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PERIODIC DECOMPOSITIONS OF FUNCTIONS

1. Let $f:\mathbb{R}\to\mathbb{R}$ be a real function and let $a_1,\dots,a_n$ be given real numbers. We say that $f=f_1+\dots+f_n$ is an $(a_1,\dots,a_n)$-__decomposition__ of f if f_i is periodic mod a_i for every $i=1,\dots,n$. If $\mathcal{F}$ is a class of real functions and each f_i belongs to $\mathcal{F}$ then we say that $f=f_1+\dots+f_n$ is an $(a_1,\dots,a_n)$-decomposition in $\mathcal{F}$.

Let Δ_a denote the difference operator, that is let

$$\Delta_a f(x)=f(x+a)-f(x) \qquad (x\in\mathbb{R};\ f:\mathbb{R}\to\mathbb{R}).$$

If $f=f_1+\dots+f_n$ is an $(a_1,\dots,a_n)$-decomposition then $\Delta_{a_i}f_i=0$ for every i and, as the operators Δ_{a_i} commute, we obtain

$$(1)\qquad \Delta_{a_1}\dots\Delta_{a_n}f=0.$$

A class $\mathcal{F}$ of real functions is said to have the __decomposition property__ (d.pr.) if, for every $f\in\mathcal{F}$ and $a_1,\dots,a_n\in\mathbb{R}$, (1) implies that f has an $(a_1,\dots,a_n)$-decomposition in $\mathcal{F}$. Neither the class of all real functions, nor $C(\mathbb{R})$, the class of all continuous functions defined on $\mathbb{R}$ has the d.pr. Indeed, if f is the identity function $f(x)=x$ then $\Delta_a\Delta_b f=0$ for every $a,b\in\mathbb{R}$. On the other hand, if, say, $a=b$ then f does not have an (a,b)-decomposition since f is not periodic.

The following result shows that $BC(\mathbb{R})$, the class of all bounded and continuous functions has the d.pr.

THEOREM 1. __Let__ $a_1,\dots,a_n$ __be real numbers and__ $f\in BC(\mathbb{R})$. __Then__ f __has an__ $(a_1,\dots,a_n)$-__decomposition in__ $C(\mathbb{R})$ __if and only if__ (1) __holds__.

A special case of the theorem above, namely when $n=2$ and a_1/a_2 is irrational, was proved by M. Wierdl in [6].

2. By the norm of the decomposition $f=f_1+\dots+f_n$ we mean $\max_{1\le i\le n} \|f_i\|_\infty$, where $\|g\|_\infty = \sup\{|f(x)| : x \in \mathbb{R}\}$. We denote by C_n the greatest lower bound of all positive numbers C with the property that whenever $f \in BC(\mathbb{R})$ satisfies (1) then f has a continuous $(a_1,\dots,a_n)$-decomposition of norm $\le C\|f\|_\infty$.

THEOREM 2. *For every* $n \ge 2$ *we have* $C_n \le 2^{n-2}$.

In certain cases better estimates can be proved.

THEOREM 3. *Suppose that* $f \in BC(\mathbb{R})$ *satisfies* (1), *where* $a_1,\dots,a_n$ *are pairwise incommensurable. Then* f *has a continuous* $(a_1,\dots,a_n)$-*decomposition with norm not exceeding* $(2-\frac{1}{n})\|f\|_\infty$.

Probably neither of the bounds 2^{n-2} and $2-\frac{1}{n}$ is sharp; the problem of finding the best constants in these theorems proves to be surprisingly difficult. The next two theorems give sharp estimates in some special cases.

THEOREM 4. *Suppose that* $f \in BC(\mathbb{R})$ *satisfies* (1), *where either* $n=2$ *or* $n=3$ *and* a_1,a_2,a_3 *are pairwise incommensurable. Then* f *has a continuous* $(a_1,\dots,a_n)$-*decomposition with norm not exceeding* $\|f\|_\infty$.

THEOREM 5. *Suppose that* $1/a_1,\dots,1/a_n$ *are linearly independent over the rationals. If* $f \in BC(\mathbb{R})$ *satisfies* (1) *then*

(i) $$\sup f = \sum_{i=1}^{n} \sup f_i \quad , \quad \inf f = \sum_{i=1}^{n} \inf f_i$$

hold for every continuous $(a_1,\dots,a_n)$-*decomposition of* f;

(ii) *there is a continuous* $(a_1,\dots,a_n)$-*decomposition* $f=f_1+\dots+f_n$ *such that* $\|f\|_\infty = \sum_{i=1}^{n} \|f_i\|_\infty$.

3. Among the (not necessarily bounded) continuous functions satisfying (1) are the polynomials of degree less than n. This observation leads to the following problem: which functions f can be written in the form $f=p+f_1+\ldots+f_n$ where p is a polynomial of degree $< n$ and $\Delta_{a_i} f_i=0$ $(i=1,\ldots,n)$. We call such a representation an $(a_1,\ldots,a_n)$-<u>quasi-decomposition</u> of f.

If $f \in C(\mathbb{R})$ has a continuous $(a_1,\ldots,a_n)$-quasi-decomposition then (1) must hold. However, it was shown by I.Z. Ruzsa and M. Szegedy that (1) is not sufficient for the existence of such a decomposition. We can give the exact condition in terms of the n-th modulus of continuity of f:

$$\delta_n(f) = \sup_{h\in\mathbb{R}} \|\Delta_h^n f\|_\infty = \sup\left\{\left|\sum_{j=0}^{n}(-1)^j\binom{n}{j}f(x+jh)\right| : x,h\in\mathbb{R}\right\}.$$

THEOREM 6. <u>A function</u> $f \in C(\mathbb{R})$ <u>has an</u> $(a_1,\ldots,a_n)$-<u>quasi-decomposition in</u> $C(\mathbb{R})$ <u>if and only if</u> (1) <u>and</u> $\delta_n(f) < \infty$ <u>hold simultaneously</u>.

As a simple application of this condition, we obtain

THEOREM 7. <u>A function</u> f <u>has an</u> $(a_1,\ldots,a_n)$-<u>quasi-decomposition in</u> $C(\mathbb{R})$ <u>with a linear</u> p <u>if and only if</u> (1) <u>holds and</u> f <u>is uniformly continuous</u>.

