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QUASICONTINUITY AND SOME CLASSES OF DARBOUX BAIRE 1 FUNCTIONS

Quasicontinuity is a generalization of the notion of continuity. It has been introduced in [Ke] and its basic properties are known (see e.g. [Bl],[LŠ], [Ma], [Th]). We shall deal with real-valued functions defined on a real non-degenerate interval I_0. Recall in this case the notion of quasicontinuity of a function at a point.

DEFINITION. A function $f\colon I_0 \to R$ (R - the real line) is said to be quasicontinuous at the point $x \in I_0$ if for each $\varepsilon > 0$ and $\delta > 0$ there exists a non-void open interval $I \subset (x - \delta, x + \delta)$ such that $|f(t) - f(x)| < \varepsilon$ holds for every $t \in I$. We denote by $Q(f)$ the set of all such points of I_0 at which the function f is quasicontinuous.

Let $f\colon I_0 \to R$ be a function. Put $d_I(f,x) = \sup_{t \in I} \{|f(t) - f(x)|\}$, where $I \subset I_0$ is a non-void open interval and $i_\delta(f,x) = \inf_{I \subset (x - \delta, x + \delta)}\{d_I(f,x)\}$ for $\delta > 0$. Obviously $i_\delta(f,x) \geq i_\eta(f,x)$ whenever $\delta < \eta$ and we can define for each $x \in I_0$

$$q_f(x) = \lim_{\delta \to 0+} i_\delta(f,x) = \sup_{\delta > 0} \{i_\delta(f,x)\}.$$

THEOREM 1. (a) A function $f\colon I_0 \to R$ is quasicontinuous at the point x if and only if $q_f(x) = 0$.

(b) If $f_n \to f$ uniformly, then also $q_{f_n} \to q_f$ uniformly.

(c) If $f_n \to f$ uniformly, then $\bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} Q(f_n) \subset Q(f)$.

THEOREM 2. Let $f: I_0 \to R$ be a Lebesgue measurable function. Then $q_f: I_0 \to R \cup \{+\infty\}$ is Lebesgue measurable.

COROLLARY 1. The set of quasicontinuity points of a Lebesgue measurable function is a Lebesgue measurable.

Further we shall deal with classes of real functions defined on the unit real interval $[0,1]$. We denote by $b\mathcal{A}$ ($b\Delta$, $b\mathcal{D}\mathcal{B}_1$) the class of bounded approximately continuous (bounded derivatives, bounded Darboux Baire 1) functions. All these classes are complete metric spaces with the metric $d(f,g) = \sup_{x\in[0,1]} \{|f(x) - g(x)|\}$. There are known some properties which hold for most of the functions of these classes in the sense of the Baire category (see e.g. a survey article [CP]). In what follows λ stands for the Lebesgue measure on $[0,1]$.

THEOREM 3. Let $\mathcal{F}$ be a Banach space of functions, $b\mathcal{A} \subset \mathcal{F} \subset b\mathcal{D}\mathcal{B}_1$, with the norm $\|f\| = \sup_{x\in[0,1]} \{|f(x)|\}$. Then the family

$$\mathcal{F}^* = \{f \in \mathcal{F} : \lambda(Q(f)) = 0\}$$

is a residual G_δ set in $\mathcal{F}$.

COROLLARY 2. The family of all $b\mathcal{A}$ ($b\Delta$) functions which are not quasicontinuous almost everywhere is a residual G_δ set in $b\mathcal{A}$ ($b\Delta$).

The last statement improves some of known results (see [BP], [KŠ]).

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Remark 1. Theorem BMJ was proved by standard methods (some explicitely defined subsets of C was shown to be nowhere dense). The original proof of the Saks' result is more sofisticated. In the first part of the proof Saks proved that the set of non-$\neg$Besicovitch functions is of the second category in every sphere of C and in the second part (which is due to Banach) he proved that this set is analytic and consequently has the property of Baire. It is now well-known that the second part of the Saks' proof is superfluous. In fact, the first part of the Saks proof can be considered as the construction of a winning strategy for the second player in the Banach-Mazur game (for the set of non--Besicovitch functions) in the metric space C . The existence of such a strategy is equivalent with the residuality of the set of non-Besicovitch functions in C (cf. [7]) .

Remark 2. Preiss' proof of Theorem P and proofs of results stated in the second part of the present article use Banach-Mazur game method. On the other hand, an interesting Garg's observation ([2] , Remark 1) shows that the Saks' result can be obtained from a Jarník's result on knot points of typical continuous functions (proved by standard methods) and from a proposition which deals with general continuous functions (e.g. Proposition 3 or Proposition 2 of [10]). This Garg's idea led Preiss to some other interesting observations :

(P1) Every $f \in C$ has an unilateral approximate derivative (finite or infinite) on a ζ-dense set (cf. [9]).

(P2) Every function of Besicovitch type has a finite approximate derivative on a set which has a positive measure in each subinterval of $(0,1)$.

