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Derivation Bases and the Hausdorff Measure

1. Introduction.

A derivation basis B is a collecton of sets β where β is a collection of interval-point pairs which are associated by a rule which determines how to choose the interval I in terms of the point x. For instance, in this paper, any positive function $\delta(\cdot)$ is used to determine the size of I; e.g. for the $D^{\#}$ derivation basis, $\beta_{\delta}^{\#} = \{(I,x): I \subset (x - \delta(x), x + \delta(x))\}$ are the elements of $D^{\#}$.

This paper answers a question posed by B. S. Thomson in "Derivation Basis on the Real Line" [5, p.164] on the relation between the Hausdorff measure and the D derivation basis. The question is stated as follows: "For $0 < p \leq 1$, let m^p denote the interval function $I \to |I|^p = m^p(I)$. Then the measure m^p evidently is related to the classical Hausdorff p-dimensional measure. More generally, if h is a monotonically increasing function on $[0,\infty)$, $h(0) = 0$, then h^* denotes the function $I \to h(m(I)) = h(|I|)$ and $h^*{}_D$ again represents a measure on R that should be related to similar ideas in the theory of Hausdorff measures. What is the exact relation here?" It will be shown that the Hausdorff measure on a set equals the measure generated by the D derivation basis (1) when the derivative of h at 0 exists and is finite, (2) when the set is countable or (3) when the sum $\sum h(|I_n|)$ over the contiguous intervals of a given closed set converges. However, it will also be shown that the symmetric derivation basis gives rise to a measure which is finite on more sets of finite Hausdorff measure than the measure from the D derivation basis.

2. Preliminary information.

The following definitions will be needed.

Definition 2.1. Hausdorff measure. Let h be a monotone increasing function on $[0,\infty)$, $h(0) = 0$, and continuous from the right such that the range of h is in $[0,\infty)$. For $\delta > 0$ and any set E, let

$$\mu_\delta^h[E] = \inf\{\textstyle\sum_{i=1}^{\infty} h(|I_i|): E \subset \bigcup_{i=1}^{\infty} I_i, \ |I_i| < \delta \text{ and } I_i \text{ is an interval}\}$$

and $\mu^h[E] = \sup_{\delta>0} \mu_\delta^h[E]$. [4, p.51]

Definition 2.2. Let h be given as above. Then for a positive function $\delta(\cdot)$ and any set E, let

$$\beta_\delta^s = \{(I,x): x \text{ is a midpoint of } I \subset (x - \delta(x), x + \delta(x))\},$$

$$V(h^*, \beta_\delta^s[E]) = \sup\{\sum_{\pi[E]} h(|I_i|): \pi = \{(I_i, x_i)\} \text{ is a partition of } [a,b] \text{ in } \beta_\delta^s \text{ and } (I,x) \in \pi[E] \text{ if } x \in E\}$$

and $V(h^*, D^s[E]) = \inf_\delta V(h^*, \beta_\delta^s[E])$. When $V(h^*, D^s[E])$ is being considered as a measure rather than a variation, we write $h^*{}_s(E) = V(h^*, D^s[E])$.

Definition 2.3. Let h and $\delta(\cdot)$ be given as above. Then let

$$\beta_\delta = \{(I,x): x \text{ is an endpoint of } I \subset (x - \delta(x), x + \delta(x))\},$$

$$V(h^*, \beta_\delta[E]) = \sup\{\sum_{\pi[E]} h(|I_i|): \pi \text{ is a partition in } \beta_\delta \text{ and } (I,x) \in \pi[E] \text{ if } x \in E\},$$

and $h^*{}_D(E) = V(h^*, D[E]) = \inf_\delta V(h^*, \beta_\delta[E])$.

Definition 2.4. Let h and $\delta(\cdot)$ be given as above. Then, let

$$\beta_\delta^\# = \{(I,x): I \subset (x - \delta(x), x + \delta(x))\},$$

$$V(h^*, \beta_\delta^\#[E]) = \sup\{\sum_{\pi[E]} h(|I_i|): \pi \text{ is a partition in } \beta_\delta^\# \text{ and } (I,x) \in \pi[E] \text{ if } x \in E\},$$

and $h^*{}_{D^\#}(E) = V(h^*, D^\#[E]) = \inf_\delta V(h^*, \beta_\delta^\#[E])$.

Definition 2.5. Let h be as given above. Let E be any set, then the lower symmetric density of E at x is $d_s(x) = \underline{\lim}_{|I|\to 0}\ \mu^h(E \cap I)/h(|I|)$ where I is symmetric about x.

Definition 2.6. Let h be as given above. Let E be any set, then the right sided lower density of E at x is $d_D(x) = \underline{\lim}_{|I|\to 0}\ \mu^h(E \cap I)/h(|I|)$ where I has x as a left hand endpoint.

We will need the following observation.

Observation 2.1. Let $E \subset [a,b]$. Then, $h^*{}_s(E) \leq h^*{}_{D^\#}(E)$ and $h^*{}_D(E) \leq h^*{}_{D^\#}(E)$. If h is concave down, then $h^*{}_s(E) \leq h^*{}_D(E)$.