4. Let S be a non-empty set and let T be a map of S into itself. A function $g:S \to \mathbb{R}$ is said to be T-periodic, if $g\circ T=g$ or, equivalently, if $\Delta_T g=0$, where $\Delta_T g = g - g\circ T$. Now let $T_1,\ldots,T_n$ be maps of S into itself and let $f=f_1+\ldots+f_n$ where f_i is T_i-periodic for every $i=1,\ldots,n$. If the maps T_i commute, i.e. $T_i\circ T_j = T_j\circ T_i$ hold for every i,j, then the operators Δ_{T_i} also commute and we have

$$\Delta_{T_1}\ldots\Delta_{T_n} f = 0. \tag{2}$$

Let $\mathcal{F}$ be a class of real valued functions defined on S. We say that $\mathfrak{F}$ has the decomposition property (d.pr.) with respect to the maps (w.r.t.) $T_1, \ldots, T_n$ if for every $f \in \mathfrak{F}$, condition (2) implies that there exists a $(T_1, \ldots, T_n)$-decomposition of f in $\mathfrak{F}$, i.e. $f = f_1 + \ldots + f_n$, where $f_i \in \mathfrak{F}$ and $\Delta_{T_i} f_i = 0$ $(i = 1, \ldots, n)$.

Suppose that the class $\mathfrak{F}$ is closed under linear operations and let T be a map of S into itself. Then $Af = f - f \circ T$ $(f \in \mathfrak{F})$ defines a linear operator on $\mathfrak{F}$ such that Ker A consists of all T-periodic functions from $\mathfrak{F}$. This observation together with the next theorem show that some Banach spaces of functions possess the d.pr. w.r.t. "reasonable" mappings.

THEOREM 8. *Let X be a linear space over $\mathbb{R}$, $\|\cdot\|$ be a norm on X and $\mathcal{T}$ be a vector topology on X such that $\{x \in X : \|x\| \leq 1\}$ is $\mathcal{T}$-compact, and if $x_k \in X$ $(k=1,2,\ldots)$ and $\|x_k\| \to 0$ then $x_k \to 0$ in $\mathcal{T}$.*

Let $A_1, \ldots, A_n$ be commuting, $\mathcal{T}$-continuous linear maps of X into itself such that

$$\|A_i - I\| \leqq 1 \quad (i=1,\ldots,n).$$

Then $\operatorname{Ker}(A_1 \ldots A_n)$, as a linear subspace of X, is spanned by the null spaces $\operatorname{Ker} A_i$ $(i=1,\ldots,n)$.

The conditions of this theorem are satisfied if X is a reflexive Banach space with $\mathcal{T}$ being the weak topology. It can be shown that the assertion of the theorem does not hold for every Banach space X and for every system of commuting linear operators A_i satisfying $\|A_i - I\| \leqq 1$.

Applying this theorem it can be proved that the $L^p(S)$ classes for $1 \leqq p < \infty$ possess the d.pr. w.r.t. commuting measurable maps which do not decrease measure, and in σ-finite spaces $L^\infty(S)$ has the d.pr. w.r.t. commuting measurable maps which do not map sets of positive measure is nowhere dense, so

$$\limsup_p ((n_{m_p} \cdot B) \cap (0,1]) \setminus \limsup_p ((n_{m_p} \cdot D) \cap (0,1])$$

is of the first category, hence $\limsup_p ((n_{m_p} \cdot D) \cap (0,1])$ is not of the first category.

Since for each $n \in N$ the set $B \cap C$ is residual in (a_n, b_n) we can choose from this set a sequence $\{x_n^1, \ldots, x_n^{l_n}\}$ fulfilling the above conditions. Take now $x_0 \in B \cap C$. Since f is strongly I_2-approximately continuous at $(x_0, 0)$, this point is a strong I_2-density point of the set

$$\{(x,y) : |f(x,y) - f(x_0,0)| < \frac{\varepsilon_0}{2}\}.$$

From Theorem 1.1 it follows that the last set is the union $G(x_0) \cup P(x_0)$, where $G(x_0)$ is open and $P(x_0)$ is nowhere dense. Moreover, $G(x_0) \cap \{(x,y) : |f(x,y) - f(0,0)| < \frac{\varepsilon_0}{2}\} = \emptyset$.

For the rest of the proof we shall need the following lemma:

LEMMA 1.6. If (x_0, y_0) is a strong I_2-density point of an open set G, then for each positive integer n there exists a number $\delta_n > 0$ such that for every $h', h'' \in (0, \delta_n]$ and for every $i, j \in \{-n+1, \ldots, 0, 1, \ldots, n\}$ we have

$$G \cap ([x_0 + \frac{i-1}{n} \cdot h', x_0 + \frac{i}{n} \cdot h'] \times [y_0 + \frac{j-1}{n} \cdot h'', y_0 + \frac{j}{n} \cdot h'']) \neq \emptyset.$$

Proof of the lemma is essentially the same as of Lemma 1, p. 170 in [16].

Applying the above lemma to $(x_n^l,0)$, $G(x_n^l)$ and n for $n \in N$, $l \in \{1,\dots,l_n\}$, we obtain two positive numbers h_n', h_n'' such that for each $i,j \in \{-n+1,\dots,0,1,\dots,n\}$ and for each $l \in \{1,\dots,l_n\}$ we have

$$G(x_n^l) \cap ([x_n + \frac{i-1}{n}\cdot h_n, x_n^l + \frac{i}{n}\cdot h_n] \times [\frac{j-1}{n}\cdot h_n'', \frac{j}{n}\cdot h_n'']) \neq \emptyset.$$

