles et metriques des ensembles cartesiennes, C.R. Acad Sci. Paris, 201, (1935), 871-873.

[S] S. Saks, Theory of the integral, Hafner, New-York, (1937).

[St] E.M. Stein, Singular integrals and differentiability properties of functions, Princeton University Press, Princeton, (1970).

[SZ] E.M. Stein-A. Zygmund, On the differentiability of functions, Studia Math., 23, (1964), 247-283.

[W] H. Whitney, Analytic extensions of differentiable functions defined in closed sets, Trans. Amer. Math. Soc., 36, (1934), 63-89.

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