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ON CONTINUITY OF DARBOUX MULTIFUNCTIONS

There are several theorems on the continuity of Darboux functions. For instance, each one–to–one Darboux function is continuous and each Darboux function with closed inverse images $f^{-1}(y)$ is continuous. Some of these theorems concern functions of many variables possessing a weak Darboux property. Generalizations of those theorems can shed light on the question of why is it that some Darboux functions are continuous. What properties of the real line as the domain and range in these theorems are essential? Can we consider multifunctions? Each generalization takes a step in the direction in clarifying the question. Some new generalizations are considered in this report.

Let X and Y be topological spaces. Throughout this paper F will denote a multifunction on X with values contained in Y. Let $E \subset Y$. The set $\{x : F(x) \cap E \neq \emptyset\}$ is denoted by $F^{-}(E)$ and is called the big inverse image of E. The set $\{x : F(x) \subset E\}$ is denoted by $F^{+}(E)$ and is called the small inverse image of E. We write $F^{-}(y)$ and $F^{+}(y)$ instead of $F^{-}(\{y\})$ and $F^{+}(\{y\})$, respectively. F is said to be upper semi–continuous at a point $x_0 \in X$ if for each open set $U \supset F(x_0)$ there exists an open set $G \subset X$ such that for each point $x \in G$, $F(x) \subset U$. We write

$F \in \mathscr{C}_{u,s}$ if F is upper semi–continuous (in brief, u.s.c.) at each point $x \in X$,

$F \in \mathscr{D}$ if $F(E) := \bigcup_{x \in E} F(x)$ is connected for every connected set $E \subset X$,

$F \in \mathscr{D}^*$ if $F(E)$ is connected for every connected open set $E \subset X$,

$F \in \mathscr{G}$ if there exists a set Y_1 dense in Y such that $F^-(y)$ is closed for each $y \in Y_1$,

$F \in \mathscr{K}$ if there exists a base $\mathscr{B}$ of open sets of Y such that $F^-(\mathrm{Fr}\ U)$ is closed for each $U \in \mathscr{B}$

$F \in \mathscr{K}^*$ if there exists a base $\mathscr{B}$ of open sets of Y such that for each finite family of sets $U_i \in \mathscr{B}$ $(i = 1,2, \dots n)$, $F^-(\mathrm{Fr} \bigcup_{i=1}^{n} U_i)$ is closed.

It is obvious that $\mathscr{C}_{u,s} \subset \mathscr{K}^* \subset \mathscr{K}$ and $\mathscr{D} \subset \mathscr{D}^*$. We identify a function $f : X \to Y$ with the multifunction F defined by $F(x) = \{f(x)\}$. Let $\mathscr{P}$ be a class of multifunctions. We shall denote by $\overline{\mathscr{P}}$ the class of all multifunctions $F \in \mathscr{P}$ such that all values of F are singletons, and identify each $F \in \overline{\mathscr{P}}$ with the function f defined by $f(x) = y$ where $F(x) = \{y\}$. Obviously, $\overline{\mathscr{C}}_{u,s}$ is identical with the class $\overline{\mathscr{C}}$ of all continuous functions defined on X.

In the sequel we assume that X is locally connected.

It was proved in our previous paper [2] that $\overline{\mathscr{D}} \cap \overline{\mathscr{G}} = \overline{\mathscr{C}}$ for $Y = \mathbb{R}$. H.W. Pu and H.H. Pu generalized this theorem showing that $\overline{\mathscr{D}} \cap \overline{\mathscr{K}} \subset \overline{\mathscr{C}}$ and, under the additional assumption that $Y = \mathbb{R}$, that $\overline{\mathscr{G}} \subset \overline{\mathscr{K}}$ J.M. Jedrzejewski established the inclusion $\overline{\mathscr{D}} \cap \overline{\mathscr{G}} \subset \overline{\mathscr{C}}$ under the assumption that Y is a connected, locally connected ordered space. (See [3] and [1].)

The purpose of this paper is to extend these results to $\mathscr{D}^*$, $\mathscr{G}$, and $\mathscr{K}^*$.

Theorem 1. Let X be a locally connected topological space and Y a topological space. If F is a multifunction on X with connected compact values contained in Y and $F \in \mathscr{D}^* \cap \mathscr{H}^*$ then $F \in \mathscr{C}_{u,s}$.

The proof is a modification of the one given for Theorem 1 in the paper [3].

Lemma. Let Y be a connected topological space with an ordered topology. Then $\mathscr{G} \subset \mathscr{H}^*$ for each topological space X.

Proof. Let the ordered topology be induced by a linear order $<$. Let $F \in \mathscr{G}$ and Y_1 be a set as mentioned in the definition of $\mathscr{G}$ We only need to find a base $\mathscr{B}$ with the properties mentioned in the definition of $\mathscr{H}^*$. It is clear that the family $\mathscr{B}$ of all sets $(\leftarrow,a) := \{x : x < a\}$, $(a,\rightarrow) := \{x := a < x\}$, and $(a,b) := \{x : a < x < b\}$, where $a,b \in Y_1$, is a base. Note that $Fr(a,b) = \{a\} \cup \{b\}$, $Fr(\leftarrow,a) = \{a\}$, and $Fr(a,\rightarrow) = \{a\}$. If $U_i \in \mathscr{B}$ $(i = 1,2, \ldots, n)$, then $Fr \bigcup_{i=1}^{n} U_i$ is a finite subset of Y_1. Also, $F^-(Fr \bigcup_{i=1}^{n} U_i)$ is closed and $F \in \mathscr{H}^*$

Theorem 2. Let X be a locally connected topological space and Y a connected space with the ordered topology. If F is a multifunction with connected compact values then $F \in \mathscr{D}^* \cap \mathscr{G}$ implies $F \in \mathscr{C}_{u,s}$.

The Theorem follows from the Lemma and Theorem 1. It is a generalization of the Jedrzejewski Theorem mentioned above. His Theorem also follows from the next

Theorem 3. Let X be a locally connected topological space and Y a connected space with the ordered topology. Then for a multifunction F on X with connected closed values contained in Y the following properties are equivalent:

$$\text{(i) } F \in \mathscr{D} \cap \mathscr{G} \quad \text{(ii) } F \in \mathscr{D}^* \cap \mathscr{G} \quad \text{(iii) } F \in \mathscr{C}_{u,s}.$$

The proof of Theorem 3 is similar to that for Theorem 1 in our previous paper [2].

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