Obviously we can choose a sequence $\{h_n''\}_{n\in N}$ to be decreasing. We can also suppose that $\frac{1}{h_n''} = k_n'' \in N$. Now take $k_m' = n_m$ and $k_m'' = \frac{1}{h_m''}$ for $m \in N$ (n_m has the same meaning as in the first part of the proof of the theorem). Both sequences $\{k_m'\}_{m\in N}$ and $\{k_m''\}_{m\in N}$ are increasing. If $\{m_p\}_{p\in N}$ is an arbitrary increasing sequence of positive integers and $r \in N$ is arbitrary, then, reasoning as before, where $B_n^l = (x_n^l - h_n, x_n^l + h_n)$ for $n \in N$ and $l \in \{1,\dots,l_n\}$, we conclude that

$$(\bigcup_{p=r}^{\infty} k_{m_p}' \cdot B)) \times [-1,1]$$

$$\setminus \bigcup_{p=r}^{\infty} ((k_{m_p}', k_{m_p}'') \cdot \bigcup_{n=1}^{\infty} \bigcup_{l=1}^{l_n} G(x_n^l))$$

is nowhere dense and so

$$(\limsup_p (k_{m_p}' \cdot B)) \times [-1,1]$$

$$\setminus \limsup_p ((k_{m_p}', k_{m_p}'') \bigcup_{n=1}^{\infty} \bigcup_{l=1}^{l_n} G(x_n^l))$$

is of the first category. Hence

$$\limsup_p \; ((k'_{m_p}, k''_{m_p}) \cdot \bigcup_{n=1}^{\infty} \bigcup_{l=1}^{l_n} G(x_n^l))$$

is not of the first category. From this we conclude that $(0,0)$ is not a strong I_2-density point of the set $\{(x,y) : |f(x,y) - f(0,0)| < \frac{\varepsilon_o}{2}\}$, since this set is disjoint from $\bigcup_{n=1}^{\infty} \bigcup_{l=1}^{l_n} G(x_n^l)$. The contradiction ends the proof.

THEOREM 1.5. If $f : R^2 \to R$ is strongly I_2-approximately continuous, then for every interval (z_1,z_2) each point of the set $f^{-1}((z_1,z_2))$ is its deep strong I_2-density point.

P r o o f. Apply Theorem 1.2, Lemma 1.2, Theorem 1.4, results of [15] stating the same as Theorem 1.5 for one-dimensional case and Theorem 1.3.

REMARK 1.2. In [2] one can find the analogous result with much easier proof because for simple I_2-density the notions of special and deep I_2-density points coincide ([2], Th. 10). In the case of strong I_2-density the situation is different, as the following example shows: if $A = [-1,1]^2 \setminus ((0,1] \times \{0\})$, then $(0,0)$ is a special strong I_2-density point of A but it is not a deep strong I_2-density point of A. The second assertion is nearly obvious. To prove the first one put $K_i = \{(x,y) : x \in (0,1),\ 2^{-i} \cdot x + 2^{-i-2} \cdot x^2 < y < 2^{-i} \cdot x + 2^{-i-2} \cdot x^2\}$ for $i \in N$ and let $B_o = \bigcup_{i=1}^{\infty} K_i$. Obviously B_o and $B = B_o \cup (R^2 \setminus [-1,1]^2)$ are open sets, moreover, $\bar{B} \supset R^2 \setminus A$ and $(0,0)$ is a strong I_2-dispersion point of B. We shall sketch the proof of this last assertion.

Let $\{k'_n\}_{n\in N}, \{k''_n\}_{n\in N}$ be increasing sequences of positive integers. We shall find an increasing sequence $\{n_p\}_{p\in N}$ of positive integers such that $\limsup_p ((k'_{n_p}, k''_{n_p})\cdot B_o) \cap [0,1]^2 \in \mathcal{I}_2$.

We shall consider three cases: $\lim_{n\to\infty} \frac{k''_n}{k'_n} = \infty$, $0 < \underline{\lim}_{n\to\infty} \frac{k''_n}{k'_n} < \infty$ and $\underline{\lim}_{n\to\infty} \frac{k''_n}{k'_n} = 0$.

In the first case let i_n for each $n \in N$ be a positive integer such that $2^{i_n} \le \frac{k''_n}{k'_n} < 2^{i_n+1}$ (we can suppose that $\frac{k''_1}{k'_1} \ge 2$). Put $b_n = 2^{-i_n} \frac{k''_n}{k'_n}$. Since $1 \le b_n < 2$ for each n, then there exists a convergent subsequence $\{b_{n_p}\}_{p\in N}$. Let $b_o = \lim_{p\to\infty} b_{n_p}$. If P_o denotes the straight line $y = b_o\cdot x$, then it is not difficult to observe that $\limsup_p ((k'_{n_p}, k''_{n_p})\cdot K_{i_{n_p}}) \cap [0,1]^2 = P_o \cap [0,1]^2$. If P_j is the straight line $y = b_o\, 2^{-j}\cdot x$ for any integer j, then a moment of reflection shows that

$$\limsup_p ((k'_{n_p}, k''_{n_p})\cdot B_o) \cap [0,1]^2 = \bigcup_{j=-\infty}^{\infty} P_j \cap [0,1]^2.$$

In the second case choose a subsequence $\{\frac{k''_{n_p}}{k'_{n_p}}\}_{p\in N}$ convergent to a positive and finite limit b_o. If P_j, $j \in N \cup \{0\}$, means the same as above, then $\limsup_p ((k'_{n_p}, k''_{n_p})\cdot B_o) \cap [0,1]^2 = \bigcup_{j=1}^{\infty} P_j \cap [0,1]^2$.

In the third case $\limsup_p ((k'_{n_p}, k''_{n_p}) \cdot B_o) \cap [0,1]^2 = \emptyset$ for any increasing sequence $\{n_p\}_{p\in N}$.

Hence regardless of behaviour of $\{k'_n\}_{n\in N}$ and $\{k''_n\}_{n\in N}$ there exists an increasing sequence $\{n_p\}_{p\in N}$ such that

$$\limsup_p ((k'_{n_p}, k''_{n_p}) \cdot B_o) \cap [0,1]^2 \in \mathcal{I}_2.$$

2. Now, we start to investigate separately $\mathcal{I}_1$-approximately continuous functions. We shall extend some of O'Malley's results from [13] concerning separately approximately continuous functions to the category case.

Let $\mathcal{T}_{xy}$ be the collection of all subsets U of R^2 with the Baire property such that for any $x,y \in R$ the sets U_x, U^y have the Baire property and are $\mathcal{I}_1$-open.

