ON ONE- AND TWO-DIMENSIONAL $\mathcal{I}$-DENSITIES AND RELATED KINDS OF CONTINUITY

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Abstract. Connections between strong $\mathcal{I}$-density (for plane sets) and one-dimensional $\mathcal{I}$-density are studied. Other results deal with separately $\mathcal{I}$-approximately continuous functions. A topology associated with separately $\mathcal{I}$-approximately continuous functions, similar to O'Malley's one is introduced and investigated. Among others it is shown that strong $\mathcal{I}$-approximate continuity implies separate $\mathcal{I}$-approximate continuity and that a separately $\mathcal{I}$-approximately continuous function is Baire 2 and need not be Baire 1. Finally, some properties of functions with $\mathcal{I}$-approximately continuous sections are studied.

0. Introduction. In [16] a category analogue of a density point (called an $\mathcal{I}$-density point) has been introduced. From that time several articles exploring this notion were published, mainly inspired by known results on metric density, approximate continuity and approximate differentiability. Our paper is a next one of that series.

Throughout the paper N denotes the set of all positive integers. The symmetric difference of sets A and B is denoted by $A \Delta B$. For $A \subset R$ or $A \subset R^2$ (where R is the real line and

R^2 - the plane) the interior and the closure of A in the natural topology are designated by $\operatorname{int} A$ and $\overline{A}$, respectively.

If $A \subset R^2$ and if $f : R^2 \to R$ is an arbitrary function, then the sections of A and f are defined in the following way:

$$A_x = \{y \in R : (x,y) \in A\}, \qquad A^y = \{x \in R : (x,y) \in A\},$$

$$f_x(y) = f(x,y), \qquad f^y(x) = f(x,y) \quad \text{for} \quad x,y \in R.$$

If (p) denotes a property of a real function of one variable, we say that a real function of two variables has property (p) separately if all sections f_x and f^y have this property.

Now let us recall basic facts about $\mathcal{I}$-density (cf. [16], [10], [15]; for the wider survey see [17]). Other necessary informations will be given successively in the sequel. Let $(X,\mathcal{S})$ be a measurable space and let $\mathcal{I} \subset \mathcal{S}$ be a proper σ-ideal of sets. We say that some property holds $\mathcal{I}$-almost everywhere (in abbr. $\mathcal{I}$-a.e.) if the set of points which have not this property belongs to $\mathcal{I}$. We say that a sequence $\{f_n\}_{n\in N}$ of $\mathcal{S}$-measurable real functions defined on X converges with respect to $\mathcal{I}$ to some $\mathcal{S}$-measurable real function f defined on X if each subsequence $\{f_{n_m}\}_{m\in N}$ of $\{f_n\}_{n\in N}$ contains a subsequence $\{f_{n_{m_p}}\}_{p\in N}$ which converges $\mathcal{I}$-a.e. to f; we use then the denotation $f_n \xrightarrow{\mathcal{I}} f$.

In this paper $\mathcal{I}_1$ (resp. $\mathcal{I}_2$) will denote the σ-ideal of sets of the first category in R (resp. R^2) and $\mathcal{S}_1$ (resp.

S_2) will denote the σ-algebra of all subsets of R (resp. R^2) having the Baire property.

Since we shall define below both linear and plane I-density, we shall use the denotations I_1- and I_2-density to distinguish these notions.

We say that 0 is an I_1-density point of a set $A \in S_1$ if the sequence $\{\chi_{(n\cdot A)\cap[-1,1]}\}_{n\in N}$ of characteristic functions (where $n\cdot A = \{nx : x \in A\}$) converges to $\chi_{[-1,1]}$ with respect to I_1. Next, $x_0 \in R$ is called an I_1-density point of $A \in S_1$ if 0 is an I_1-density point of the set $A - x_0 = \{x - x_0 : x \in A\}$. We say that x_0 is an I_1-dispersion point of $A \in S_1$ if it is an I_1-density point of $R \setminus A$. Similarly, we define right- and left-hand I_1-density and I_1-dispersion points. The family of all sets $A \in S_1$ such that each point of A is its I_1-density point (these sets will be named I_1-open) forms a topology (essentially stronger than the natural topology) called the I_1-density topology. Real functions continuous with respect to that topology are called I_1-approximately continuous.