Proof. Let $\delta\colon R \to R^+$. Then, $\beta_\delta^s[E] \subset \beta_\delta^\#[E]$ and $\beta_\delta[E] \subset \beta_\delta^\#[E]$. Therefore, $V(h^*, \beta_\delta^s[E]) \leq V(h^*, \beta_\delta^\#[E])$ and $V(h^*, \beta_\delta[E]) \leq V(h^*, \beta_\delta^\#[E])$. Hence $h^*{}_s(E) \leq h^*{}_{D^\#}(E)$ and $h^*{}_D(E) \leq h^*{}_{D^\#}(E)$. Suppose h is concave down. Let $\pi \subset \beta_\delta^s$ and let $\pi[E] = \{(I_i,x_i)\}_{i=1}^n$. Then, $\sum_{i=1}^n h(|I_i|) \leq \sum_{i=1}^n 2h(|I_i|/2)$. Let $\pi' \subset \beta_\delta$ such that $\pi'[E] = \{(I_i', x_i), (I_i'', x_i)\}_{i=1}^n$ where I_i', I_j'' are closed intervals such that x_i is a right hand endpoint of I_i' and the left hand endpoint of I_i'' and $I_i' \cup I_i'' = I_i$. Then, $\pi'[E]$

$\subset \beta_\delta[E]$. Since $\sum_{i=1}^n 2h(|I_i|/2) = \sum_{i=1}^n h(|I_i'|) + \sum_{i=1}^n h(|I_i''|)$, $V(h^*, \beta_\delta^s[E]) \leq V(h^*, \beta_\delta[E])$. Therefore $h^*_s(E) \leq h^*_D(E)$.

The most frequently considered Hausdorff measures are the α-dimensional measures, where $0 < \alpha < 1$, obtained by letting $h(x) = x^\alpha$. The h^*_s and h^*_D measures have two properties in common with the Hausdorff measures when $h(x) = x^\alpha$; namely, those given in the next two observations.

Observation 2.2. h^*_s and h^*_D are translation invariant.

Observation 2.3. Let $\alpha \in (0,1)$ and $h(x) = x^\alpha$. Then, $h^*_s(kE) = k^\alpha h^*_s(E)$ and $h^*_D(kE) = k^\alpha h^*_D(E)$ where $k \geq 0$, $kE = \{kx : x \in E\}$.

We have the following theorem for closed sets and the D derivation basis.

Theorem 2.1. *Let* E *be a closed set in* $[a,b]$. *Then* $\mu^h(E) \leq h^*_D(E)$.

Proof. If $h^*_D(E) = \infty$ there is nothing to prove. Assume $h^*_D(E) < \infty$. Let $\varepsilon > 0$ be given. Let $\delta: R \to R^+$ be such that $(x - \delta(x), x + \delta(x)) \subset [a,b] \setminus E$, if $x \in [a,b] \setminus E$, $[a, a + \delta(a)) \subset [a,b] \setminus E$ if $x = a$, similarly if $x = b$, and

$V(h^*, \beta_\delta[E]) < h^*_D(E) + \varepsilon$. First assume $\mu^h(E) < \infty$. Let δ_o be such that $\mu^h(E) - \varepsilon < \mu^h_{\delta_o}(E)$. Let $\delta_1(x) = \min\{\delta(x), \delta_o\}$. Let $\pi \subset \beta_{\delta_1}$ where $\pi[E] = \{(I_i, x_i)\}_{i=1}^n$ and $x_i \in E$. Then, $E \subset \bigcup_{i=1}^n I_i$. Therefore $\mu^h_{\delta_o}(E) \leq \sum_{i=1}^n h(|I_i|) \leq V(h^*, \beta_{\delta_1}[E])$. Hence $\mu^h(E) - \varepsilon < V(h^*, \beta_{\delta_1}[E]) \leq V(h^*, \beta_\delta[E]) < h^*_D(E) + \varepsilon$. Thus $\mu^h(E) \leq h^*_D(E)$.

3. The lower right derivate of h is finite at 0.

Now we consider the case where the lower right derivate of h at 0 is finite. In this case the Hausdorff measure is a multiple of the Lebesgue measure. $\underline{D}^+h(0)$ is the corresponding factor.

Theorem 3.1. *Let* E *be a closed set in* $[a,b]$. *Let* $r = \underline{D}^+h(0) < \infty$. *Then* $\mu^h(E) = r|E|$.