THEOREM 2.1. The collection $\mathcal{T}_{x,y}$ forms a topology on R^2. The continuous functions relative to this topology are precisely the separately $\mathcal{I}_1$-approximately continuous functions.

P r o o f. To prove the first assertion we shall only show that for each index set A, if U_α belongs to $\mathcal{T}_{xy}$ for all $\alpha \in A$, then $U = \bigcup_{\alpha\in A} U_\alpha$ has the Baire property (the rest is easy).

For any $D \in \mathcal{S}_2$ we denote

$$\varphi_x(D) = \{(t,y) \in R^2 : t \text{ is an } \mathcal{I}_1\text{-density point of } A \text{ for some } A \in \mathcal{S}_1,\ A \subset D^y\},$$

$$\varphi_y(D) = \{(x,t) \in R^2 : t \text{ is an } \mathcal{I}_1\text{-density point of } A \text{ for some } A \in \mathcal{S}_1,\ A \subset D_x\}.$$

Note that $\varphi_x(D) \Delta D$, $\varphi_y(D) \Delta D$ are of the first category (cf. [2], th.4). Consequently, $\varphi_y(\varphi_x(D)) \Delta D$ is of the first category.

Let B be a Baire kernel of U (i.e. a subset of U having the Baire property such that each subset of $U \setminus B$ having the Baire property is of the first category). We may assume that B is a Borel set. It is enough to prove that $U \subset \varphi_y(\varphi_x(B))$ since this inclusion together with the known facts $B \subset U$, $\varphi_y(\varphi_x(B)) \setminus B \in I_2$ imply that $U \in S_2$. So, let $(x_o,y_o) \in U$. Then $(x_o,y_o) \in U_\alpha$ for some $\alpha \in \Lambda$. From the properties of a Baire kernel, it follows that $U \setminus B \in I_2$. By the Kuratowski-Ulam theorem, (see for example [14]), there is a residual set $E \subset R$ such that $(U_\alpha)^y \setminus B^y \in I_1$ for all $y \in E$. Since $U_\alpha \in T_{xy}$, therefore x_o is an I_1-density point of $(U_\alpha)^y$ for all $y \in (U_\alpha)_{x_o}$ Thus it follows that x_o is an I_1-density point of B^y for all $y \in (U_\alpha)_{x_o} \cap E$. Consequently, $(U_\alpha)_{x_o} \cap E \subset (\varphi_x(B))_{x_o}$ Hence $(U_\alpha)_{x_o} \setminus (\varphi_x(B))_{x_o}$ is included in $R \setminus E$, so it is of the first category. This implies that y_o is an I_1-density point of $(U_\alpha)_{x_o} \cap (\varphi_x(B))_{x_o}$ since, by $U \in T_{xy}$, it is an I_1-density point of $(U_\alpha)_{x_o}$. Thus $(x_o,y_o) \in \varphi_y(\varphi_x(B))$.

The second assertion follows immediately from the definition of T_{xy} and from the fact that a separately I_1-approximately continuous function has the Baire property (cf. [18]).

THEOREM 2.2. If $(x_o,y_o) \in U \in T_{xy}$, then (x_o,y_o) is not a strong I_2-dispersion point of U.

P r o o f. We may assume that $(x_o, y_o) = (0,0)$. At first, we shall construct sequences $\{h'_n\}, \{h''_n\}$ of real positive numbers tending decreasingly to 0 such that for any $n \in N$ and $i,j \in \{-n+1,\dots,n\}$ we have

$$(**) \qquad \left(\left[\frac{i-1}{n} h'_n, \frac{i}{n} h'_n\right] \times \left[\frac{j-1}{n} h''_n, \frac{j}{n} h''_n\right]\right) \cap U \notin \mathcal{I}_2.$$

Let $n \in N$. From $U \in \mathcal{T}_{xy}$ it follows that 0 is an $\mathcal{I}_1$-density point of U^o. Therefore, by Lemma 0.1, there exists $\delta_n > 0$ such that for any $h \in (0,\delta_n)$ and $i \in \{-n+1,\dots,n\}$ we have

$$\left[\frac{i-1}{n} h, \frac{i}{n} h\right] \cap U^o \notin \mathcal{I}_1.$$

Choose $h'_n \in (0, \min\{1/n, \delta_n\})$. Fix any $i \in \{-n+1,\dots,n\}$. Observe that, since $U \in \mathcal{T}_{xy}$, therefore 0 is an $\mathcal{I}_1$-denisty point of U_t for each $t \in \left[\frac{i-1}{n} h'_n, \frac{i}{n} h'_n\right] \cap U^o$. So, by Lemma 0.1, we can choose $\delta(t) > 0$ such that for any $h \in (0,\delta(t))$ and $j \in \{-n+1,\dots,n\}$ we have

$$\left[\frac{j-1}{n} h, \frac{j}{n} h\right] \cap U_t \notin \mathcal{I}_1.$$

Consequently,

$$\left[\frac{i-1}{n} h'_n, \frac{i}{n} h'_n\right] \cap U^o = \bigcup_{p=1}^{\infty} A_p$$

where $A_p = \{t : t \in \left[\frac{i-1}{n} h'_n, \frac{i}{n} h'_n\right] \cap U^o, \ \delta(t) \geq \frac{1}{p}\}$ for $p \in N$. Then we can choose $p_o = p_o(i) \in N$ such that $A_{p_o(i)} \notin \mathcal{I}_1$. So, for each $t \in A_{p_o(i)}$ and for any $h \in (0, 1/p_o(i))$ and $j \in \{-n+1,\dots,n\}$, we obtain

$$[\frac{j-1}{n} h, \frac{j}{n} h] \cap U_t \notin I_1.$$

Thus, by the converse to the Kuratowski-Ulam theorem ([14]) for any $j \in \{-n+1,\dots,n\}$ and $h \in (0,1/p_o(i))$ we have

$$[\frac{i-1}{n} h_n', \frac{i}{n} h_n'] \times [\frac{j-1}{n} h, \frac{j}{n} h] \cap U \notin I_2.$$

Choose $h_n'' \in (0,\min\{1/n,1/p_o(-n+1),\dots,1/p_o(n)\}$. Then (**) evidently holds. Proceeding inductively we can easily get $h_{n+1}' < h_n'$, $h_{n+1}'' < h_n''$ for all $n \in N$.