Now let us pay some attention to I_2-density (cf.[2]). For $m,n \in N$ and $A \subset R^2$ we denote $(m,n)\cdot A = \{(mx,ny) : (x,y) \in A\}$. We say that $(0,0)$ is a strong I_2-density point of a set $A \in S_2$ if for any increasing sequences $\{k'_n\}_{n\in N}, \{k''_n\}_{n\in N}$ of positive integers the sequence $\{\chi_{((k'_n,k''_n)\cdot A)\cap[-1,1]^2}\}_{n\in N}$ converges to $\chi_{[-1,1]^2}$ with respect to I_2. Next, $(x_0,y_0) \in R^2$ is called a strong I_2-density point of $A \in S_2$ if $(0,0)$ is a strong I_2-density point of $A - (x_0,y_0) = \{(x - x_0, y - y_0) : (x,y) \in A\}$. We say that (x_0,y_0) is a strong I_2-dispersion point of $A \in S_2$ if it is a strong I_2-density point of $R^2 \setminus A$. The family of all sets $A \in S_2$ such that each point of A is its strong I_2-density point forms a topology which will be denoted by $T^s_{I_2}$. Real functions continuous with respect to that topology are called strongly I_2-approximately continuous (compare [2], where also ordinary I_2-density points involving only sets of the form $(n,n)\cdot A$ are studied).

Observe that the above definition of a strong I_2-density point reads as follows: $(0,0)$ is a strong I_2-density point of $A \in S_2$ if and only if for any increasing sequences $\{k'_n\}_{n\in N}$, $\{k''_n\}_{n\in N}$ of positive integers and for any increasing sequence $\{n_m\}_{m\in N}$ of positive integers there exists a subsequence $\{n_{m_p}\}_{p\in N}$ such that the sequence $\{\chi_{((k'_{n_{m_p}},k''_{n_{m_p}})\cdot A)\cap[-1,1]^2}\}_{p\in N}$ converges to $\chi_{[-1,1]^2}$ I_2-almost everywhere. As we shall see one of subsequences can be eliminated.

THEOREM 0.1. $(0,0)$ is a strong I_2-density point of $A\in S_2$ if and only if for any increasing sequences $\{k'_n\}_{n\in N}, \{k''_n\}_{n\in N}$ of positive integers there exists an increasing sequence $\{n_p\}_{p\in N}$ of positive integers such that the sequence $\{\chi_{((k'_{n_p},k''_{n_p})\cdot A)\cap[-1,1]^2}\}_{p\in N}$ converges to $\chi_{[-1,1]^2}$ I_2-almost everywhere.

P r o o f. Suppose that $(0,0)$ is a strong I_2-density point of $A \in S_2$. Taking $\{n\}_{n\in N}$ as $\{n_m\}_{m\in N}$ we obtain the thesis.

Suppose now that the last condition is fulfilled. Since it holds for any $\{k'_n\},\{k''_n\}$, then it also holds for $\{k'_{n_m}\}_{m\in N}$ and

$\{k''_{n_m}\}_{m\in N}$ which means that $(0,0)$ is a strong I_2-density point of A.

THEOREM 0.2. $(0,0)$ is a strong I_2-density point of $A\in S_2$ if and only if for any increasing sequences $\{t'_n\}_{n\in N}, \{t''_n\}_{n\in N}$ of real numbers tending to infinity there exists an increasing sequence $\{n_p\}_{p\in N}$ of positive integers such that the sequence $\{\chi_{((t'_{n_p}, t''_{n_p})\cdot A)\cap[-1,1]^2}\}_{p\in N}$ converges to $\chi_{[-1,1]^2}$ I_2-almost everywhere.

P r o o f. It follows immediately from Lemma 1 in [2].

In the sequel we shall make use of the above facts. We shall also need the following characterization of a right-hand I_1-density point for closed sets (see [11]):

LEMMA 0.1. A point x_0 is a right-hand I_1-density point of a closed set $F\subset R$ if and only if for each $n\in N$ there exist $k\in N$ and $\delta>0$ such that for any $h\in(0,\delta)$ and $i\in\{1,\dots,n\}$ there exists $j\in\{1,\dots,k\}$ such that

$$[x_0+\frac{(i-1)\cdot k+j-1}{n\cdot k}\cdot h,\ x_0+\frac{(i-1)\cdot k+j}{n\cdot k}\cdot h]\subset F.$$

1. In this section we continue investigations of [2] concerning plane I_2-density points and related kinds of continuity. We shall study, among others, the interrelations between so-called special and deep strong I_2-density points. The notions of special (deep) I_1- and I_2-density points were considered in [15] and [2], respectively. They play an essential role in the characterization of coarsest topologies for I_1- and I_2-approximate-

ly continuous functions. The coarsest topology for strongly I_2-
-approximately continuous functions has not been yet characterized.