Proof. Let ε and δ be positive numbers and let $A = \{t \in (0,\delta): h(t)/t < r+\varepsilon\}$. Further let $J_1, J_2, \ldots$ be intervals such that $E \subset \bigcup J_i$ and $\sum(r+\varepsilon)|J_i| < (r+\varepsilon)|E| + \varepsilon$. Subdividing the intervals J_i we get intervals $I_1, I_2, \ldots$ such that $E \subset \bigcup I_s$ and that $|I_s| \in A$ for each s. Then $\mu^h{}_\delta \leq \sum h(|I_s|) < \sum(r+\varepsilon)|I_s| < (r+\varepsilon)|E| + \varepsilon$ whence $\mu^h(E) \leq r|E|$. For the other inequality, let $\varepsilon > 0$ be given. Then there exists a $\delta_o > 0$ such that $r-\varepsilon < \inf_{0<a<\delta} h(a)/a$ for all $\delta < \delta_o$. Let $\{I_i\}$ be any sequence of open intervals such that $E \subset \bigcup_{i=1}^{\infty} I_i$, $|I_i| < \delta < \delta_o$. Then $\sum_{i=1}^{\infty} (r-\varepsilon)|I_i| < \sum_{i=1}^{\infty} h(|I_i|)$. Therefore $\inf\{\sum_i (r-\varepsilon)|I_i| : E \subset \bigcup I_i, \ |I_i| < \delta\} \leq \mu_\delta^h(E)$ which implies that $(r-\varepsilon)|E| < \mu^h(E)$. Since $\varepsilon > 0$ was arbitrary, $r|E| \leq \mu^h(E)$. Thus $r|E| = \mu^h(E)$ where $r = \underline{D}^+h(0) < \infty$.

When $\overline{D}^+h(0) < \infty$, then $h^*{}_s(E)$, $h^*{}_D(E)$ and $h^*{}_{D}\#(E)$ are a multiple of the Lebesgue measure and the corresponding factor is $\overline{D}^+h(0)$.

<u>Theorem</u> 3.2. <u>Let</u> $R = \overline{D}^+h(0) < \infty$. <u>Then for</u> E <u>measurable</u>, $R|E| = h^*{}_s(E) = h^*{}_D(E) = h^*{}_D\#(E)$.

<u>Proof</u>. First we show that $h^*{}_D\#(E) \leq R|E|$. Let $R = \overline{D}^+h(0) = \inf_{\delta>0} \sup_{0<a<\delta} h(a)/a < \infty$. Let $\varepsilon > 0$ be given. Then there exists a $\delta_o > 0$ such that if $\delta < \delta_o$, $\sup_{0<a<\delta} h(a)/a < R+\varepsilon$. Then, there exists a sequence $\{I_i\}_{i=1}^{\infty}$ of open intervals such that no three intervals intersect (i.e., no interval is contained in a union of others) and $\sum_{i=1}^{\infty} (R+\varepsilon)|I_i| < (R+\varepsilon)|E| = \varepsilon$.

Define δ: $[a,b] \to R^+$ such that $(x - \delta(x), x + \delta(x)) \subset I_i$, if $x \in I_i$, and $\delta(x) < \delta_o$ for all x. Let π be a partition of $[a,b]$, such that $\pi \subset \beta_\delta^\#$ and let $\pi[E] = \{(J_i, x_i)\}_{i=1}^m$. Then, $\sum_{i=1}^m h(|J_i|) < \sum_{i=1}^m (R+\varepsilon)|J_i| \leq \sum_{i=1}^\infty (R+\varepsilon)|I_i|$. Therefore $V(h^*, \beta_\delta^\#[E]) \leq \sum_{i=1}^\infty (R+\varepsilon)|I_i|$. Hence ${h^*}_{D^\#}(E) \leq (R+\varepsilon)|E| + \varepsilon$. Thus, ${h^*}_{D^\#}(E) \leq R|E|$.

We now prove the inequality $R|E| \leq {h^*}_s(E)$. Let $\varepsilon > 0$ be given. Then, for each $t > 0$, there exists an $a_t < t$ such that $R-\varepsilon < h(2a_t)/2a_t$. Therefore $(R-\varepsilon)2a_t < h(2a_t)$ for each $t > 0$. Let $\delta(\cdot)$ be any positive function. Then for any $\varepsilon > 0$, there exists a $t > 0$ such that $[x-a_t, x+a_t] \subset (x-\delta(x), x+\delta(x))$ and $2a_t < \varepsilon$. Hence by the Vitali Covering Theorem, there are points $x_i \in E$ and positive numbers a_i such that

$$|E \setminus \bigcup_{i=1}^\infty [x_i - a_i, x_i + a_i]| = 0$$

and the intervals $[x_i - a_i, x_i + a_i]$ are disjoint. Therefore $(R-\varepsilon)|E| < \sum_{i=1}^\infty (R-\varepsilon)|[x_i - a_i, x_i + a_i]| + \varepsilon < \sum_{i=1}^n h(2a_i) + 2\varepsilon \leq V(h^*, \beta_\delta^s[E]) + 2\varepsilon$. Since $\delta(\cdot)$ was arbitrary, $(R-\varepsilon)|E| \leq {h^*}_s(E) + 2\varepsilon$ which implies that $R|E| \leq {h^*}_s(E)$.

The proof of the inequality $R|E| \leq {h^*}_D(E)$ follows as above with $2a_t$ replaced by a_t and the two-sided interval by one-sided.

It can therefore be concluded that when the derivative of h at 0 exists and is finite then the Hausdorff measure agrees with the measure generated by the symmetric derivation basis, the D derivation basis, and the $D^{\#}$ derivation basis. However, if $\underline{D}^{+}h(0) < \overline{D}^{+}h(0) < \infty$, none of the measures generated by the three derivation bases agree with the Hausdorff measure. Nonetheless, the results given above suggest a close relationship between these measures. That this is not so is shown by the results which follow.

