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STABILITY OF CHAOTIC AND NON-CHAOTIC MAPS OF THE INTERVAL

In the sequel we consider only continuous maps $I \longrightarrow I$ where I is a compact real interval. A map f is called <u>ε-chaotic</u> for some $\varepsilon > 0$ if there is a non-empty perfect set S such that for any $x, y \in S$, $x \neq y$, and any periodic point p of f ($p \in Per(f)$),

(1) $$\limsup_{n \to \infty} |f^n(x) - f^n(y)| \geq \varepsilon$$

(2) $$\liminf_{n \to \infty} |f^n(x) - f^n(y)| = 0$$

(3) $$\limsup_{n \to \infty} |f^n(x) - f^n(p)| \geq \varepsilon$$

where f^n denotes the n-th iterate of f. In this case, S is called an ε-scrambled set for f . We recall that this concept is equivalent to the original concept of chaos by Li and Yorke (see [1]).

Any continuous map f which is not chaotic, has the following property (see [5]): For any $x \in I$ and any $\varepsilon > 0$ there is a $p \in Per(f)$ with $\limsup_{n \to \infty} |f^n(x) - f^n(p)| < \varepsilon$. In other words, any map f is either chaotic or has every trajectory approximable by cycles. In practice, the second type of behavior cannot be distinguished from asymptotical periodicity of the trajectories and the corresponding maps can serve as predictible deterministic mathematical models of certain real processes.

In modelling the problem of stability is very important: The proper map f is usually replaced by its approximation g and the problem is to find conditions under which both f and g give models with similar properties (provided $\|f - g\|$ is small). We present here some results of this type.

Definition 1. (See [4].) A non-chaotic map f is stable if for any $\varepsilon > 0$, any map g sufficiently near to f has every trajectory ε-approximable by cycles (i.e., for any x there is some $p \in \mathrm{Per}(g)$ with $\limsup_{n \to \infty} |g^n(x) - g^n(p)| < \varepsilon$).

Definition 2. An ε-chaotic map is stable if for any ε' with $0 < \varepsilon' < \varepsilon$, any map g sufficiently near to f is ε'-chaotic.

Theorem 1. (See [4].) A non-chaotic map f is stable iff the following two conditions are satisfied:

(i) $\mathrm{Per}(f)$ is nowhere dense;

(ii) for any infinite ω-limit set $L_f(x)$ there is a sequence $\{I_n\}$ of closed periodic intervals such that the minimal period of I_n is 2^n and

$$L_f(x) = \bigcap_{n=1}^{\infty} \bigcup_{i=1}^{2^n} f^i(I_n)$$

This generalizes earlier results [8] and [9]. Note that for any map f with zero topological entropy (i.e., for any f without cycles of period $\neq 2^n$, $n = 0, 1, 2, \ldots$; in particular, for any non-chaotic map) and any infinite $L_f(x)$ there is a sequence $\{I_n\}$ of periodic intervals of the above described type with

$$L_f(x) \subset \bigcap_n \bigcup_i f^i(I_n)$$

(see [5]) but the converse inclusion can be false.

Using this we can prove

Theorem 2. In the space of maps with zero topological entropy with the uniform metric, the stable non-chaotic maps form a dense subset.

As a consequence we get

Theorem 3. The non-chaotic maps are generically stable.

Concerning stability of chaotic maps, the situation is more complicated but also here a characterization is possible. Every chaotic map f with zero topological entropy is by Theorem 2 non-stable (a simple example is given in [3]). On the other hand, if f has a positive topological entropy then there are disjoint closed intervals $I_0, I_1 \subset I$ and a positive integer m such that

$$f^m(I_0) \cap f^m(I_1) \supset I_0 \cup I_1$$

(f has a horseshoe). In this case, f is ε-chaotic where $\varepsilon = $ $=$ dist $(I_0, I_1)/2$ (see [1]) and any g sufficiently near to f is ε-chaotic, too. It can be proved that the unstable chaotic maps are the maps with only small horseshoes and with restrictions that represent maps of zero topological entropy with large chaos. A more precise description of stable chaotic maps is based on the following result.

Theorem 4. Let for some $x \neq y$, (1) and (2) be satisfied. Then f is ε-chaotic if f has zero topological entropy (see [2]) and is $\varepsilon/2$-chaotic otherwise (see [6]).

As a consequence we get (see also a related paper [7])

<u>Theorem 5</u>. (See [6].) The chaotic maps are generically stable.

References

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