Let us start with some definitions. A point (x_0,y_0) is called a special strong I_2-density point of $A \in S_2$ if there exists an open set $B \subset R^2$ such that $\bar{B} \supset R^2 \setminus A$ and (x_0,y_0) is a strong I_2-dispersion point of B. A point (x_0,y_0) is called a deep strong I_2-density point of $A \in S_2$ if there exists an open set $B \subset R^2$ such that $B \supset R^2 \setminus A$ and (x_0,y_0) is a strong I_2-dispersion point of B. Obviously, if (x_0,y_0) is a deep strong I_2-density point of $A \in S_2$, then it is a special strong I_2-density point of A. Let $T^{1s}_{I_2} = \{G \cup P : G$ is open, P is nowhere dense and each point of $G \cup P$ is its strong I_2-density point$\}$. Obviously, P is a part of the frontier of G. It is easily observed that $T^{1s}_{I_2}$ forms a topology and that $T^{1s}_{I_2} \subset T^{s}_{I_2}$.

THEOREM 1.1. Each strongly I_2-approximately continuous function $f : R^2 \to R$ is continuous with respect to $T^{1s}_{I_2}$.

P r o o f. The proof is analogous as in the one-dimensional case (see page 172 in [16], part of the proof of Th. 8).

THEOREM 1.2. If $f : R^2 \to R$ is strongly I_2-approximately continuous, then for every interval (z_1,z_2) each point of the set $f^{-1}((z_1,z_2))$ is its special strong I_2-density point.

P r o o f. See [2], Theorem 9.

Now we proceed to detailed study of special (deep) strong I_2-density points.

LEMMA 1.1. If $A \in \mathcal{T}^{1s}_{I_2}$, then $R^2 \setminus A = G_2 \cup P_2$, where G_2 is open and P_2 is nowhere dense.

P r o o f. Observe that if $A = G_1 \cup P_1$, then the sets $G_2 = \operatorname{int}(R^2 \setminus A)$ and $P_2 = (R^2 \setminus A) \setminus G_2$ fulfil all requirements, moreover, P_1 and P_2 are complementary parts of the frontier of G_1 (see also [2], Lemma 3).

REMARK 1.1. In the denotation of the above lemma, the definitions of special or deep strong I_2-density points for sets in $\mathcal{T}^{1s}_{I_2}$ can be stated in the following form: A point (x_o,y_o) is a special (deep) strong I_2-density point of $A \in \mathcal{T}^{1s}_{I_2}$ if there exists an open set B such that $\bar{B} \supset P_2$ ($B \supset P_2$, respectively) and (x_o,y_o) is a strong I_2-dispersion point of B.

LEMMA 1.2. If $(0,0)$ is a special (deep) strong I_2-density point of $A \in \mathcal{T}^{1s}_{I_2}$, then (in the denotation of the previous lemma) for each pair $\{k'_n\}_{n\in N}, \{k''_n\}_{n\in N}$ of increasing sequences of positive integers there exists an increasing sequence $\{n_p\}_{p\in N}$ of positive integers such that for each rectangle $(a',b')\times(a'',b'')$ $\subset [-1,1]^2$ there exists a rectangle $(c',d')\times(c'',d'') \subset (a',b')$ $\times (a'',b'')$ and a natural number r such that

$$\bigcup_{p=r}^{\infty} ((k'_{n_p},k''_{n_p})\cdot P_2) \cap ((c',d')\times(c'',d'')) = \emptyset.$$

P r o o f. Suppose that this condition does not hold. So there exist two increasing sequences $\{k_n\}_{n\in N}, \{k''_n\}_{n\in N}$ of positive integers such that for each increasing sequence $\{n_p\}_{p\in N}$ there exists a rectangle $(a',b')\times(a'',b'') \subset [-1,1]^2$ such that for each rectangle $(c',d') \times (c'',d'') \subset (a',b') \times (a'',b'')$ and for each $r \in N$ we have $\bigcup_{p=r}^{\infty} ((k'_{n_p}, k''_{n_p}) \cdot P_2) \cap ((c',d') \times (c'',d'')) \neq \emptyset$. Obviously, we can suppose that the rectangle $(a',b') \times (a'',b'')$ is included in one quarter of the plane. Assume, for definiteness, that $a' > 0$ and $a'' > 0$. It follows that for each $r \in N$ the set $\bigcup_{p=r}^{\infty} ((k'_{n_p}, k''_{n_p}) \cdot P_2)$ is dense in $(a',b') \times (a'',b'')$ for all sequences $\{n_p\}_{p \in N}$. If B is any open set such that $\overline{B} \supset P_2$ ($B \supset P_2$, resp.) then it is easy to see that for every positive integer r the set $\bigcup_{p=r}^{\infty} (k'_{n_p}, k''_{n_p}) \cdot B$ is dense in this rectangle. Hence $\limsup_p ((k'_{n_p}, k''_{n_p}) \cdot B) \cap [-1,1]^2$ is residual in $(a',b') \times (a'',b'')$ and, by virtue of Remark 1.1, $(0,0)$ is not a special (deep, resp.) strong $\mathcal{I}_2$-density point of A.