4. The right derivate of h is infinite at 0.

One question that remains is what is the relation when h has an infinite derivate at 0. For the $D^{\#}$ derivation basis, the answer is trivial.

__Theorem__ 4.1. __Suppose__ $\overline{D}^{+}h(0) = \infty$. __Then, if__ E __is any non-empty set in__ $[a,b]$, $h^{*}{}_{D^{\#}}(E) = \infty$.

__Proof__. Let M be any positive integer, let $\delta: [a,b] \to R^{+}$ be any positive function, and let $x \in E$. Let $\varepsilon > 0$ be such that $[x-\varepsilon, x+\varepsilon] \subset (x-\delta(x), x+\delta(x))$. Since $\overline{D}^{+}h(0) = \infty$, there exists ε' with $0 < \varepsilon' < \varepsilon$ and a positive integer n such that $nh(2\varepsilon'/n) > M$. Let π be a partition in $\beta_\delta^{\#}$ such that $\{(I_i, x_i)\}_{i=1}^{n} \subset \pi$, $|I_i| = 2\varepsilon'/n$, I_i are non-overlapping and $\bigcup_{i=1}^{n} I_i \subset [x-\varepsilon, x+\varepsilon]$. Then $\sum_{\pi[E]} h(|I|) \geq \sum_{i=1}^{n} h(|I_i|) = nh(2\varepsilon'/n) > M$. Hence $V(h^*, \beta_\delta^{\#}[E]) > M$. Since $\delta(\cdot)$ was arbitrary, $h^*{}_D\#(E) \geq M$. Since M was arbitrary, $h^*{}_D\#(E) = \infty$.

Sufficient conditions are given below for the $h^*{}_D$ measure of a set to be zero. The conditions are those given by Besicovitch and Taylor [3] for a set to be of μ^h measure zero. (That not all sets of μ^h measure zero are of $h^*{}_s$ measure zero will be shown later.)

Theorem 4.2. *Let* $E \subset [0,1]$ *with* $\overline{E}$ = *the closure of* E *of Lebesgue measure zero and let* $\{J_i\}_{i=1}^{\infty}$ *be the contiguous intervals of* $\overline{E}$. *Let* h *be concave down. Suppose* $\sum_{i=1}^{\infty} h(|J_i|) < \infty$. *Then* $h^*{}_D(E) = 0$.

Proof. Let $\varepsilon > 0$ be given and choose N so that $\sum_{i=N+1}^{\infty} h(|J_i|) < \varepsilon$. Let $\{\varepsilon_n\}$ satisfy $\sum_{n=1}^{\infty} h(\varepsilon_n) < \varepsilon$. Let $\delta(x) = \mathrm{dist}(x, \overline{E})$ if $x \notin \overline{E}$, $\delta(x) = \varepsilon_n/2$ if x is an endpoint of J_n, and for all other points let $\delta(x) = \mathrm{dist}(x, \bigcup_{i=1}^{N} J_i)$. Then, if π is any partition of $[0,1]$ with $\pi \subset \beta_\delta$, $\pi = \{I_j\}$, $\sum_{x_j \in E} h(|I_j|) =$

$\sum^1 h(|I_j|) + \sum^2 h(|I_j|)$ where the first sum is over those intervals I_j which have their endpoint x_j an endpoint of a contiguous interval to $\overline{E}$ and the second sum is that of the remaining intervals. Thus, $\sum^1 h(|I_j|) \leq \sum h(e_n) < \varepsilon$. Because $|\overline{E}| = 0$, for each j, $|I_j| = \sum_{i=1}^{\infty} |J_i \cap I_j|$. Since each I_j from $\sum^2$ does not meet $\bigcup_{i=1}^{N} J_i$ and since each J_i, $i > N$, can intersect at most two of the I_j, $\sum^2 h(|I_j|) \leq \sum_{i=N+1}^{\infty} h(|J_i|)$ which follows from the concavity of h. Thus, $\sum_{x_j \in E} h(|I_j|) < 3\varepsilon$ and since $\varepsilon > 0$ was arbitrary, $h^*{}_D(E) = 0$.

An example of an application of this theorem is given by E, the Cantor Ternary Set, in $[0,1]$ where $h(x) = x^{\alpha}$, $\alpha = \log b/\log 3$ and $2 < b < 3$. For if $\{J_i\}_{i=1}^{\infty}$ are the contiguous intervals of E, then $\{|J_n|\}_{n=1}^{\infty} = \{1/3, 1/9, 1/9, 1/27, 1/27, 1/27, 1/27, \ldots, 1/3^n, \ldots, 1/3^n, \ldots\}$ where there are 2^n intervals of length $1/3^{n+1}$. Therefore, $\sum_{n=1}^{\infty} |J_n|^{\alpha} = 1/b + 1/b^2 + 1/b^2 + 1/b^3 + 1/b^3 + 1/b^3 + 1/b^3 + \ldots + 1/b^n + \ldots + 1/b^n + \ldots = \sum_{n=0}^{\infty} 2^n(1/b^{n+1}) < \infty$ since $\sum_{n=0}^{\infty} (2/b)^n$ is a geometric series with $(2/b) < 1$. Hence by the theorem $\mu^h(E) = h^*{}_D(E) = 0$.