Now, put $t_n' = 1/h_n'$, $t_n'' = 1/h_n''$, $n \in N$. Then $\{t_n'\}$, $\{t_n''\}$ are increasing sequences of real numbers tending to infinity. The set U has the Baire property, so there is an open set G such that $U \Delta G \in I_2$. Then, by (**), for any $n \in N$ and $i,j \in \{-n+1,\dots,n\}$, we have

$$(***) \qquad ((t_n',t_n'') \cdot G) \cap ([\frac{i-1}{n}, \frac{i}{n}] \times [\frac{j-1}{n}, \frac{j}{n}]) \neq \emptyset.$$

Denote $E_n = ((t_n',t_n'') \cdot G) \cap [-1,1]^2$ for $n \in N$. Let $\{n_p\}$ be an arbitrary increasing sequence of positive integers. For each $S \in N$ the set $\bigcup_{p=S}^{\infty} E_{n_p}$ is open and, by (***), dense in $[-1,1]^2$. Thus $\limsup_p E_{n_p}$ is residual in $[-1,1]^2$. Since $U \Delta G \in I_2$, the same holds for $\limsup_p ((t_{n_p}',t_{n_p}'') \cdot U) \cap [-1,1]^2$. Consequently, the sequence $\{\chi_{((t_{n_p}',t_{n_p}'')\cdot U)\cap[-1,1]^2}\}_{p\in N}$ does not converge I_2-a.e. to 0. Thus, by Theorem 0.2, we get the assertion.

We say that $(0,0)$ is an upper strong I_2-density point of $A \in S_2$ if there exist increasing sequences $\{t'_n\}_{n\in N}, \{t''_n\}_{n\in N}$ of real numbers tending to infinity such that

$$\{\chi_{((t'_n,t''_n)\cdot A)\cap[-1,1]^2}\}_{n\in N}$$

converges I_2-a.e. to $\chi_{[-1,1]^2}$. The respective definition for an arbitrary point can be formulated by using the standard translation trick.

PROBLEMS. Must each point of $U \in T_{xy}$ be its upper strong I_2-density point? Is there, for any $y \in R$ and a separately I_1-approximately continuous function f, a set $E \in I_1$ such that f is strongly I_2-approximately continuous at (x,y) for all $x \in R \setminus E$? Note that the measure analogues of these questions are answered in affirmative (cf. [13], Th. 2 and 3).

3. In this section, we show that a separately I_1-approximately continuous function is Baire 2 and need not be Baire 1 (our result is analogous to that for separately approximately continuous functions, cf. [3]).

At first, recall some facts from [10].

For $x \in R$ let $P(x)$ be the family of all intervals $[a,b]$ such that $x \in (a,b)$ and of all interval sets P of the form $P = \bigcup_{n=1}^{\infty} [a_n,b_n] \cup \bigcup_{n=1}^{\infty} [c_n,d_n] \cup \{x\}$ (where $a_n < b_n < a_{n+1} < x < d_{n+1} < c_n < d_n$ for all n and $\lim_n a_n = \lim_n d_n = x$) such that x is an I_1-density point of P.

Let τ be the family of all I_1-open sets such that for each nonempty $U \in \tau$ and each $x \in U$ there is $P \in \mathcal{P}(x)$ contained in $\{x\} \cup \operatorname{int} U$. Then τ forms a topology which is The coarsest one for I_1-approximately continuous functions.

For $A \subset R$ let $\Delta(A)$ denote the set of all $x \in R$ such that $P \cap A \setminus \{x_0\} \neq \emptyset$ for each $P \in \mathcal{P}(x_0)$.

Here we shall say that sets $A, B \subset R^2$ have property (ds) if and only if for each set $X \subset R^2$, conditions $\overline{A \cap X} = X$, $\overline{B \cap X} = X$ imply $X = \emptyset$.

LEMMA 3.1. Let $A, B \subset R^2$.

(a) If A, B have property (ds) and $A_1 \subset A$, $B_1 \subset B$, then A_1, B_1 have property (ds).

(b) If A_i, B have property (ds) for $i = 1, 2, \ldots, n$, then $\bigcup_{i=1}^{n} A_i, B$ have property (ds).

P r o o f. (a) Observe that $\overline{A_1 \cap X} = X$, $\overline{B_1 \cap X} = X$ imply $\overline{A \cap X} = X$, $\overline{B \cap X} = X$.

(b) Suppose that there is $X \neq \emptyset$ such that $\bigcup_{i=1}^{n} \overline{A_1 \cap X} = X$, $\overline{B \cap X} = X$. Then there are an open set G and a number $k \in \{1, \ldots, n\}$ such that $\emptyset \neq G \cap X \subset \overline{A_k \cap X}$. Then we have

$$\overline{A_k \cap \overline{G \cap X}} = \overline{A_k \cap G \cap X} = \overline{G \cap X},$$

$$\overline{B \cap \overline{G \cap X}} = \overline{B \cap G \cap X} = \overline{G \cap X},$$

which gives a contradiction.

For a function $f : R^2 \to R$ and $\alpha \in R$ we denote

$$E_\alpha = \{(x,y) : f(x,y) < \alpha\}, \quad E^\alpha = \{(x,y) : f(x,y) > \alpha\}.$$

LEMMA 3.2. If $f : R^2 \to R$ is separately $\mathcal{I}_1$-approximately continuous, then for any $r_1, r_2 \in R$, $r_1 < r_2$, there exist disjoint sets $H_{r_1 r_2}$, $K_{r_1 r_2}$ of type $G_{\delta\sigma}$ such that $E_{r_1} \subset H_{r_1 r_2}$ and $E^{r_2} \subset K_{r_1 r_2}$.