In the sequel we shall need the following fact for linear sets:

LEMMA 1.3. If x_0 is not an $\mathcal{I}_1$-dispersion point of an open set $A \subset R$, then there exists an interval set $D \subset A$ (it is a set of the form $\bigcup_{n=1}^{\infty} (a_n, b_n)$, where intervals (a_n, b_n) are pairwise disjoint, and a sequence $\{a_n\}_{n \in N}$ is monotone and converges to x_0) such that x_0 is not a dispersion point of D.

P r o o f. For simplicity assume that $x_0 = 0$ and that 0 is not a right $\mathcal{I}_1$-dispersion point of A. So there exists an increasing sequence $\{n_m\}_{m \in N}$ of positive integers such that for each subsequence $\{n_{m_p}\}_{p \in N}$ the set $\limsup_p ((n_{m_p} \cdot A) \cap [0,1])$ is not of the first category.

Let $A_n = A \cap (\frac{1}{n+1}, \frac{1}{n})$ for each n. Choose a set $B_n \subset A_n$ which is a finite union of open intervals and has the property that for each $x \in A_n$ there exists $x' \in B_n$ such that $|x - x'| < \frac{1}{n^2}$. Put $D = \bigcup_{n=1}^{\infty} B_n$. Obviously D is an interval set. We shall show that 0 is not an I_1-dispersion point of D. Let $\{n_{m_p}\}_{p\in N}$ be an arbitrary subsequence of the above mentioned sequence $\{n_m\}_{m\in N}$. Because $\limsup_p ((n_{m_p} \cdot A) \cap [0,1])$ has the Baire property and is not of the first category, there exists a subinterval $[a,b] \subset (0,1]$ in which this set is residual. We shall show that also $\limsup_p ((n_{m_p} \cdot D) \cap [0,1])$ is residual in $[a,b]$. Indeed, observe that for each $r \in N$ the open set $\bigcup_{p=r}^{\infty} (n_{m_p} \cdot A)$ is dense in $[a,b]$. Now we shall show that also $\bigcup_{p=r}^{\infty} (n_{m_p} \cdot D)$ is dense in $[a,b]$ for each $r \in N$, which is enough because the last set is also open. First observe that for each point $x \in (n_{m_p} \cdot A) \cap [0,1] = n_{m_p} \cdot (A \cap [0, \frac{1}{n_{m_p}}])$ the distance from x to $(n_{m_p} \cdot D) \cap [0,1]$ is less than $\frac{1}{n_{m_p}}$ (from the definition of D). Let $r \in N$ and let $\bar{x} \in [a,b]$ be an arbitrary number. We can find a sequence $\{x_k\}_{k\in N}$ tending to $\bar{x}$ and such that $x_k \in \bigcup_{p=r+k}^{\infty} (n_{m_p} \cdot A) \cap [0,1]$. Let $n_{m_{p_k}} > n_{m_r}$ be such positive integer that $x_k \in n_{m_{p_k}} \cdot A$. Obviously $n_{m_{p_k}} \xrightarrow[k\to\infty]{} \infty$.

For each $k \in N$ there exists a number $x' \in n_{m_{p_k}} \cdot D$ such that $|x_k - x_k'| < \frac{1}{n_{m_{p_k}}}$. Hence x_k' also converges to $\overline{x}$, so $\bigcup_{p=r}^{\infty} (n_{m_p} \cdot D)$ is dense in $[a,b]$.

LEMMA 1.4. Under the assumption of Lemma 1.3 there exists also a "closed" interval set $D \subset A$ $(D = \bigcup_{n=1}^{\infty} [a_n,b_n])$ such that x_o is not an I_1-dispersion point of D.