Corollary 4.1. *A countable union of sets* E_n *of measure zero satisfying* $\sum_{i=1}^{\infty} h(|J_i^n|) < \infty$, *where* J_i^n *are the contiguous intervals of* E_n, *is of* h^*_D *and* h^*_s *measure zero and hence all countable sets are of* h^*_D *and* h^*_s *measure zero.*

The following example shows that the conditions in Corollary 4.1 are not necessary and sufficient.

Example 4.1. *Let* $h(x) = x^{\alpha}$ *where* $\alpha \in (0,1)$. *Then a set* E *can be constructed such that* $h^*_s(E) = 0$ *and the contiguous intervals of* E, $\{I_n\}$ *in* $[0,1]$ *satisfy* $\sum_n h(|I_n|) = \infty$.

Construction. Let A be a non-empty closed set where $h^*_s(A) = 0$, $A \subset [0,1]$. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} a_n = 1$ and $\sum_{n=1}^{\infty} h(a_n) = \infty$. Let $\{\varepsilon_n\}$ be a sequence of positive numbers chosen such that $\varepsilon_n < a_n$ and $\sum h((1-\varepsilon_n)a_n) = \infty$. By Observation 2.3, $h^*_s(\varepsilon_n A) = 0$ for each n. Let $\{I_n\}$ be a sequence of non-overlapping intervals such that $|I_n| = a_n$ and $\bigcup_n I_n = [0,1]$. In the left hand side of each I_n, put the set $\varepsilon_n A$. Let this be D_n. Let $E = \bigcup_n D_n$. Then, $h^*_s(E) \leq \sum_n h^*_s(\varepsilon_n A) = 0$ and the contiguous intervals of E in sum is larger than $\sum_n ((1-\varepsilon_n)a_n) = \infty$.

The following is an example of a set of Hausdorff measure zero and h^*_s measure infinity.

Example 4.2. Let h and k be increasing functions on $[0,\infty)$, $h(0) = k(0) = k(0+) = 0$, $h'^+(0) = \infty$. Then there is a compact set E such that $h^*_s(E) = \infty$ and $\mu^k(E) = 0$.

Proof. Let $p_o = \lambda_o = 1$, $L_1{}^o = [0,1]$. We proceed by induction. Suppose that n is a natural number and that disjoint closed intervals L_j^{n-1} $(j = 1, \ldots, p_{n-1})$ of length λ_{n-1} are given. Since $h'^+(0) = \infty$, there is a natural number q such that $qh(\lambda_{n-1}/2q) > n$. Partition each interval L_j^{n-1} into $2q$ intervals of equal length. We obtain intervals J_i^n $(i = 1, \ldots, 2qp_{n-1})$. There is an $\eta \in (0, \lambda_{n-1}/2q)$ such that $2qnp_{n-1}k(\eta) < 1$. For each i let L_i^n be the closed interval of length η with the same center at J_i^n. Then $|J_i^n| = \lambda_{n-1}/2q > |L_i^n|$ so that $L_i^n \subset J_i^n$. We set $p_n = 2qp_{n-1}$, $\lambda_n = \eta$. Obviously $\lambda_n \leq 2^{-n}$ and

$$(!) \qquad \textstyle\sum_i k(|L_i^n|) = p_n k(\eta) < 1/n \quad \text{for each } n.$$

In the rest of the proof we write $q_n = q$.

Let $E = \bigcap_n(\bigcup_i L_i^n)$ and let $\delta\colon [0,1] \to R^+$, $M \in R^+$. Since $[0,1] = \bigcup_n\{x\colon \delta(x) > 1/n\}$, there is a natural number N such that $E \cap \{x\colon \delta(x) > 1/N\}$ is dense in some portion of E, say in $E \cap L_j^{n-1}$, where $N\lambda_{n-1} < 1$ and $n > M$. There are numbers $y_o, \ldots, y_{2q}$ such that $L_j^{n-1} = [y_o, y_{2q}]$ and that $\{[y_{t-1}, y_t]\}_{t=1}^{2q}$

$= \{J_i^n \colon J_i^n \subset J_j^{n-1}\}$. For $v = 1, \ldots, q$ choose a number $x_v \in E \cap (y_{2v-1}, y_{2v})$. Let S_v be the closed interval with center x_v of length $\lambda_{n-1}/2q$ and let $\pi = \{S_v\}_{v=1}^{q}$. It is easy to see that the intervals S_v are disjoint. Since $\delta(x_v) > 1/n > \eta_{n-1}/q$, we have $\pi \subset \beta_\delta^S(E)$. Thus, $V(h^*, \beta_\delta^S[E]) \geq \sum h(|S_v|) = qh(\lambda_{n-1/2q}) > n > M$ so that $h^*{}_S(E) = \infty$. The relation $\mu^k(E) = 0$ follows at once from (!).

The author wishes to thank one of the referees for this statement and proof of Example 4.2.