P r o o f. Let $\alpha, \beta \in R$ and $\alpha < \beta$ For any $n, k \in N$, $h > 0$, $i \in \{1,\ldots,n\}$, $j \in \{1,\ldots,k\}$ let

$$A_{nhi} = \{(x,y) \in E_\alpha : \{x\} \times [y + \frac{i-1}{n} h, y + \frac{i}{n} h] \subset E_\alpha\},$$

$$B_{nkhij} = \{(x,y) \in E^\beta : \{x\}$$

$$\times [y + \frac{(i-1)k + j - 1}{nk} h, y + \frac{(i-1)k + j}{nk} h] \subset E^\beta\}.$$

Observe that

$$E_\alpha = \bigcup_{n=1}^{\infty} \bigcap_{h \in (0,\frac{1}{n})}^{\infty} \bigcup_{i=1}^{\infty} A_{nhi}.$$

Indeed, consider an arbitrary $(x_0, y_0) \in E_\alpha$. Then there exists a closed set $F \subset E_\alpha$ (see the definition of τ) such that y_0 is an $\mathcal{I}_1$-density point of F_{x_0}. Then, by Lemma 0.1, there exist $n_0 \in N$ and $\delta > 0$ such that for each $h \in (0,\delta)$ there exits $i_0 \in \{1,\ldots,n_0\}$, $i_0 = i_0(h)$, such that

$$\{x_0\} \times [y_0 + \frac{i_0 - 1}{n_0} h, y_0 + \frac{i_0}{n_0} h] \subset F \subset E_\alpha.$$

Choose $l \in N$ such that for $n_1 = n_0 l$ we have $n_1 > 1/\delta$. For each $h \in (0,1/n_1)$ put $i_1 = i_0(h)l$. Then $(x_0,y_0) \in A_{n_1hi_1}$. So, we have shown that

$$(x_0,y_0) \in \bigcup_{n=1}^{\infty} \bigcap_{h\in(0,\frac{1}{n})} \bigcup_{i=1}^{n} A_{nhi}$$

and now the desired equation is clear. In a similar way, we can prove that for each $n \in N$

$$E^{\beta} = \bigcup_{k=1}^{\infty} \bigcap_{h\in(0,\frac{1}{k})} \bigcap_{i=1}^{n} \bigcup_{j=1}^{k} B_{nkhij}.$$

Consider fixed $n \in N$, $k \in N$, $h \in (0, \min\{\frac{1}{n}; \frac{1}{k}\})$, $i \in \{1,\dots,n\}$, $j \in \{1,\dots,k\}$. We shall prove that A_{nhi}, B_{nkhij} have property (ds). Suppose to the contrary that there is $X \neq \emptyset$ such that

$$\overline{A_{nhi} \cap X} = X \qquad \text{and} \qquad \overline{B_{nkhij} \cap X} = X.$$

Choose a rectangle $[a,b] \times [c,d]$ such that $d - c < h/nk$ and the set $X_1 = ([a,b] \times [c,d]) \cap X$ is nonempty. Put

$$y_0 = \frac{c+d}{2} + \frac{(i-1)k + j - 1}{nk} h + \frac{1}{2nk} h.$$

Since $d - c < h/nk$, we easily observe that

$$(*) \qquad y_0 \in [y + \frac{(i-1)k + j - 1}{nk} h,\ y + \frac{(i-1)k + j}{nk} h]$$

for each $y \in [c,d]$.

For $(x,y) \in R^2$ let $pr(x,y) = x$ and denote

$$P = pr\,(X_1), \quad P_A = pr\,(A_{nhi} \cap X_1),$$

$$P_B = pr\,(B_{nknij} \cap X_1).$$

Then P is closed since X_1 is compact. By the property that for any continuous mapping the image of a dense subset of the domain is dense in the image of the domain, P_A, P_B are dense in P. Next, observe that

$$P_A \times \{y_o\} \subset E_\alpha, \qquad P_B \times \{y_o\} \subset E^\beta.$$

Indeed, to show the first inclusion, consider any $x \in P_A$. Then $(x,y) \in A_{nhi} \cap X_1$ for some y. Then $y \in [c,d]$, so, by $(*)$, we deduce that

$$y_o \in [y + \frac{i-1}{n} h,\ y + \frac{i}{n} h].$$

Hence $(x,y) \in A_{nhi}$ implies that $(x,y_o) \in E_\alpha$. Thus we have shown that $P_A \times \{y_o\} \subset E_\alpha$. Similarly, one can prove that $P_B \times \{y_o\} \subset E^\beta$.

The function f_{y_o} is I_1-approximately continuous and therefore Baire 1 (see [16]), so the restriction $f_{y_o}|P$ has a point of continuity $x_o \in P$. Since $P = \overline{P}_A$, there is a sequence $\{x_n\}_{n \in N} \subset P_A$ which tends to x_o. Since $(x_n,y_o) \in P_A \times \{y_o\} \subset E_\alpha$, therefore $f(x_n,y_o) \leq \alpha$, $n \in N$, and thus $f(x_o,y_o) \leq \alpha$. Analogously, from $P = \overline{P}_B$ and $P_B \times \{y_o\} \subset E^\beta$ it follows that $f(x_o,y_o) \geq \beta$. We have obtained a contradiction. Thus, A_{nhi}, B_{nkhij} have property (ds).

Now, for any $n,k \in N$ let

$$C_n = \bigcap_{h\in(0,\frac{1}{n})} \bigcup_{i=1}^{n} A_{nhi}$$

and

$$D_{nk} = \bigcap_{h\in(0,\frac{1}{n})} \bigcap_{i=1}^{n} \bigcup_{j=1}^{k} B_{nkhij}.$$

then

$$C_n \subset \bigcap_{h\in(0,\min\{\frac{1}{n},\frac{1}{k}\})} \bigcup_{i=1}^{n} A_{nhi}$$

and

$$D_{nk} \subset \bigcap_{h\in(0,\min\{\frac{1}{n},\frac{1}{k}\})} \bigcap_{i=1}^{n} \bigcup_{j=1}^{k} B_{nkhij}.$$

By the previous part of the proof and Lemma 1 (b), the sets

$$A_{nhi}, \qquad \bigcup_{j=1}^{k} B_{nkhij}$$

have property (ds) for $i = 1,2,\dots,n$. Next, by Lemma 3.1 (b), the sets

$$\bigcup_{i=1}^{n} A_{nhi}, \qquad \bigcup_{j=1}^{k} B_{nkhij}$$

have property (ds), and, finally, by Lemma 3.1 (a), the sets C_n, D_{nk} have that property.