(P3) Almost all trajectories of the one-dimensional Brownian motion are not of Besicovitch type.

A point $x \in (0,1)$ is said to be a knot point of a $f \in C$ if $D^{+}f(x) = D^{-}f(x) = \infty$ and $D_{+}f(x) = D_{-}f(x) = -\infty$. V.Jarník [3] proved that for a.a. $f \in C$ the set N_f of points which are not knot points of f has measure zero (and consequently is also of the first category , as was shown by Garg in [2]).

G.Petruska (an oral communication) has proved that we can assert in the above statement that N_f is σ-bilateraly strongly porous. It is proved independently in [12] that N_f is σ-porous in a stronger sense (σ-[g]- totally porous). Theorem 1 below is a further (and in the view of Theorem 2 possibly the best) improvement of these results.

Definition 1. (cf.[5]) Let $x \in R$, $y \in \overline{R}$, $d \geq 0$. We shall say that y is a derived number of $f: R \to R$ at x with a density d (lower density d, upper density d, right lower density d, symmetrical upper density d, ...) if there exist a set $E \subset R$ such that the density (lower density, upper density, ...) of E at x equals to d and

$$\lim_{t \to x, t \in E} (f(t) - f(x))(t-x)^{-1} = y .$$

If y is a derived number of f at x with a right (left) upper density 1, we say that y is a right (left) essential derived number of f at x (cf.[11]). We shall say that x is an essential knot point of f if each extended real number $y \in \overline{R}$ is a bilateral (i.e. simultaneously right and left) essential derived number of f at x.

A $y \in \overline{R}$ is said to be a preponderant derivative (right preponderant derivative) if y is a derived number of f at x with a lower density (rigt lower density) $d > 1/2$.

The following Jarník's results seem to be not commonly known (except (i)).

Theorem J. The following properties of $f \in C$ are typical :

(i)([11]) Almost all $x \in (0,1)$ are essential knot points of f.

(ii) ([4]) For each point $x \in (0,1)$ at least one from numbers $\infty, -\infty$ is a right essential derived number and at least one from numbers $\infty, -\infty$ is a left essential derived number of f.

(iii)([4]) At each point $x \in (0,1)$ both numbers $\infty, -\infty$ are derived numbers of f at x with a symmetrical upper density $d \geq 1/2$.

(iv)([5]) At each point $x \in (0,1)$ there exists a side (s) (right or left) such that at least two from three numbers $-\infty, 0, \infty$

are derived numbers of f at x with (s) - side upper density $d \geqq 1/4$.

Corollary 1. A typical $f \in C$

(i) has not a preponderant derivative ,

(ii) has not a finite approximate (preponderant) one-sided derivative ,

(iii) has not both one-sided approximate derivative

at each point $x \in (0,1)$.

2. New results. By a "figure" we shall mean a nonempty set of the form $F = [a_1,b_1] \cup \dots \cup [a_n,b_n]$, where $0 \leqq a_1 < b_1 < a_2 < b_2 < \dots < b_n \leqq 1$. The "norm" of the figure F is defined as $n(F) = \max(a_1, b_1-a_1, a_2-b_1, \dots, b_n-a_n, 1-b_n)$.

Definition 2. Let $\mathcal{N}$ be a σ-ideal of subsets of $[0,1]$. We define a $\mathcal{N}$-game , an infinite game between two players (F-player and ε-player) , as follows. In the first step the ε-player choose an $\varepsilon_1 > 0$. In the second step the F-player choose a figure F_1 such that $n(F_1) \leqq \varepsilon_1$. Generally, in the $(2n-1)$th step the ε-player choose an $\varepsilon_n > 0$ and in the $(2n)$th step the F-player choose a figure F_n such that $n(F_n) \leqq \varepsilon_n$. If $\liminf F_n = \bigcup_{k=1}^{\infty} \bigcap_{n=k}^{\infty} F_n \in \mathcal{N}$, then the F-player wins. If $\liminf F_n \notin \mathcal{N}$, then the ε-player wins.

Remark 3.

A. If $\mathcal{N}$ is the system of all σ-bilaterally strongly porous sets (or the system of all σ-[g]- totally porous sets [12]), then the F-player has a winning strategy in the $\mathcal{N}$-game.

B. If μ is a σ-finite Borel measure on $[0,1]$ and $\mathcal{N}$ is the system of all μ-null sets, then the F-player has a winning strategy as well.

C. Let $\mathcal{N}$ be the system of all σ-superporous sets. (A set $S \subset R$ is superporous iff $S \cup P$ is a porous set whenever P is porous. Superporous sets a precisely discrete sets in the deep $\mathcal{J}$-density topology - the weak topology induced by the family of all functions which are continuous with respect to the Wilczyński's $\mathcal{J}$-density topology.) Then the $\mathcal{E}$-player has a winning strategy in the $\mathcal{N}$-game.

