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A CHARACTERIZATION OF EXTENDABLE CONNECTIVITY FUNCTIONS

Stallings [9] asked the question: "If one considers $I = [0,1]$ embedded in I^2 as $I \times 0$, can a connectivity function $I \to X$ be extended to a connectivity function $I^2 \to X$?" Cornette [3] and Roberts [8] each gave a negative answer to this question by constructing a connectivity function $I \to I$ that is not almost continuous.

In [4] we constructed an almost continuous function $I \to I$ that has a perfect road at no point and showed in [5] and [6] that it can not be extended to a connectivity function $I^2 \to I$.

Recently, Brown, Humke, and Laczkovich [1] showed that a connectivity function $I \to I$ of Baire class 1 can be extended to a connectivity function $I^2 \to I$. Thus for Baire class 1 functions, extendable connectivity functions have been characterized by a number of properties, [2].

The purpose of this paper is to give a necessary and sufficient condition for a connectivity function $I \to I$ to be extendable to a connectivity function $I^2 \to I$.

A function $f:X \to Y$ is said to be a connectivity function provided that if C is a connected subset of X, then the graph of f restricted to C is a connected subset of $X \times Y$. The function f is said to be an almost continuous function provided that every open set containing the graph of f contains the graph of a continuous function with the same domain. Also the function f is said to be peripherally continuous provided that for each x and for each pair of open sets U and V containing x and $f(x)$, respectively, there exists an open set W such that $x \in W \subset U$ and $f(bd(W)) \subset V$ where bd = boundary.

If $f: I \to I$ is an almost continuous function, then f is a connectivity function, [9]; and if $f: I \to I$ is a connectivity function, then f is a peripherally continuous function. However if $f: I^n \to I$, $n \geq 2$, then connectivity functions and peripherally continuous functions are the same, [7]. But if $f: I^n \to I$, $n \geq 2$, is a connectivity function, then f is an almost continuous function, [9].

We now state the condition and prove that it is both necessary and sufficient for a function $I \to I$ to be extendable to a connectivity function $I^2 \to I$.

Definition. Let $f: I \to I$ be a function. A family of peripheral intervals for f consists of a sequence of ordered pairs (I_n, J_n) of subintervals of I such that

(1) I_n is open in I and the length of I_n converges to 0;

(2) for each $x \in I$ and for any $\varepsilon > 0$ there exits (I_n, J_n) such that $x \in I_n$, $f(x) \in J_n$, and the lengths of I_n and J_n are less than ε;

(3) both endpoints of I_n map into J_n; and

(4) if I_n and I_m have points in common but neither is a subset of the other, then J_n and J_m have points in common.

Theorem 1. If a family of peripheral intervals exists for $f: I \to I$, then f is the restriction of a connectivity function $g: I^2 \to I$ such that g is continuous on the complement of $I \times 0$ where I is embedded in I^2 as $I \times 0$.

Proof. First rearrange the sequence (I_n, J_n) if necessary so that the lengths of I_n are non-increasing. This can be done by part (1) of the defintion. We may also assume that no two I_n's are equal, for otherwise we can replace J_n by the intersection of all the corresponding J_n's. This will be a non-empty interval by part (3) of the definition. If $[0,p)$ and $(0,p)$ are in the family of peripheral intervals, let $[0,p)$ precede $(0,p)$. Similarly $(p,1]$ will precede $(p,1)$, if they are in the family of peripheral interval.

Next we will associate with each I_n a trapezoid T_n whose base is the closure of I_n and whose heights strictly decreases to 0. We construct the family T_n in sequence requiring that

(1) if I_n contains neither 0 nor 1 and I_n is a subset of I_m, then the part of T_n not on $I \times 0$ is inside T_m;

(2) if I_n contains neither 0 nor 1, then the two non-parallel sides of T_n have no points off $I \times 0$ in common with previous trapezoids;

(3) if I_n contains neither 0 nor 1, then the two non-parallel sides of T_n makes acute angles with I_n;

(4) if I_n contains 0 or 1, then the left side or right side, respectively, of T_n makes a right angle with I_n and its opposite side makes an acute angle with I_n; and

(5) the height of T_n is less than the height of any previous trapezoid and is less than $1/n$.

Thus the intersection of two trapezoids T_n and T_m consist of points in $I \times 0$ together with either

(1) $\emptyset$, if the intervals I_n and I_m do not intersect;

(2) $\emptyset$, if one interval is a subset of the other and the small interval contains neither 0 nor 1;

(3) a single point on top of the lower trapezoid, if I_n and I_m have points

in common but neither is a subset of the other; or

(4) a side of the lower trapezoid T_n, if I_n contains 0 or 1 and I_n is a subset of I_m.

