TYPICAL COMPACT SETS IN THE HAUSDORFF METRIC ARE POROUS

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The idea of set porosity was introduced by E. P. Dolženko [3] as a way of describing the "thinness" of a set. Since then, many papers have been published which show that porous and σ-porous sets are actually the typical set encountered in many situations; e.g., see [1], [2], [4] and [6]. This paper contains another such result. We show that, in a sense, the typical compact subset of $\mathbf{R}$ is porous. To be more precise, it is shown that if the compact subsets of $\mathbf{R}$ are given the Hausdorff metric, then the porous sets are a dense $\mathbf{G}_\delta$ subset of this complete metric space.

First, we introduce some notation and definitions. If $S \subset \mathbf{R}$ and $x \epsilon A$, then the *right-hand porosity* of A at x is defined to be

$$\limsup_{h \to 0+} \frac{\lambda(A, x, h)}{h},$$

where $\lambda(A, x, h)$ is the length of the longest interval contained in $(x, x+h) \cap A^c$. A set is *right porous* if it has positive right-hand porosity at each of its points and it is *right α-porous* if its porosity is not less than α at each of its points. A set is *strongly right porous* if its right-hand porosity is 1 at each of its points. The left-hand and bilateral versions of these ideas are defined similarly. A set is *σ-porous* if it is the countable union of porous sets.

Let C be the collection of all compact subsets of $\mathbf{R}$. For $A, B \epsilon C$, let

$$\bar{\rho}_B^A = \inf\{\varepsilon > 0 : B \subset \bigcup_{x \epsilon A} B(x, \varepsilon)\},$$

where $B(x, \varepsilon) = \{y : |x - y| < \varepsilon\}$. The *Hausdorff distance* between A and B is

$$\rho(A, B) = \max\{\bar{\rho}_B^A, \bar{\rho}_A^B\}.$$

It can be shown ([5]) that $\mathcal{K} = (C, \rho)$ is a complete metric space and that if C is restricted to the closed subsets of any compact subset of $\mathbf{R}$, then the corresponding $\mathcal{K}$ is compact.

We denote

$$P^+(\alpha) = \{F \epsilon \mathcal{K} : F \textit{ is right } \alpha - \textit{porous}\}.$$

The meanings of $P^-(\alpha)$ and $P(\alpha)$ are analogous. From the definitions, it is easy to see that if $0 < \alpha < \beta \leq 1$, then

$$P^+(\alpha) \supset P^+(\beta) \supset P^+(1) \supset P(1) \quad \text{and} \quad P^+(\beta) = \bigcap_{\lambda < \beta} P^+(\lambda).$$

Similar relations hold for $P^-(\alpha)$ and $P(\alpha)$.

Theorem 1. *$P^+(1)$ is a dense $\mathbf{G}_\delta$ subset of $\mathcal{K}$.*

The main idea in the proof of this theorem is that in the complete metric space, $\mathcal{K}$,

$$P_n^+(\eta) = \{F \epsilon \mathcal{C} : \forall x \epsilon F,\ \exists r \epsilon (x, x + 1/n) \text{ and}$$

$$\text{a closed interval } I \subset (x, x + r) \setminus F \text{ such that } |I|/r > \eta\}.$$

is an open set for each $\eta \epsilon (0,1)$ and each $n \epsilon \mathbf{N}$. Using this, $P^+(\alpha)$ can be built by using

$$P^+(\alpha) = \bigcap_{n=1}^{\infty} \bigcap_{m=1}^{\infty} P_n^+(\alpha - 1/m)$$

and

$$P^+(1) = \bigcap_{n=1}^{\infty} P^+(1 - 1/n).$$

Then, an easy consequence of this is the following theorem.

Theorem 2. *The strongly porous sets form a dense $\mathbf{G}_\delta$ subset of $\mathcal{K}$ and therefore are residual in $\mathcal{K}$.*

Since every strongly porous set is σ-porous, the following corollary is a trivial consequence of Theorem 2.

Corollary 1. *The σ-porous sets are residual in $\mathcal{K}$.*

This leads at one to the following corollaries.

Corollary 2. *The measure zero compact sets are typical.*

Corollary 3. *The first category compact sets are typical.*

A logical question to ask at this point is whether some of the known theorems about porous sets can be attained as a consequence of Theorems 1 or 2. Since the level sets of a continuous function with a compact domain are in $\mathcal{K}$, likely candidates are the following theorems.

Theorem T1. *(B. S. Thomson [6]) If f is a nowhere constant continuous function, then the typical level set of f is in $P(1)$.*

(A level set of f is defined as $f^{-1} \circ f(x)$).

Theorem T2. *(B. S. Thomson [6]) The typical continuous function has every level set in $P(1)$.*

These theorems can be rephrased in light of Theorems 1 and 2 as follows.

Theorem T1*. *If $f : [a,b] \to \mathbf{R}$ is a nowhere constant continuous function, then the typical level set of f is typical.*

Theorem T2*. *Every level set of the typical continuous function defined on a compact interval is typical.*

We have been unable to do this, however. So, the following question remains open.

Question. *Is Theorem T1 or Theorem T2 a consequence of Theorem 2?*

In relation to this, it is possible to prove that the mapping

$$\lambda : x \epsilon [a, b] \to f^{-1} \circ f(x) \epsilon \mathcal{K}$$

is continuous on a residual subset of $[a, b]$. However, it is easy to find examples of functions f where this mapping is badly discontinuous, or nowhere constant and continuous at an x where $\lambda(x)$ is not strongly porous.

Bibliography

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