A PERRON TYPE INTEGRAL IN AN ABSTRACT SPACE

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We introduce a Perron type integral in a measure space based on the definition of the extreme derivates with respect to a sequence of nets. This integral is used to establish some generalizations of uniqueness theorems for certain orthogonal series.

Let $(X,\mathfrak{M},\mu)$ be a topological measure space, μ being a complete regular measure, $\mu(X)<\infty$, and $\mu(G)>0$ if G is open. Let $\mathfrak{N}=\{\mathfrak{N}_n\}_{n=1}^{\infty}$ be a monotone sequence of nets, each $\mathfrak{N}_n$ being a finite class $\mathfrak{N}_n=\{N_1^n,N_2^n,\dots,N_{j_n}^n\}$ of disjoint sets $N_j^n\in\mathfrak{M}$, called cells, $\bigcup_{j=1}^{j_n}N_j^n=X$ ($\mathfrak{N}_1=\{X\}$). Denote by Γ_j^n the boundary of N_j^n.

We assume that

1° $\mu(N_j^n)>0$, $\mu(\Gamma_j^n)=0$, $n=1,2,\dots$, $j=1,2,\dots,j_n$;

2° given a neighbourhood U_x of any $x\in X$, there exist a cell N_j^m such that $x\in\overline{N}_j^m\subset U_x$ ($\overline{N}$ is the closure of N).

For every closed set $L\subset X$ we will write $L^m=\bigcup_{j:\,\overline{N}_j^m\cap L\neq\emptyset}N_j^m$

Now let $\sigma(E)$ be a finitely additive set function defined at least on the algebra $\mathfrak{U}$ generated by the class of all cells N_j^n, $n=1,2,\dots$, $j=1,2,\dots,j_n$.

Given a closed set L, $\mu(L)=0$, and a cell N, we say that $\sigma(E)$ is $\mathfrak{N}$-continuous on L inside N if

$$\lim_{m\to\infty} \sigma(L^m \cap N) = 0.$$

In the usual way we can define the upper and lower derivates with respect to $\mathcal{N}$ for each point $x \in X$ denoting them, respectively, by $\overline{D}_{\mathcal{N}}\sigma(x)$ and $\underline{D}_{\mathcal{N}}\sigma(x)$.

The construction of the integral depends on the following result.

Lemma. Let σ be additive on $\mathcal{U}$ and $\underline{D}_{\mathcal{N}}\sigma(x) \geqslant 0$ everywhere on X except on the set $\bigcup K_i$ where each K_i is closed, $\mu(K_i) = 0$; let also $\sigma(E)$ be $\mathcal{N}$-continuous inside each cell N on its boundary Γ and on each set K_i, $i = 1,2,\ldots$. Then σ is non-negative on $\mathcal{U}$.

Next, we define $\mathcal{N}$-major and $\mathcal{N}$-minor functions for a real-valued function f on X.

An additive set function M defined on $\mathcal{U}$ is said to be an $\mathcal{N}$-major function of f on X if

(i) $\underline{D}_{\mathcal{N}}M(x) \geqslant f(x)$ almost everywhere on X;

(ii) $\underline{D}_{\mathcal{N}}M(x) > -\infty$ everywhere except on a set $\bigcup_{i=1}^{\infty} K_i$ where each K_i is closed and $\mu(K_i) = 0$;

(iii) M is $\mathcal{N}$-continuous inside each cell N on its boundary Γ and on each set K_i.

An additive set function m defined on $\mathcal{U}$ is said to be an $\mathcal{N}$-minor function of f on X if $-m$ is an $\mathcal{N}$-major function of $-f$ on X.

Using the lemma above one can prove that for all such M and m the function $M - m$ is non-negative set function on $\mathcal{U}$.

Then we define the $P_{\mathcal{N}}$-integral of f over X as the common value of sup $m(X)$ and inf $M(X)$ where the sup is taken over all m, and the inf is taken over all M.

This construction for some special choices of X and $\mathfrak{N}$ can be applied to recover the coefficients, for certain orthogonal systems, from the sum of pointwise convergent orthogonal series, i.e. to solve the problem of finding an integral wide enough to make the usual Fourier formulae valid for such series.

We give two example of such applications.

I. Let (1) $\sum a_n h_n(x)$ where $n = (n_j)_{j=1}^k$,

$x = (x_j)_{j=1}^k$ be a multiple Haar series which is rectangularly convergent to a finite sum f everywhere on the unit cube I except on a set $\bigcup K_i$ where each K_i is the intersection of I and a hyperplane of the form $\{x: x_j = a_i\}$. Suppose that for all $x \in I$

$$a_n h_n(x) = \bar{o}(n_j) \text{ for fixed } n_s, \quad s = 1,2,\dots,k, \quad s \neq j.$$

Then (1) is a $P_{\mathfrak{N}}$-Fourier series of f where $P_{\mathfrak{N}}$-integral is defined on $X = I$, $\mathfrak{N}$ being the sequence of dyadic nets and the exceptional set $\bigcup K_i$ in the definition of major and minor functions is as above.

II. As the second example, we consider the series with respect to the Vilenkin multiplicative system $\{\chi_i\}$ which is the system of characters of the group $G = \prod_{j=1}^{\infty} Z_{p_j}$, a countable product of cyclic groups with orders p_j.

In this case, let $X = G$, suppose the measure μ is the Haar measure on G, the net $\mathfrak{N}_n$ is constituted by all cosets of the subgroup

$$G_n = \left\{ x=(x_j) \in G : \ x_1 = x_2 = \dots = x_n = 0 \right\},$$

the exceptional set $\bigcup K_i$ in the definition of major and minor functions being a countable subset of G .

We then get a $P_{\mathfrak{N}}$-integral wide enough for any series $\sum a_n \chi_n(x)$ convergent everywhere on G except on a countable set to be a $P_{\mathfrak{N}}$-Fourier series of its sum.

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