We now consider sets of finite, nonzero Hausdorff measure and give some necessary conditions for the symmetric derivation basis and the D derivation basis measures to be infinite (respectively finite).

Theorem 4.3. Let E be a nowhere dense, closed set of Lebesgue measure zero. Suppose $0 < \mu^h(E) < \infty$. Then if

$$\int_E d_D^{-1}(x)\, d\mu^h(x) = \infty, \quad h^*{}_D(E) = \infty.$$

Proof. Note that $d_D^{-1}(x)$ is defined to be ∞ if $d_D(x) = 0$. Assume $\int_E d_D^{-1}(x)\, d\mu^h(x) = \infty$. Let M be a positive integer.

Then, there exists a simple function $s(x) = \sum_{n=1}^{r} a_n \chi_{E_n}(x)$ such that $d_D^{-1}(x) > s(x)$ for all $x \in E$ and $\sum_{n=1}^{r} a_n \mu^h(E_n) > M$ where the E_n are pairwise disjoint. Let $\varepsilon > 0$ be given. Because μ^h is a regular measure, there exists a closed set $F_n \subset E_n$ such that $\mu^h(F_n) > \mu^h(E_n) - \varepsilon/a_n 2^n$, $n = 1, 2, \ldots, r$. Let $\delta(x)$ be any positive function. Let $\delta_1(x)$ be a positive function such that $\delta_1(x) \leq \delta(x)$ for all $x \in [0,1]$, $(x-\delta_1(x), x+\delta_1(x)) \subset [0,1] \setminus E$ if $x \in [0,1] \setminus E$, and $\delta_1(x) < (1/3)\mathrm{dist}(F_n, \bigcup_{m=1, m\neq n}^{r} F_m)$ for $x \in F_n$, $n = 1, \ldots, r$. Since $1/a_n > d_D(x)$ for $x \in E_n$, for each $\varepsilon < \delta_1(x)$, there exists an I such that x is a left hand endpoint of I, $|I| < \varepsilon$ and

$$(*) \qquad 1/a_n > \mu^h(E \cap I)/h(|I|).$$

For each n, by the Vitali Covering Theorem for Hausdorff measures, there exists intervals $\{I_s^n\}_{s=1}^{Nn}$ satisfying (*) which are disjoint and $\sum_{s=1}^{Nn} \mu^h(F_n \cap I_s^n) > \mu^h(F_n) - \varepsilon/a_n 2^n$. The choice of $\delta_1(x)$ implies that all I_s^n are pairwise disjoint. Therefore $\sum_{n=1}^{r} \sum_{s=1}^{Nn} h(|I_s^n|) > \sum_{n=1}^{r} a_n \sum_{s=1}^{Nn} \mu^h(E \cap I_s^n)$

$$\geq \sum_{n=1}^{r} a_n \sum_{s=1}^{Nn} \mu^h(F_n \cap I_s^n) > \sum_{n=1}^{r} a_n(\mu^h(F_n) - \varepsilon/a_n 2^n)$$

$$= \sum_{n=1}^{r} a_n \mu^h(F_n) - \sum_{n=1}^{r} \varepsilon/2^n > \sum_{n=1}^{r} a_n \mu^h(E_n) - 2\sum_{n=1}^{r} \varepsilon/2^n$$

$> M - 2\varepsilon$. Therefore. $V(h^*, \beta_{\delta 1}[E]) \geq M - 2\varepsilon$. Hence, $h^*_D(E) \geq M$ since $V(h^*, \beta_\delta[E]) \geq V(h^*, \beta_{\delta 1}[E])$ and $\delta(x)$ was arbitrary. Since M was arbitrary, $h^*_D(E) = \infty$.

Note. If E is the Cantor set and $h(x) = x^\alpha$ where $\alpha = \log 2/\log 3$, Besicovitch [1] proved that $d_D(x) = 0$ on E. Therefore $h^*_D(E) = \infty$ by the above theorem.

The analogous theorem holds for the symmetric derivation basis.

Theorem 4.4. *Let E be a nowhere dense, closed set of Lebesgue measure zero. Assume* $0 < \mu^h(E) < \infty$. *Then, if* $\int_E d_s^{-1}(x)\, d\mu^h(x) = \infty$, $h^*_s(E) = \infty$.

Proof. The proof is the same as that for Theorem 2.6 with $d_s(x)$ replacing $d_D(x)$.

The following example, given by Besicovitch [1], is that of a set E of finite Hausdorff measure which has $d_s(x) = 0$ at almost every point $x \in E$. For this set $h^*_s(E) = \infty$ by the previous theorem.

Example 4.3. Let $E = \{x\colon x = a_1/n^2 + a_2/n^{2^2} + a_k/n^{2^k} + \ldots$

where a_k takes the values $0, 1n^{2^{k-1}}, 2n^{2^{k-1}}, 3n^{2^{k-1}}, \ldots, (n^{2^{k-1}}-1)n^{2^{k-1}}$ for all $k\}$. Besicovitch notes that if $n = 10$ and $h(x) = x^{1/2}$, then $0 < \mu^h(E) < \infty$ and $d_s(x) = 0$ a.e. μ^h. Therefore $h^*_s(E) = \infty$.

