Descriptive Set Theory and the Structure of Sets of Uniqueness for Trigonometric Series

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Summary

A subset $P \subseteq \mathbb{T}$ of the unit circle is called a *set of uniqueness* if every trigonometric series converging to 0 off P is identically 0. It is a *set of extended uniqueness* if this is true for trigonometric series which are Fourier-Stieltjes transforms of measures on $\mathbb{T}$.

We first survey some of the results of the classical and modern theory of sets of uniqueness and discuss a number of basic open problems in this area.

The main part of the talk is devoted to an exposition of the recently discovered connections and applications of descriptive set theory to this area of analysis. Most of these results deal with the global structure of the classes U, U_0 of closed sets of uniqueness, resp. extended uniqueness.

Results discussed include: 1) the classification of U, U_0 as co-analytic non-Borel sets in the space $K(\mathbb{T})$ of closed sebsets of $\mathbb{T}$ and its application to the Characterization Problem for U - and U_0−sets (Solovay, Kaufman); 2) Study of ordinal rankings on U and U_0 and their use in proving decomposition theorems for U−and U_0−sets and in particular the existence of a Borel basis for U_0 (Piatetski-Shapiro, Kechris-Louveau); 3) Structure theorems for co-analytic σ-ideals of closed sets in compact metrizable spaces (Kechris-Louveau-Woodin); 4) Applications of the preceding methods and results to the proof that every Borel set of extended uniqueness (and thus also of uniqueness) is of the first category (Bary's problem) and, together with Körner's Theorem on Helson sets of multipilicity, to the proof of the non-existence of a Borel basis for U (Debs-Saint Raymond).