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Differentiable Restrictions of Real Functions

We review some of the known theorems about differentiable and monotonic restrictions of continuous and arbitrary real functions and present some new results of this type. We adopt the convention that differentiable functions are assumed to have finite valued derivatives, and when we mean differentiability in the extended sense (allowing $f'(x) = +\infty$ or $-\infty$), we put quotation marks on "differentiability".

The first result of the type we are conisdering would be the following.

Theorem 1: For every continuous $f : [0,1] \to R$, there exists a perfect subset P of [0,1] such that $f|P$ is differentiable.

We don't know when this result was first discovered, but it certainly follows from Lebesgue's Theorem together with the monotonicity results of Minakshisundaram [13], Padmavally [15], Marcus [12], and Garg [7]. The set P in the conclusion of Theorem 1 cannot be made to have positive measure because of the existence of nowhere approximately differentiable continuous functions (see the paper of Jarnik [9] or [3, Ch. 16]).

In order to improve the conclusion of Theorem 1 (i.e. to obtain that $f|P$ is C^1 or twice differentiable or better), it is clear that one needs similar theorems for functions with domains arbitrary perfect sets other than [0,1]. A theorem of this type for monotonic restrictions was established by Filipczak in 1966 [6].

Theorem 2: If P is a perfect subset of [0,1], then for every continuous $f : P \to R$, there exists a perfect subset Q of P such that $f|Q$ is monotonic.

Then, a "differentiable" restriction result was established by Bruckner, Ceder, and Weiss in 1969 [4].

Theorem 3: If P is a perfect subset of [0,1], then for every continuous $f : P \to R$, there exists a perfect subset Q of P such that $f|Q$ is "differentiable" and monotonic.

Morayne gave a simplied proof of Theorem 3 in [14].

Theorem 1 was drastically improved by Laczkovich in 1984 [11], and a second remarkable result was obtained by Agronsky, Bruckner, Laczkovich, and Preiss in 1985 [1].

Theorem 4: If P is a perfect subset of [0,1] and P is of positive measure, then there exists a perfect subset Q of P such that

1) $f|Q$ is C^{∞} (relative to Q) [11], and

2) $f|Q$ extendable to a C^1 $g : [0,1] \to R$ [1].

and 3) $f|Q$ is monotonic.

Of course, once you get $f|Q$ differentiable, it follows that f can be extended to a differentiable $g : [0,1] \to R$, because **every** differentiable function with domain a perfect set is so extendable (see [10] or [16]). However, C^1 functions with perfect domains are not necessarily extendable to C^1 functions $g : [0,1] \to R$. In order to obtain 2) of Theorem 4, it is necessary to show that the conditions of the "Whitney Extension Theorem" [17] are satisfied.

The following is an improvement of Theorem 3 which is in the spirit of 2) of Theorem 4. We say that a function f with domain a perfect set P is "C^1" (with quotation marks) if f is "differentiable" and the extended real valued function f' is continuous.

Theorem 5: If P is a perfect subset of [0,1], then for every continuous $f : P \to R$, there exists a perfect subset Q of P such that $f|Q$ is monotonic and extendable to a "C^1" $g : [0,1] \to R$.

The proof of Theorem 5 calls upon a variation of the Whitney Extension Theorem for "C^1" extensions.

It follows from the following theorem that the function $f|Q$ of conclusion 2) of Theorem 4 cannot necessarily be extended to a function $g : [0,1] \to R$ which is twice differentiable or even to a $g : [0,1] \to$ which is "twice differentiable" (i.e. g is differentiable and g' is "differentiable").

Theorem 6 [1]: For every $\epsilon > 0$, there exists a perfect subset P of [0,1] of measure at least $1-\epsilon$ and a continuous $f : P \to R$ such that a) $\{x \epsilon P : f(x)=g(x)\}$ is finite for every twice differentiable $g : [0,1] \to R$, and

b) $\{x \epsilon P : f(x)=g(x)\}$ has at most finitely many limit points for every "twice differentiable" $g : [0,1] \to R$.

Actually, only a) was proved in [1], but it takes only a little more argument to prove b). As a matter of contrast, the following theorem holds.

Theorem 7: If A is a subset of [0,1] of positive outer measure, then for every f : A -> R, there exists an infinite closed subset B of A such that f|B is extendable to a monotonic "twice differentiable" g : [0,1] -> R.

Since this is a theorem about arbitrary functions, it is a variant of Blumberg's Theorem.

In 1969 Ceder [5] proved the following "differentiability" variant of Blumberg's Theorem.

Theorem 8: If A is an uncountable subset of [0,1], then for every f : A -> R, there exists a bilaterally dense in itself subset B of A such that f|B is "differentiable" and monotonic.

It should be noted that the monotonicity of f|Q in the conclusions of Theorems 3-5 can be obtained after all the other desired properties have been established by just applying Filipczak's Theorem 2. Monotonicity of f|B in the proof of Theorem 6 must be obtained essentially simultaneously with "differentiability" because there is no theorem analogous to Filipczak's Theorem for functions with a bilaterally dense in itself domain. In fact, there was an error in Ceder's original proof concerning the monotonicity of f|B, but clarification was given in [2] and [8].

The following is the "Blumberg variant" of Theorem 5 and represents an improvement in Theorem 6.

Theorem 9: If A is an uncountable subset of [0,1], then for every f : A -> R, there exists a bilaterally dense in itself subset B of A such that f|B is monotonic and extendable to a "C^1" g : [0,1] -> R.

The following theorem is the "Blumberg variant" of 2) and 3) of Theorem 4.

Theorem 10: If A is a subset of [0,1] of positive outer measure, then for every $f : A \to R$, there exists a bilaterally dense in itself subset B of A such that $f|B$ is monotonic and extendable to a C^1 $g : [0,1] \to R$.

Open Problem: We have as yet been unable to determine if the set B of Theorem 8 can be chosen so that $f|B$ is C^∞ (relative to B).

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