These trapezoids will be the boundaries of open sets about points on $I \times 0$ and we will require that g map T_n into J_n. This, together with the continuity of g on the complement of $I \times 0$ implies that g is peripherally continuous and therefore is a connectivity function.

Consider the trapezoid T_n. For notation purposes t_n will denote the top, l_n will denote the left side, r_n will denote the right side, and h_n will denote the height. Also I_n is the base and $h_n < 1/n$.

Consider the pair (I_1,J_1). Define $g:I \times [h_1,1] \to I$ to be continuous so that $g(t_1) \subset J_1$. Now consider the pair (I_2,J_2). We have the following cases.

(a) Suppose $I_1 \cap I_2 = \emptyset$. We extend g to $I \times [h_2,1]$ to be continuous as follows:

$$g(t_2) \subset J_2,$$
$$g(l_1 \cap (I \times [h_2,h_1])) \subset J_1, \text{ and}$$
$$g(r_1 \cap (I \times [h_2,h_1])) \subset J_1.$$

Now g is defined on a closed subset of $I \times [h_2,1]$ and we can extend g to the rest of $I \times [h_2,1]$ so that this extension is continuous.

(b) Suppose $I_2 \subset I_1$ and I_2 contains neither 0 nor 1. Extend g as in case (a).

(c) Suppose $I_2 \subset I_1$ and I_2 contains 0. Since 0 is an endpoint of I_1 and I_2, $f(0) \in J_1 \cap J_2$. So $J_1 \cap J_2 \neq \emptyset$. Let p be the point of $l_1 \cap t_2$. Define

$$g(p) \in J_1 \cap J_2,$$
$$g(l_1 \cap (I \times [h_2,h_1])) \subset J_1, \text{ and}$$

$$g(t_2) \subset J_2$$

continuously. Now extend g continuously to the rest of $I \times [h_2, 1]$. Similarly for $I_2 \subset I_1$ and I_2 contains 1.

(d) Lastly, suppose I_1 and I_2 have interior points in common but neither is a subset of the other. Let p be the point on t_2 that is in T_1 and define $g(p) \in J_1 \cap J_2$. Let

$$g(t_2) \subset J_2,$$

$$g(l_1 \cap (I \times [h_2, h_1])) \subset J_1, \text{ and}$$

$$g(r_1 \cap (I \times [h_2, h_1])) \subset J_1$$

in a continuous manner. Now extend g continuously to the rest of $I \times [h_2, 1]$.

Now assume g is defined and continuous on $I \times [h_n, 1]$. We now show how to extend it to $g: I \times [h_{n+1}, 1] \to I$. By the decreasing property of the height, the extension will involve points of only a finite number of trapezoids. First define it on intersection points of pairs of trapezoids. This can be done by condition (4) of the definition and the fact that no such intersection lies on the top of two trapezoids, and the number of such intersections is finite.

Now extend g to the rest of the trapezoid T_k that lies in $I \times [h_{n+1}, 1]$ so that on T_k, g has values in J_k where $k \leq n+1$ and so that g is continuous. This defines g on a closed subset of $I \times [h_{n+1}, 1]$. Now extend g continuously to the rest of $I \times [h_{n+1}, 1]$. If this is done for each n, it defines g and completes the proof.

Theorem 2. The existence of a family of peripheral intervals is both necessary and sufficient that a function $f: I \to I$ be the restriction of a connectivity function $g: I^2 \to I$ where I is embedded in I^2 as $I \times 0$.

Proof. Sufficiency follows from theorem 1.

Assume that f is the restriction of a connectivity function $g:I^2 \to I$. Let n be a positive integer. Then for each $x \in I \times 0$ there exists a connected open set $U_{n,x}$ with connected boundary $bd(U_{n,x})$ such that $x \in U_{n,x}$, $g(bd(U_{n,x})) \subset (g(x) - 1/16n , g(x) + 1/16n)$, and the diameter of $U_{n,x}$ is less than $1/n$.

Let $I_{n,x}$ be the open interval of $U_{n,x} \cap (I \times 0)$ containing x and let $J_{n,x}$ be an interval of the form $[i/2^k , (i+4)/2^k]$ such that $(g(x) - 1/16n , g(x) + 1/16n) \subset [i/2^k , (i+4)/2^k] \subset (g(x) - 1/n , g(x) + 1/n)$. Note that for a fixed n, as x varies the collection of $I_{n,x}$ may be uncountable but $J_{n,x}$ is finite. Also the lengths of both $I_{n,x}$ and $J_{n,x}$ are less than $1/n$. Also note that the interval $[i/2^k , (i+4)/2^k]$ will be paired with other $I_{p,x}$ such that $2^{k-3} \le p \le 2^{k-2}$.