The above Theorem 4.4 does not give necessary and sufficient conditions for $h^*_s(E)$ to be finite. For let E_1 be a set such that $\mu^h(E_1) = 0$ and $h^*_s(E_1) = \infty$. Let E be a set such that $0 < \mu^h(E) < \infty$ and $\int_E d_s^{-1}(x)\, d\mu^h(x) < \infty$. Then $\int_{E_1 \cup E} d_s^{-1}(x)\, d\mu^h(x) < \infty$ and $h^*_s(E_1 \cup E) = \infty$.

A necessary condition for $h^*_s(E)$ to be finite when μ^h is finite is given by the following theorem.

<u>Theorem</u> 4.4. <u>Let</u> $E = \bigcup_n E_n$ <u>where</u> E <u>and</u> E_n <u>are measurable sets and the</u> E_n <u>are pairwise disjoint</u>. <u>If</u> $d_s(x) \geq d_n > 0$ <u>for each</u> $x \in E_n$ <u>and</u> $\sum_{n=1}^{\infty} d_n^{-1}\mu^h(E_n) < \infty$, <u>then</u> $h^*_s(E) < \infty$.

<u>Proof</u>. For each $x \in E$, let $\delta(x)$ be so small that if $|I| < \delta(x)$ and $x \in E_n$, then $\mu^h(E_n \cap I)/h(|I|) > (1/2)d_n$. Then, for any partition $\pi \subset \beta_\delta^s$, where $\pi[E] = \{(I_i, x_i)\}_{i=1}^m$,

$\sum_{i=1}^m h(|I_i|) < \sum_{i=1}^m \sum_{n=1}^{\infty} 2d_n^{-1}\mu^h(E_n \cap I_i) \leq 2\sum_{n=1}^{\infty} d_n^{-1}\mu^h(E_n) < \infty$.

Therefore $V(h^*, \beta_\delta^s[E]) \leq 2\sum_{n=1}^{\infty} d_n^{-1}\mu^h(E_n)$ and $h^*{}_s(E) \leq 2\sum_{n=1}^{\infty} d_n^{-1}\mu^h(E_n) < \infty$.

<u>Corollary</u> 4.2. <u>If at each point of</u> E, $d_s(x) > d > 0$, <u>then</u> $h^*{}_s(E) < \infty$.

The same theorems hold for the D derivation basis.

Even given the necessary conditions for a set to be of finite (or infinite) measure with respect to the symmetric derivation basis (respectively, the D derivation basis) it does not follow that when $h^*{}_s(E)$ is finite that it is equ~~q~~al to the Hausdorff measure. In the following example a set E has finite measure with respect to the symmetric derivation basis measure, but this measure is not equal to the Hausdorff measure of the set.

<u>Example</u> 4.4. <u>The Cantor Ternary Set</u> E <u>with</u> $h(x) = x^\alpha$ <u>where</u> $\alpha = \log 2/\log 3$ <u>satisfies</u> $\mu^h(E) = 1 < h^*{}_s(E) < \infty$. (It is known that $\mu^h(E) = 1$, $\mu^h(E \cap [0, 1/3]) = 1/2$ etc.)

<u>Proof</u>. Let J be any symmetric interval about a point $x \in E$. Let I be the largest contiguous interval of E in [0,1] in J. Without loss of generality, assume I is to the right of x in J. The $h(|I|) \leq \mu^h(J \cap E)$ since J contains at least one

portion of E of diameter $|I|$ to the left of I in J and each such portion has μ^h measure $h(|I|)$. For each interval $I = [a,b]$ contiguous to E there are two intervals contiguous to E on opposite sides of I one of these intervals having length three times the length of I. It follows that $|J|$ can be no larger that 12 times $|I|$; for otherwise, a larger interval than I would be contained in J. Thus $h(|J|) \leq h(12|I|) = h(12)h(|I|) \leq h(12)\mu^h(J \cap E)$. Hence $\mu^h(J \cap E)/h(|J|) \geq \mu^h(J \cap E)/h(12)\mu^h(J \cap E) = 1/h(12)$. Therefore $d_s(x) \geq 1/h(12)$ for all $x \in E$. Hence $h^*{}_s(E) < \infty$. Again, let J be any interval with center $x \in E$. Let I be the largest contiguous interval of E in $[0,1]$ contained in J. Assume I is to the right of x in J. Fix $I = [a,b]$ with $|I| = 1/3^n$. Fix y in $[b, b + 1/3^n] \cap E$. Let x be in $[a - 1/3^n, a] \cap E$. Then $\mu^h(E \cap J)$ does not depend on the choice of x because $E \cap J = E \cap ([a - 1/3^n, a] \cup [b,y])$ no matter which x in $[a - 1/3^n, a] \cap E$ is chosen. However, $|J|$ is minimized when $x = a$. Therefore fix $x = a$. Now let y vary in $[b, b + 1/3^n]$. Since the left hand endpoints of the contiguous intervals in $[b, b + 1/3^n]$ are dense in $[b, b + 1/3^n] \cap E$, it suffices to prove that $h(|J|) \geq h(2)(2/3)\mu^h(E \cap J)$ when y is in one of the contiguous intervals of $[b, b + 1/3^n] \cap E$.