For $n,k \in N$ let U_{nk}, V_{nk} be disjoint sets of type G_δ such that $C_n \subset U_{nk}$, $D_{nk} \subset V_{nk}$ (see [9], Chapter I, § 12, III, 1°, p. 65). Let $W_{\alpha\beta} = \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} U_{nk}$. Then $W_{\alpha\beta}$ is of type $G_{\delta\sigma}$.

Now, we observe that $E_\alpha \subset W_{\alpha\beta}$ and $E^\beta \subset \bigcap_{s=1}^{\infty} \bigcup_{p=1}^{\infty} V_{sp}$. Indeed, for any $n,k \in N$ we have $C_n \subset U_{nk}$ therefore $C_n \subset \bigcap_{k=1} U_{nk}$ for each $n \in N$ and thus

$$E_\alpha = \bigcup_{n=1}^{\infty} C_n \subset \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} U_{nk}.$$

Since

$$E^\beta = \bigcup_{p=1}^{\infty} D_{sp} \subset \bigcup_{p=1}^{\infty} V_{sp}$$

for each $s \in N$, therefore

$$E^\beta \subset \bigcap_{s=1}^{\infty} \bigcup_{p=1}^{\infty} V_{sp}.$$

Now, it follows that $W_{\alpha\beta} \cap E^\beta = \emptyset$ since

$$W_{\alpha\beta} \cap E^\beta \subset \bigcup_{n=1}^{\infty} \bigcap_{k=1}^{\infty} U_{nk} \cap \bigcap_{s=1}^{\infty} \bigcup_{p=1}^{\infty} V_{sp}$$

$$\subset \bigcup_{n=1}^{\infty} \left(\bigcap_{k=1}^{\infty} U_{nk} \cap \bigcup_{p=1}^{\infty} V_{np} \right)$$

$$= \bigcup_{n=1}^{\infty} \bigcup_{p=1}^{\infty} \left(\bigcap_{k=1}^{\infty} U_{nk} \cap V_{np} \right)$$

$$\subset \bigcup_{n=1}^{\infty} \bigcup_{p=1}^{\infty} (U_{np} \cap V_{np}) = \emptyset.$$

In a similar way we can show that there exists a set $V_{\alpha\beta}$ of type $G_{\delta\sigma}$ such that $V_{\alpha\beta} \supset E^\beta$ and $V_{\alpha\beta} \cap E_\alpha = \emptyset$.

Now, we are able to get the assertion. Let $r_1, r_2 \in R$ and

$r_1 < r_2$. Choose any r_3, r_4 such that $r_1 < r_3 < r_4 < r_2$. We know that there exist sets $W_{r_1 r_3}$, $V_{r_4 r_2}$ of type $G_{\delta\sigma}$ such that

$E_{r_1} \subset W_{r_1 r_3}$, $E^{r_3} \cap W_{r_1 r_3} = \emptyset$, $E^{r_2} \subset V_{r_4 r_2}$, $E_{r_4} \cap V_{r_4 r_2} = \emptyset$.

Let $K_{r_1 r_2} = W_{r_1 r_3}$, $H_{r_1 r_2} = V_{r_4 r_2}$. Then $K_{r_1 r_2}$, $H_{r_1 r_2}$ are sets of type G_δ such that $E_{r_1} \subset K_{r_1 r_2}$, $E^{r_2} \subset H_{r_1 r_2}$ and

$$K_{r_1 r_2} \cap H_{r_1 r_2} = W_{r_1 r_3} \cap V_{r_4 r_2} \subset (R^2 \setminus E^{r_3}) \cap (R^2 \setminus E_{r_4})$$

$$= R^2 \setminus (E^{r_3} \cup E_{r_4}) = \emptyset.$$

THEOREM 3.1. If $f : R^2 \to R$ is separately $\mathcal{I}_1$-approximately continuous then is Baire 2.

P r o o f. The function f will be proved to be Baire 2 if we show that E_c and E^c are of type $G_{\delta\sigma}$ for every c. Let sets K, H with the respective indices have the meaning as in Lemma 3.2. We then have

$$E_c = \bigcup_{n=1}^{\infty} H_{c-\frac{1}{n}, c-\frac{1}{n+1}}, \qquad E^c = \bigcup_{n=1}^{\infty} K_{c+\frac{1}{n+1}, c+\frac{1}{n}}.$$

Hence E_c and E^c are of type $G_{\delta\sigma}$ and the theorem is proved.

LEMMA 3.3. Let $x_0 \notin \Delta(F)$ where F is a closed subset of R. Then there exists an $\mathcal{I}_1$-approximately continuous function $g : R \to [0,1]$ such that $g(x_0) = 1$ and $\overline{\{x : g(x) \neq 0\}} \cap F \setminus \{x_0\} = \emptyset$.

P r o o f. Since $x_0 \notin \Delta(F)$, there is a set $P \in \mathcal{P}(x_0)$ such that $P \cap F \setminus \{x_0\} = \emptyset$. Assume that

$$P = \bigcup_{n=1}^{\infty} [a_n, b_n] \cup \bigcup_{n=1}^{\infty} [c_n, d_n] \cup \{x_0\}$$

where $a_n < b_n < a_{n+1} < x_0 < d_{n+1} < c_n < d_n$ for all n and $\lim_n a_n = \lim_n b_n = x_0$. Let $G_0 = \bigcup_{n=1}^{\infty} (a_n', b_n') \cup \bigcup_{n=1}^{\infty} (c_n', d_n')$ be such that $a_n' < a_n < b_n < b_n' < a_{n+1}' < x_0 < d_{n+1}' < c_n' < c_n < d_n' \; d_n$ and $\overline{G}_0 \cap F \setminus \{x_0\} = \emptyset$. Let $G = \operatorname{int}(R \setminus G_0)$. Put

$$g(x) = \begin{cases} 1 & \text{if } x = x_0 \\ \dfrac{\rho(x,\overline{G})}{\rho(x,\overline{G}) + \rho(x,P)} & \text{if } x \neq x_0. \end{cases}$$

Then g is I_1-approximately continuous, $g(x_0) = 1$ and

$$\overline{\{x : g(x) \neq 0\}} \cap F \setminus \{x_0\} = \overline{(R \setminus \overline{G}) \cup \{x_0\}} \cap F \setminus \{x_0\}$$

$$= \overline{G}_0 \cap F \setminus \{x_0\} = \emptyset.$$

In the case when $P = [a,b]$ and $x_0 \in [a,b]$, we choose $G_0 = (a', b')$ such that $a' < a < b < b'$ and $\overline{G}_0 \cap F \setminus \{x_0\} = \emptyset$. The rest is the same as above.

