Włodzimierz A. Ślęzak, Instytut Matematyki, Wyższa Szkoła Pedagogiczna, 85-064 Bydgoszcz, Chodkiewicza 30, Poland

ON EXTENSION OF RESTRICTIONS OF BAIRE 1 VECTOR-VALUED MAPS

Let (X,T) be a topological space and $C(X)$ the lattice of continuous real functions on X. The family

$P(X) := \{U \subset X : U = \{x \in X : f(x) > 0\}$ for some $f \in C(X)\}$

of cozero sets of $C(X)$ creates a paving, viz. $P(X)$ is closed under finite intersections and countable unions, in particular $\emptyset \in P(X)$, $X \in P(X)$. A multifunction $F: X \to Y$, where Y denotes an arbitrary topological space, will be called z-lower semicontinuous / briefly z-lsc / iff for each subset G open in Y we have

$$F^{-}(G) := \{x \in X : F(x) \cap G \neq \emptyset\} \in P(X),$$

in other words, iff F is lower $P(X)$- measurable .

In case $Y=R$, $F(x) := [f(x), g(x)]$, F is z-lsc iff f is z-usc and g is z-lsc in the meaning of [4]/ cf. also [20], [14]/. If card $F(x) = 1$ for all $x \in X$, i.e. $F(x) = \{f(x)\}$, then F is z-lsc if and only if f is continuous on X as a single-valued function. It may be proved, that the paving $P(X)$ of cozero sets of an arbitrary space X is $(\aleph_1, \infty)$ - paracompact / see [16] for the definition/. Thus, using [16] we obtain the following very general selection theorem:

THEOREM 1. Let X be topological space and $(Y, |\cdot|)$ - separable Fréchet space. Each z-lsc multifunction $F: X \to Y$ with closed and convex values has continuous selector, i.e. continuous map $f \in C(X,Y)$ such that $f(x) \in F(x)$ for all $x \in X$.

In case where X is perfectly normal, z-lower semicontinuity reduces to lsc and theorem 1 reduces to celebrated Michael's selection theorem [17]. In our theorem 1 the values of F

may even belong to the family $D(Y)$ defined in [17], in particular may be convex and finite-dimensional without being closed.
It is also possible to consider separable metric space with suitable kind of generalized convexity /e.g. S-contractibles ones/ instead of Fréchet space Y and to replace $C(X)$ in the definition of $P(X)$ by some others lattices of functions.
As an easy corollary we obtain the simple proof of theorem 2 from 12 :

THEOREM 2 ([12]) Let (X,d,m) be a Chaika measure metric space /see [6] or [12] for definition / with nonatomic measure m.
Then the following conditions are equivalent:

/i/ for each Baire 1 $g: X \to Y$ there is approximately continuous map $f: X \to Y$ such that

/§§§/ $\{x \in X : f(x) = g(x)\} \supset A$, $A \subset X$

/ii/ $m(A) = 0$

For the history of theorem 2 see [19], [9], [10], [1], [21], [15].
All of listed paper deal with scalar-valued functions.
To prove the theorem 2 it suffices to observe that the multifunction defined by formula

$$F(x) := \begin{cases} g(x) & , \ x \in A \\ \mathrm{cl\ conv}\ g(A) & \text{otherwise} \end{cases}$$

is z-lsc and take as f the selector existing by virtue of theorem 1.
Besides the topology of density we may consider in (X,d,m) another topology $T_{a.e.}$ consisting of all subset $U \subset X$ for which U is open in the density topology and $U = G \cup Z$ where G is metricaly open and $m(Z) = 0$; see [18] , [13] .
Theorem 1 leads to a simple solution of a problem 13-a from [11].
Namely we obtain the following generalization of theorem 3 from [12]:

THEOREM 3 . Let X be the same as in theorem 2. The following conditions are equivalent:

/i/ for each Baire 1 map $g: X \to Y$ there is $T_{a.e.}$ - continuous /= approximately continuous and m-a.e. continuous/ map $f: X \to Y$ such that the inclusion /§§§/ holds

/ii/ $m\left(\mathrm{cl}\ A\right) = 0$.

Note that the method used in theorem 3 in 12 does not carry over the present case: (it essentially relly on the fact that X is a subset of the one-dimensional line).

In Chaika space X on may also consider the r-modification of the density topology / cf.[15]/. We obtain rather unsatisfactory result:

THEOREM 4. In the framework of theorem 2 the following conditions are equivalent:

/i/ for each Baire 1 map $g: X \to Y$ there exists a r-continuous map $f: X\ \ Y$ such that /§§§/ holds

/ii/ $m\left(r\text{-}\mathrm{cl}\ \ A\right) = 0$.

The sign r- cl stands here for the closure operator in the r-modification of the density topology on X, while cl in theorem 3 stands for the closure operator in the metrical topology. It is an open question to prove or disprove the unequivalence of the above conditions and the following:

/iii/ $m(A) = 0$ and A is nowhere dense.

The implication /iii/$\Rightarrow$/ii/ is obvious.

From theorem 3 we deduce the following solution of problem 12-a posed in [11]:

THEOREM5 .There is a function $f: R^2 \to R$ ordinarily approximately continuous and m_2- a.e. continuous such that the set

$D(f) := \{(x,y)\in R^2 : f^x$ fails to be approximately continuous at y or f_y fails to be approximately continuous at $x\}$

is uncountable.

The characterization of the $P(X)$ in case when considered topology fails to have Lusin-Menshoff property (e.g. for the density topology on the plane with respect to the differentiation base of all rectangles) is unknown to the author. Thus the following Grande's conjecture from [11] is still an open question:

CONJECTURE([11]). The following conditions are equivalent:

/i/ for each Baire 1 function $g: R^2 \to R$ there is $f: R^2 \to R$ strongly approximately continuous and m_2-a.e. continuous such that /§§§/ holds

/ii/ $m_1([\mathrm{cl}\ A]_x) = 0$ and $m_1([\mathrm{cl}\ A]^y) = 0$ for all $(x,y) \in R^2$

For other facts on extension theorems see [2], [3], [4], [7], [8], [14], [15], [17], [21]. Note that the methods used in [21], [3] to reprove results of [19], [1]/ howewer almost identical with [14]/ relly heavilly on the fact that the range space has the total order, and thus are not applicable in case of Fréchet-space valued mappings.

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