P r o o f. Essentially the same proof works.

LEMMA 1.5. If (x_o,y_o) is a deep strong I_2-density point of $A \in S_2$, then x_o is a deep I_1-density point of the linear set A^{y_o} and y_o is a deep I_1-density point of A_{x_o}

P r o o f. We shall prove the first part for $(x_o,y_o) = (0,0)$. Suppose that if it is not the case. Let $B \supset R^2 \setminus A$ be an open set such that $(0,0)$ is a strong I_2-dispersion point of B. Suppose that 0 is not a deep I_1-density point of A^o (A^{y_o} for $y_o = 0$). Then 0 is not an I_1-dispersion point of B^o, say, from the right. Using Lemma 1.4 we can construct a "closed" interval set $D = \bigcup_{j=1}^{\infty} [a_j,b_j] \subset B^o$ Then there exists a sequence $\{h_j\}_{j \in N}$ of positive numbers such that $\bigcup_{j=1}^{\infty} ([a_j,b_j] \times [-h_j,h_j]) \subset B$.

From the fact that 0 is not a right I_1-dispersion point of D it follows that there exists an increasing sequence $\{k_n'\}_{n \in N}$ of positive integers such that for each subsequence $\{k_{n_p}'\}_{p \in N}$ the set $\limsup_p (k_{n_p}' \cdot D)$ is not of the first category. Now take k_1''

sufficiently big to assure that $h_j \cdot k''_j \geq 1$ for all j such that that $k'_1 \cdot b_j \geq 2^{-1}$. Next take $k''_2 \geq k''_1$ sufficiently big to assure that $h_j \cdot k''_2 \geq 1$ for all j such that $k'_2 \cdot b_j \geq 2^{-2}$. Generally, put $k''_n \geq k''_{n-1}$ sufficiently big to assure that $h_j \cdot k''_2 \geq 1$ for all j such that $k'_n \cdot b_j \geq 2^{-n}$ (only finite number of j's fulfils this inequality). Now it is not difficult to observe that for each subsequence $\{n_p\}_{p\in N}$ we have

$$\limsup_p ((k'_{n_p}, k''_{n_p})\cdot B) \cap [-1,1]^2 \supset$$

$$\limsup_p ((k'_{n_p}, k''_{n_p}) \cdot (\bigcup_{j=1}^{\infty} [a_j,b_j] \times [-h_j,h_j])$$

$$\cap [-1,1]^2 = (\limsup_p (k'_{n_p}\cdot D)) \times [-1,1]$$

and the last set is not of the first category as a plane set - a contradiction with the choice of B.

THEOREM 1.3. If $A \in S_2$, $(0,0)$ is a strong I_2-density point of A, 0 is a deep I_1-density point of A^0 and A_0, and, moreover, for each pair of increasing sequences $\{k'_n\}_{n\in N}$, $\{k''_n\}_{n\in N}$ of positive integers there exists an increasing sequence $\{n_p\}_{p\in N}$ of positive integers such that for each rectangle $(a',b') \times (a'',b'') \subset [-1,1]^2$ there exists a rectangle $(c',d') \times (c'',d'') \subset (a',b') \times (a'',b'')$ and a positive integer r such that $\bigcup_{p=r}^{\infty} ((k'_{n_p}, k''_{n_p})\cdot P_2) \cap ((c',d') \times (c'',d'')) = \emptyset$ (where P_2 is the set associated with A as described in Lemma 1.1), then $(0,0)$ is a deep strong I_2-density point of A.

P r o o f. We shall construct an open set $B \supset R^2\setminus A$ such that $(0,0)$ is a strong I_2-dispersion point of B.

Let $B_{(x)}, B_{(y)}$ be open sets such that $B_{(x)} \supset R \setminus A^o$, $B_{(y)} \supset R \setminus A_o$ and 0 is an I_1-dispersion point of $B_{(x)}$ and $B_{(y)}$. Put $B_1 = B_{(x)} \times R$, $B_2 = R \times B_{(y)}$. It is not difficult to see that $(0,0)$ is a strong I_2-dispersion point of both B_1 and B_2.

