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ON QUADRATIC FUNCTIONALS AND SOME PROPERTIES OF HAMEL BASES

This talk is based on a joint paper with Zbigniew Gajda (Katowice, Poland) recently submitted for publication .

A mapping $q : R \to R$ is called a quadratic functional if $q(x+y) + q(x-y) = 2q(x) + 2q(y)$ holds for all x and y in R (the real numbers) . An extensive study of quadratic functionals on linear spaces may be found in [4] , [9] and [11] . We restrict ourselves to functionals defined on the real line only . In this case quadratic functionals share several regularity properties of additive functions , i.e. maps $a : R \to R$ satisfying the Cauchy functional equation : $a(x+y) = a(x) + a(y)$, $x,y \in R$.

For instance, both in the case of additive functions and quadratic functionals (two-sided) boundedness on a set of positive inner Lebesgue measure or on a second category Baire set implies their continuity on the whole real line (see e.g. [1] , [3] , [8] , [9] and [10]) .

It is also known that any additive function with a continuous restriction to an analytic set containing a Hamel basis of R must be continuous everywhere . This result is due to F.B. Jones [5] (also see [7]) . Since every second category Baire subset of R contains a Hamel basis (cf. [1] [6] , [8]) , we infer from Jones's theorem that any additive function whose restriction to a Baire set of the second category is continuous must be continuous on R. This observation gives rise to the question whether the same result is true for quadratic functionals. The answer to this question is affirmative (see Theorem 3).

Definition. $S(A) := \{x \in R : x = \frac{1}{2}(a+b), \; a, b \in A\}$.

It was pointed out by R. Ger [4] that any function prescribed arbitrarily on the set $S(H)$ (where H is a Hamel basis of R) can be uniquely extended to a quadratic functional. In this regard the set $S(H)$ plays the same role for quadratic functionals as the Hamel basis plays for additive functions. This analogy motivates the following.

THEOREM 1. Let $T \subset R$ be an analytic set and suppose that there exists a Hamel basis H such that $S(H) \subset T$. If $q : R \to R$ is a quadratic functional whose restriction to T is continuous, then q is continuous on R.

In addition the following two results are true.

THEOREM 2. If T is a second category Baire subset of R, then, under the continuum hypothesis, there exists a Hamel basis H such that $S(H) \subset T$.

THEOREM 3. Let $T \subset R$ be a Baire set of the second category and assume the continuum hypothesis. Then every quadratic functional $q : R \to R$ with a continuous restriction to T is continuous on R.

REMARK. Martin's Axiom, which is weaker than the continuum hypothesis, implies that the union of less than c, the cardinal of the continuum, first category sets is a first category set and that the union of less than c sets of measure zero is a set of measure zero (see: D.A. Martin and R.M.

Solovay, Internal Cohen extensions, Ann. Math. Logic 2 (1970), 143-178). Moreover, the hypothesis "the union of less than c first category sets is first category and the union of less than c sets of measure zero is a set of measure zero" is even weaker than Martin's Axiom. Theorems 2 and 3 can be proved using this weaker axiom in place of the continuum hypothesis.

REFERENCES

[1] Z. Ciesielski, W. Orlicz, Some remarks on the convergence of functionals on bases, Studia Math. 16(1958), 335-352 .

[2] Z. Gajda, On some properties of Hamel bases connected with the continuity of polynomial functions, Aequationes Math. 27(1984), 57-75.

[3] R. Ger, Convex functions of higher orders in Euclidean spaces, Ann. Polon. Math. 25(1972), 293-302.

[4] R. Ger, Some remarks on quadratic functionals (submitted to Glasnik Mat.).

[5] F.B. Jones, Measure and other property of a Hamel basis, Bull. Amer. Math. Soc. 48(1942), 472-481.

[6] Z. Kominek, On the sum and difference of two sets in topological vector spaces, Fund. Math. 71(1971), 165-169.

[7] Z. Kominek, On the continuity of Q-convex functions and additive functions, Aequationes Math. 23(1981), 146-150.

[8] M. Kuczma, An Introduction to the Theory of Functional Equations and Inequalities, Polish Scientific Publishers and Silesian University Press, Warszawa-Krakow?Katowice , 1985 .

[9] S. Kurepa, On the quadratic functional, Acad. Serbe Sci. Publ. Inst. Math. 13(1959), 57-72.

[10] A. Ostrowski, Uber die funktionalgleichung der exponentialfunktion und verwandte funktional-gleichungen, Jber. Deutch Math. Verein 38(1929), 54-62.

[11] J. Rätz, Quadratic functionals satisfying a subsidiary inequality, General Inequalities 1 (Edited by E.F. Beckenbach) ISNM41, Birkhauser Verlag, Basel und Stuttgart, 1978, 261-270.