First, let $y \geq b + 1/3^n$. Then $h(|I|) = (1/2)\mu^h(E \cap J)$ and $|J| \geq 4|I|$. Therefore, $h(|J|) \geq h(4)(1/2)\mu^h(E \cap J) =$

$h(2)h(2/3)\mu^h(E \cap J) > h(2)(2/3)\mu^h(E \cap J)$. The principle of mathematical induction will be used to prove that $h(|J|) \geq h(2)(2/3)\mu^h(E \cap J)$ when y is in one of the contiguous intervals of $[b, b + 1/3^n] \cap E$. Let $m = 1$. Let $y \in [b + 1/3^{n+1}, b + 2/3^{n+1}]$. Then, $h(|I|) = (2/3)\mu^h(E \cap J)$ and $|J| \geq 2|I| + (2/3)|I| = [(6 + 2)/3]|I|$. Therefore $h(|J|) \geq h((6 + 2)/3)(2/3)\mu^h(E \cap J) = h(6 + 2)(1/3)\mu^h(E \cap J) = h(2)h(3 + 1)(1/3)\mu^h(E \cap J) > h(2)h(3)(1/3)\mu^h(E \cap J) = h(2)(2/3)\mu^h(E \cap J)$. Assume $h(|J|) > h(2)(2/3)\mu^h(E \cap J)$ for y in the contiguous intervals I' in $[b, b + 1/3^n] \cap E$ where $|I'| = 1/3^{n+K}$ for all $K < m$. Let y be in one of the contiguous intervals I'' in $[b, b + 1/3^n] \cap E$ where $|I''| = 1/3^{n+m}$. Then $b + \sum_{s=1}^{m-1} j_s/3^{n+s} + 1/3^{n+m} \leq y \leq b + \sum_{s=1}^{m-1} j_s/3^{n+s} + 2/3^{n+m}$ where $j_s = 0$ or 2 and $(1 + r/2^m)h(I) = \mu^h(E \cap J)$ where $r = \sum_{s=1}^{m-1} i_s 2^s + 1$ where $i_s = 0$ if $j_s = 0$ and $i_s = 1$ if $j_s = 2$. Note that $|J| \geq 2|I| + 2(k/3^m)|I|$ where $k = \sum_{s=1}^{m-1} j_s 3^s + 1$. Therefore $h(|J|)$ $\geq h(2|I| + 2(k/3^m)|I|) = h(2)h((3^m+k)/3^m)(2^m/(2^m+r))\mu^h(E \cap J)$ $= h(2)h(3^m+k)(1/(2^m+r))\mu^h(E \cap J)$. Hence, it suffices to show that $h(3^m+k)/(2^m+r) \geq 2/3$. Now, $3h(3^m + \sum_{s=1}^{m-1} j_s 3^s + 1) =$

$3h([3^m 2 + 2\sum_{s=1}^{m-1} j_s 3^s + 2]/2) > 3[h(3^m 2) + \sum_{s=1}^{m-1} h(2j_s 3^s) + h(2)]/2$

$= 3h(2/3)h(3^m) + 3\sum_{s=1}^{m-1} h(2j_s 3^s)/2 + 3h(2)/2 > 2^m 2 +$

$(3/2)h(2)\sum_{s=1}^{m-1} i_s 2^s + 3h(2/3) > 2^m 2 + 2\sum_{s=1}^{m-1} i_s 2^s + 2 = 2^m 2 + 2r.$

Thus, $h(3^m+k)/(2^m+r) \geq 2/3$ and $h(|J|) > h(2)(2/3)\mu^h(E \cap J)$. By the principle of mathematical induction, $h(|J|) > h(2)(2/3)\mu^h(E \cap J)$ for any y in the contiguous intervals of $[b, b + 1/3^n] \cap E$. Therefore, if $\{J_i\}$ is any finite collection of non-overlapping intervals in β_δ^s that covers E, $\sum_i h(|J_i|) \geq \sum_i h(2)(2/3)\mu^h(E \cap J_i) > h(2)(2/3)\mu^h(E) = h(2)(2/3)$. Thus, $V(h^*, \beta_\delta^s[E]) > h(2)(2/3)$ for all $\delta(x)$ which implies that $h^*{}_s(E) \geq h(2)(2/3)$. It remains to show that $h(2)(2/3) > 1$. Now $h(2)(2/3) = h(6)/3$, so it suffices to show that $h(6) > 3$. But, $h(6) = h((9 + 3)/2) > [h(9) + h(3)]/2 = (4 + 2)/2 = 3$.

In example 4.2 a set was found to have Hausdorff measure zero and symmetric derivation basis measure infinity. It is unknown as to whether a Borel set of infinite symmetric derivation basis measure contains a set of finite symmetric derivation basis measure [2]. Assuming it does, the above examples show that when $\underline{D}^+h(0) = \infty$, for a closed set E, all possible combinations of zero, finite and infinite values can occur subject only to the condition that $\mu^h(E) \leq h^*{}_s(E) \leq h^*{}_D(E)$ where the second inequality is valid when h is concave.

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