As the first approximation to a family of peripheral intervals consider the family of pairs $(I_{n,x}, J_{n,x})$. This family will satisfy (2), (3), and (4) but may not satisfy (1). The family of $I_{n,x}$ may be uncountable and contain subsequences such that the length of the intervals in the subsequence does not converge to 0. Note that the collection of $J_{n,x}$'s is countable and the length of the $J_{n,x}$'s converge to 0. Rearrange the sequence $J_{n,x}$ so that the lengths of $J_{n,x}$ are decreasing, if necessary. Let $\{J_k: k = 1,2,3,...\}$ be this arrangement.

Select any J_k and let it be fixed. Now $J_k = J_{n(k),x}$ for some $n(k)$. Consider the family $S_m(J_k)$ of $I_{n(k),x}$ having length at least $1/m$ for some positive integer m that are paired with J_k. Note that $S_m(J_k)$ may be empty, since the length of each member of $S_m(J_k)$ is less than $1/n(k)$.

We now replace $S_m(J_k)$ by a finite family satisfying (3) and (4) for each m. Let $S'_m(J_k)$ consist of $S_m(J_k)$ together with all limits of intervals of $S_m(J_k)$. By limit we mean the intervals (a_n, b_n) converge to the interval (a,b), if a_n converges to a and b_n converges to b. Let $S'_r(J_i)$ and $S'_s(J_j)$ be two such families, and $E_{r,i}$ and $E_{s,j}$ be members of these families, respectively, such that they have points in common but neither is a subset of the other. Then there exist $I_{r,i}$ and $I_{s,j}$ in $S_r(J_i)$ and $S_s(J_j)$, respectively, such that $I_{r,i}$ and $I_{s,j}$ have points in common but neither is a subset of the other. Thus J_i and J_j have points in common and (4) holds.

Now consider (3). Take open sets $U_{n(k),x}$ as above for intervals $I_{n(k),x} \in S_m(J_k)$ where $I_{n(k),x}$ converges to $E_{m,k}$. We have that $g(\mathrm{bd}(U_{n(k),x})) \subset J_k$. Let z be a boundary point of $E_{m,k}$. Then z is in $\overline{\bigcup_{n=1}^{\infty} \mathrm{bd}(U_{n(k),x})}$. Suppose that $g(z) \notin J_k$. Let $N(z)$ be a neighborhood of z with diameter $<1/m$ such that the complement, $\widetilde{N(z)}$, of $N(z)$ is connected. Then $A = \widetilde{N(z)} \cup (\bigcup_{n=1}^{\infty} \mathrm{bd}(U_{n(k),x})) \cup \{z\}$ is connected but $g|A$ is not connected, since $(z,g(z))$ is not a limit point of the rest of $g|A$. Therefore $g(z) \in J_k$ and (3) holds.

Select a finite subfamily $F_m(J_k)$ of $S'_m(J_k)$ so that $\bigcup_m F_m(J_k)$ containing all x for which $I_{n(k),x}$ was in the original family $S_m(J_k)$. Start with an interval of $S_m(J_k)$ having minimal left endpoint. At each stage select either

(1*) an interval which overlaps the last interval in an interval of length at most $1/2m$,

(2*) an interval intersecting the last interval and having maximal right endpoint but strictly beyond that of the last interval if (1*) cannot happen, or

(3*) if no interval having larger right endpoint intersects the last interval, the disjoint interval having minimal left endpoint beyond the right endpoint of the last interval.

If this set is infinite, some infinite sequence satisfying (2*) occurs since (1*) and (3*) can happen at most finitely often by measure (length of intervals). Suppose the set is infinite. Select a subsequence I_i of this set which converges to E. Then for all i larger than some $i(1)$, E will intersect I_i since the right endpoints are strictly increasing. But the right endpoint of E is beyond that of $I_{i(1)+1}$. This contradicts (2*). So the family $F_m(J_k)$ is finite and (3) and (4) hold.

Let $F = \bigcup_{m=1}^{\infty}(\bigcup_{k=1}^{\infty} F_m(J_k))$. This is the desired family of peripheral intervals. We now need only show that (1) and (2) hold.

Choose any $x \in I \times 0$ and select any $\varepsilon > 0$. Now select any positive integer n such that $1/n < \varepsilon$. There exists a J_n such that $(f(x) - 1/16n\ ,\ f(x) + 1/16n) \subset J_n \subset (f(x) - 1/n\ ,\ f(x) + 1/n) \subset (f(x) - \varepsilon\ ,\ f(x) + \varepsilon)$ and there exists an $I_{n(k)} \in F$ such that $x \in I_{n(k)}$ and the length of $I_{n(k)} < 1/n < \varepsilon$. So (2) holds.

Select any $\varepsilon > 0$. Then there exist a positive integer m such that $1/m < \varepsilon$. By construction of F there exists only a finite number of intervals with length at least $1/m$. So (1) holds.

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