Now let us represent $R^2 \setminus A$ in the form $G \cup P_2$, using Lemma 1.1. Put $P = P_2 \setminus \{(x,y) : x \cdot y = 0\}$. Let

$$B_3 = \bigcup_{(x,y)\in P} (x - |x|^2, x + |x|^2) \times (y - |y|^2, y + |y|^2).$$

We shall show that $(0,0)$ is a strong I_2-dispersion point of B_3. Let $\{k'_n\}_{n\in N}, \{k''_n\}_{n\in N}$ be arbitrary increasing sequences of positive integers. From the assumption it follows that there exists an increasing sequence $\{n_p\}_{p\in N}$ of positive integers such that for each rectangle $(a',b') \times (a'',b'') \subset [-1,1]^2$ there exists a rectangle $(c',d') \times (c'',d'') \subset (a',b') \times (a'',b'')$ and a positive integer r such that $\bigcup_{p=r}^{\infty} ((k'_{n_p}, k''_{n_p}) \cdot P) \cap ((c',d') \times (c'',d''))$ $= \emptyset$ (since $P \subset P_2$). Observe that for each $(\bar{x},\bar{y}) \in (c',d')$ $\times (c'',d'')$ there exists r (depending on $(\bar{x},\bar{y})$) such that $(\bar{x},\bar{y}) \notin \bigcup_{p=r}^{\infty} (k'_{n_p}, k''_{n_p}) \cdot B_3$. It means that $((c',d') \times (c'',d''))$ $\cap \limsup_p ((k'_{n_p}, k''_{n_p}) \cdot B_3) = \emptyset$, hence $\limsup_p ((k'_{n_p}, k''_{n_p}) \cdot B_3)$ $\cap [-1,1]^2$ is nowhere dense. Observe also that $(0,0)$ is a strong I_2-dispersion point of G, since $G \subset R^2 \setminus A$.

Finally, put $B = B_1 \cup B_2 \cup B_3 \cup G$. We have obviously $B \supset R^2 \setminus A$ and $(0,0)$ is a strong I_2-dispersion point of B, as required.

THEOREM 1.4. Suppose that $f : R^2 \to R$ is strongly I_2-approximately continuous. Then f is separately I_1-approximately continuous.

P r o o f. We shall prove I_1-approximate continuity of f^o at $x = 0$ (for remaining points the proof is the same). Suppose that f^o is not I_1-approximately continuous at 0. It means that there exists $\varepsilon_o > 0$ such that 0 is not an I_1-dispersion point of the set $C = \{x : |f(x,0) - f(0,0)| \geq \varepsilon_o\}$. Since f is Baire 1 (see [2], th. 8'), C has the Baire property, so $C = G \Delta P$, where G is open and P is of the first category. Obviously 0 is not an I_1-dispersion point of G. Using Lemma 1.3 we construct an interval set $B = \bigcup_{n=1}^{\infty} (a_n, b_n) \subset G$ such that 0 is not an I_1-dispersion point of B (assume from the right).

It means that there exists an increasing sequence $\{n_m\}_{m\in N}$ of positive integers such that for each subsequence $\{n_{m_p}\}_{p\in N}$ the set $\limsup_p ((n_{m_p} \cdot B) \cap [0,1])$ is not of the first category.

If for each $n \in N$ $\{x_n^1, \ldots, x_n^{l_n}\} \subset (a_n, b_n)$ is an increasing sequence of numbers such that $x_n^1 - a_n < \frac{1}{n} \cdot a_n$, $b_n - x_n^{l_n} < \frac{1}{n} \cdot a_n$ and $x_n^l - x_n^{l-1} < \frac{1}{n} \cdot a_n$ for $l \in \{2, \ldots, l_n\}$; $B_n^l \subset (a_n, b_n)$ is any neighbourhood of x_n^l for $n \in N$ and $l \in \{1, \ldots, l_n\}$ and $D = \bigcup_{n=1}^{\infty} \bigcup_{l=1}^{l_n} B_n^l$, then for each $r \in N$

$$\left(\bigcup_{p=r}^{\infty} (n_{m_p} \cdot B) \setminus \bigcup_{p=r}^{\infty} (n_{m_p} \cdot D)\right) \cap [0,1]$$

Theorem J implies easily the following

Corollary 3. For a.a. $f \in C$ the following holds : If for $x \in (0,1)$ both $[\underline{D}_{ap}^{-}f(x), \overline{D}_{ap}^{-}f(x)] \neq \overline{R}$ and $[\underline{D}_{ap}^{+}f(x), \overline{D}_{ap}^{+}f(x)] \neq \overline{R}$, then each $y \in \overline{R}$ is a derived number of f at x with an upper one-sided density $d \geq 1/4$.

Theorem 4 (iii) shows that a typical $f \in C$ has at some points no finite derived number with a positive upper density.

R e f e r e n c e s

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