Theorem 1. Let the F-player has a winning strategy in a $\mathcal{N}$-game. Then for a.a. $f \in C$ the set of points $x \in (0,1)$ which are not essential knot points of f belongs to $\mathcal{N}$.

Remark 4. Also a sharper version of Theorem 1 is true : instead of essential knot points we can consider $[g]$-knot points. This sharper version improves the main result of [12] .

Theorem 2. Let the $\mathcal{E}$-player has a winning strategy in a $\mathcal{N}$-game. Then for a.a. $f \in C$ the following statement holds: If D^+, D_+, D^-, D_- are extended real numbers for which
$\max(|D^+|, |D_+|) = \max(|D^-|, |D_-|) = \infty$ and $[D_-, D^-] \cup [D_+, D^+] = [-\infty, \infty]$, then there exists a set $P \subset (0,1)$, $P \notin \mathcal{N}$ such that $D^+f(x) = D^+$, $D_+f(x) = D_+$, $D^-f(x) = D^-$, $D_-f(x) = D_-$ for all $x \in P$.

Theorem 3. If we write $\overline{D}^+_{ap}f(x)$, $\underline{D}^+_{ap}f(x)$, $\overline{D}^-_{ap}f(x)$, $\underline{D}^-_{ap}f(x)$ instead of $D^+f(x)$, $D_+f(x)$, $D^-f(x)$, $D_-f(x)$ in Theorem BMJ and Theorem P , we obtain new correct theorems.

Theorem 4. A typical $f \in C$ has the following properties:

(i) For all $x \in (0,1)$, there exists a bilateral essential number of f at x .

(ii) There exists a $\mathcal{C}$-dense set $P \subset (0,1)$ such that ∞ is a derived number of f at x with a lower density $d \geq 1/2$ for each $x \in P$.

(iii) There exists a $\mathcal{C}$-dense set $Q \subset (0,1)$ such that $\lim\limits_{t \to x} \text{ap} \left| (f(t) - f(x))(t-x)^{-1} \right| = \infty$ for each $x \in Q$.

Theorem J and Theorem 4, (i) imply the following

Corollary 2. A typical $f \in C$ has both one-sided preponderant derivatives at no point $x \in (0,1)$.

into sets of measure zero. Also, the class of all bounded functions defined on S has the d.pr. w.r.t. every commuting system of maps.

As for classes of real functions, we have the following immediate corollary.

THEOREM 9. *Let $\mathfrak{F}$ be a translation-invariant normed space of $\mathbb{R}\to\mathbb{R}$ functions. Suppose that there is a translation-invariant vector topology $\mathcal{T}$ on $\mathfrak{F}$ such that $\{f\in\mathfrak{F} : \|f\|\leq 1\}$ is $\mathcal{T}$-compact, and whenever $f_n\in\mathfrak{F}$ and $\|f_n\|\to 0$ then $f_n\to 0$ in $\mathcal{T}$. Then $\mathfrak{F}$ has the d.pr. (w.r.t. translations).*

Making use of this condition, one can prove that each of the following classes has the d.pr.

$$\text{b-BV}^1 = \{f:\mathbb{R}\to\mathbb{R} : f \text{ is bounded and } \sup_x V(f;[x,x+1])<\infty\}$$

$$\text{b-Lip} = \{f:\mathbb{R}\to\mathbb{R} : f \text{ is bounded and Lipschitz}\}$$

$$\text{b-Lip}^k = \{f:\mathbb{R}\to\mathbb{R} : f \text{ is bounded, } f^{(k-1)} \text{ exists everywhere and is Lipschitz}\}.$$

We remark that the d.pr. of the class $BC(\mathbb{R})$ does not follow from Theorem 9. It was proved by V. Totik, that there does not exist a vector topology on $BC(\mathbb{R})$ satisfying the conditions of Theorem 9.

5. We conclude with the following problem:

Is every bounded, continuous solution of a homogeneous difference equation

$$(3) \qquad \sum_{i=1}^{n} c_i f(x+a_i) = 0$$

necessarily uniformly continuous?

(We remark that if we replace (3) by the more general convolution equation $\mu * f=0$ then the answer is negative; see [3]. We also point out the connection of this problem with the investigations of S. Bochner and others concerning continuous solutions of difference equations; see [2].)

If the answer to this problem is affirmative, it provides a simple proof of our Theorem 1. We note first that (1) is a homogeneous difference equation. Now, if $f \in BC(\mathbb{R})$ is uniformly continuous and satisfies (1) then an elementary construction gives a continuous $(a_1,\dots,a_n)$-decomposition of f via the Arzela-Ascoli lemma. Another approach is the following. Any solution of (3) is mean-periodic, and any bounded and uniformly continuous mean periodic function is uniformly almost periodic (see [3], p.43). Then we also can find an $(a_1,\dots,a_n)$-decomposition of f using the Fourier series of f (see [1]).

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