THEOREM 3.2. There exists a separately I_1-approximately continuous function $f : R^2 \to R$ which is not Baire 1.

P r o o f. Let C be the perfect nowhere dense set constructed in [1]. Denote by $D = \{x_n : n \in N\}$ the set of all end-points of components of $[0,1] \setminus C$. By the construction, $x_n \notin \Delta(C)$ for all n. We shall define a separately I_1-approximately continuous function $f : R^2 \to R$ such that $f(x,x) = 1$ for $x \in D$ and $f(x,x) = 0$ for $x \in C \setminus D$; it follows from the well-known Baire's criterion that f is not Baire 1.

By Lemma 3.3 (applied with $F = C$) there is an I_1-approximately continuous function $g_1 : R \to [0,1]$ such that $g_1(x_1) = 1$ and $C \setminus \{x_1\}$ does not intersect the set

$$F_1 = \overline{\{x : g_1(x) \neq 0\}}.$$

By the definition of Δ, we have $\Delta(F_1) \subset \overline{F}_1 = F_1$, so $x_2 \notin F_1$ implies $x_2 \notin \Delta(F_1)$. Consequently, we easily get $x_2 \notin \Delta(C \cup F_1)$. By Lemma 3.4 again (applied with $F = C \cup F_1$) there is an I_1-approximately continuous function $g_2 : R \to [0,1]$ such that $g_2(x_2) = 1$ and $(C \cup F_1) \setminus \{x_2\}$ does not intersect the set $F_2 = \overline{\{x : g_2(x) \neq 0\}}$. Next, we proceed by induction.

Finally, observe that the function f defined on R^2 by

$$f(x,y) = \sum_{n=1}^{\infty} g_n(x) g_n(y)$$

fulfils the assertion.

4. In the proof of Theorem 3.1 we do not use the fact, shown in [18], that each separately I_1-approximately continuous function has the Baire property. It makes a contrast with the paper [3] of Davies who while proving that a separately approximately continuous function is Baire 2 uses his own result that this function is measurable. Now, we shall describe further properties of functions with I_1-approximately continuous sections,

which strengthen the above mentioned theorem of [18].

Let X and Y be topological spaces. A mapping $f : X \to Y$ is called quasicontinuous at a point $x \in X$ if for each neighbourhood U of x and each neighbourhood V of $f(x)$ there exists an open nonempty set $U_o \subset U$ such that $f(U_o) \subset U$ (cf. [8], [12]). If f is quasicontinuous at each point of X, it is called quasicontinuous.

From the definition we get

LEMMA 4.1. A mapping $f : X \to Y$ is quasicontinuous at a point x if and only if for any neighbourhoods U,V of x, $f(x)$, respectively, $f^{-1}(V) \cap U$ has the nonempty interior.

Further, we shall consider quasicontinuity only in the cases when $X = R$ or R^2 and $Y = R$, and the natural topologies in X,Y are considered.

In [16] (the proof of Th. 8) the following property is observed

LEMMA 4.2. If $f : R \to R$ is I_1-approximately continuous, then for any interval (a,b) there is an open set G included in $f^{-1}((a,b))$ and dense in $f^{-1}((a,b))$.

From Lemmas 4.1 and 4.2 we easily deduce

THEOREM 4.1. If $f : R \to R$ is I_1-approximately continuous, then it is quasicontinuous.

COROLLARY 4.1. If $f : R^2 \to R$ is separately I_1-approximately continuous then it is quasicontinuous.

P r o o f. The assertion follows from Proposition 4.3 and the fact that each separately quasicontinuous function is quasicontinuous (see [8]).

COROLLARY 4.2. If $f : R^2 \to R$ has all sections f_x $\mathcal{I}_1$-approximately continuous and all sections f^y with the Baire property, then f has the Baire property.

P r o o f. Grande showed in [7] that if $f : R^2 \to R$ has all sections f_x quasicontinuous and all sections f^y with the Baire property, then f has the Baire property. So, Theorem 4.1 yields the assertion.

REMARK 4.1.

(a) Each quasicontinuous function has the Baire property (even its set of discontinuity points is of the first category) but the converse is false (see [12]).

(b) Comparing Theorem 3.3 and Corollary 4.4, let us recall that any implication between quasicontinuity and being a Baire 2 function does not hold (see [12]).

(c) The measure analogue of Theorem 4.1 is false. Indeed, since the density topology is completely regular ([5]), there exists an approximately continuous function $g : R \to [0,1]$ such that $g(\sqrt{2}) = 0$ and $g(x) = 1$ for each rational x. Let $U = g^{-1}([0,1/2))$, $V = R$. Then, by Lemma 4.1, g is not quasicontinuous. If we put $f(x,y) = g(x)$ for $x,y \in R^2$, we observe that the measure analogue of Corollary 4.1 is false.

(d) By assuming Continuum Hypothesis it is proved in [4] that there exists a function $f : R^2 \to R$ with approximately continuous sections f_x and measurable sections f^y which is not measurable. So, then the measure analogue of Corollary 4.2 becomes false.

PROBLEMS. By Corollary 4.1 and Remark 4.1 (a) the set of discontinuity points of a separately I_1-continuous function is of the first category. It would be interesting to obtain an exact characterization of this set. Grande in [6] proved that there is a separately approximately continuous function $f : [0,1]^2 \to R$ whose set of discontinuity points contains the diagonal. Is this possible for separately I_1-approximately continuous functions? (Note that this is impossible for separately continuous functions; cf [6]).

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