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NEAR INTERSECTION CONDITIONS FOR PATH DERIVATIVES

1. Introduction.

In recent years, the notion of "Path Derivative" has proved useful in clarifying why certain generalized derivatives are good substitutes for the ordinary derivative when the latter is not known to exist. When the system of paths satisfies certain conditions, the functions that are differentiable with respect to that system, as well as their derivatives with respect to that system, will exhibit many of the desirable properties of functions differentiable in the ordinary sense and of their ordinary derivatives. Similarly, the extreme path derivates will serve as good substitutes for the ordinary extreme derivates. Various results of this nature can be found in the recent articles [A], [BJ], [BOT], [BTa], [BT], and [Ta].

Prominent among the conditions a system of paths may satisfy are so-called "intersection conditions" and "porosity conditions". These two types of conditions are of different natures. An intersection condition involves interactions of the paths within the system, while porosity conditions involve the paths individually. Nonetheless, appropriate porosity conditions can sometimes compensate for lack of an intersection condition [BTa], [LP], [Ta]. In other cases, such conditions do not suffice [BLPT].

In the present paper we introduce notions of "Near intersection conditions." With any system of paths, E, we can associate certain functions F such that the pair (E,F) satisfy a near intersection condition. We study the extent to which various near intersection conditions on a pair (E,F) can compensate for the lack of a (full) intersection condition. We focus on <u>external</u> near intersection conditions. A development of other forms of near intersection conditions, together with a more complete development of the external version will appear in [C].

2. Preliminaries.

In this section we present four conclusions that follow when a path system satisfies the External Intersection Condition (EIC). In the next section we will indicate how the EIC can be weakened without losing these conclusions.

Definition: A path system $E = \{E_x | x \in \mathbb{R}\}$ satisfies the EIC if there is a positive function δ so that if

$0 < y - x < \min\{\delta(y), \delta(x)\}$ then

(*) $\qquad E_x \cap E_y \cap [2x-y, x] \neq \emptyset$ and $E_x \cap E_y \cap [y, 2y-x] \neq \emptyset$.

In order to make the connection between the EIC and the conditions of the next section more clear note that the statement (*) is equivalent to the following: there are points a_1, a_2 which satisfy

(i) $\quad a_1, a_2 \in E_x \cap E_y$ and (ii) $a_1 \in [2x-y, x]$, $a_2 \in [y, 2y-x]$.

Theorem 0. Suppose E and E^* are path systems that satisfy the EIC.

(1) If F is E-continuous (i.e. for each x, $\lim_{\substack{t\to x \\ t\in E_x}} F(t) = F(x)$),

then F is a Baire one function. ([BOT], 5.2)

(2) If F is E-differentiable on a set X, then F is ACG on X. ([BOT], 5.4)

(3) If F is E-differentiable, then F'_E is a Baire one function. ([BOT], 6.3)

(4) If F is both E and E^* differentiable on a set X then $F'_E = F'_{E^*}$ n.e. on X. ([C])

Conclusion (4) is an EIC version of the result ([BOT], 7.8) for the Intersection condition.

Here, as usual, the "n.e." (nearly everywhere) indicates that the equality holds except possibly on a denumerable subset of X.

3. Results.

In this section we will define a sequence of conditions which weaken the EIC yet still give the conclusions discussed in section 2. It is important to keep in mind that while the EIC is a condition on the path system, the following are conditions on both the function and the path system. The proofs of the results and the examples discussed may all be found in [C].

In the remainder of this section φ denotes a positive non-decreasing function defined on the positive real numbers with $\varphi(0) = 0$.

(3.1) Definition: The function F and the path system E together satisfy the condition E_1 if there is a function φ continuous at zero and a positive function δ so that whenever $0 < y - x < \min\{\delta(y), \delta(x)\}$ there are points x_1, x_2, y_1, y_2 for which the following hold:

(i) $x_1, x_2 \in E_x$ and $y_1, y_2 \in E_y$.

(ii) $x_1, y_1 \in [2x-y, x]$ and $x_2, y_2 \in [y, 2y-x]$.

(iii) $|F(x_i) - F(y_i)| \leq \varphi(y-x)$, for $i = 1,2$.

Notice that this condition does not guarantee that the paths E_x and E_y intersect, only that there are points in E_x where F assumes values which are near (in terms of φ) values assumed at points in E_y.

(3.2) Theorem: If F and E satisfy E_1 and F is E-continuous, then F is a Baire one function.

Now we define a property for functions which is similar to the properties VBG and ACG.

(3.3) Definition: A function F is generalized Lipshitz (denoted LG) on a set X if X may be expressed as a countable union of sets on each of which F satisfies a Lipschitz condition. If each of the sets may be taken as closed then F is [LG].

Remark: LG implies ACG but the converse does not hold. Many results which conclude that a function is ACG can be strengthed to get LG with only minor modifications of the proofs. For example, this is true for the result (2) given in the previous section.

(3.4) Definition: The function F and the path system E together satisfy the condition E_2 if there is a function φ with finite derivative at zero and a positive function δ so that whenever $0 < y - x < \min\{\delta(y), \delta(x)\}$ there are points x_1, x_2, y_1, y_2 which satisfy:

(i) $x_1, x_2 \in E_x$ and $y_1, y_2 \in E_y$

(ii) $x_1, y_1 \in [2x-y, x]$ and $x_2, y_2 \in [y, 2y-x]$

(iii) $|F(x_i) - F(y_i)| \leqslant \varphi(y-x)$ for $i = 1,2$.

The only difference between E_1 and E_2 is that we require φ to be differentiable at the origin for E_2.

(3.5) Theorem: If F and E satisfy E_2 and F is E differentiable on a set X, then F is LG on X. If the set X is closed then F is [LG].

We will now describe an example which shows that the condition E_1 is not strong enough to get the above conclusion. Begin with an absolutely continuous function F that has an infinite derivative everywhere on a nowhere dense perfect set P. Next, define a function G so that $G = F$ on P but so that G has zero as a bilateral derived number at each point of P. The path system E can be chosen so $G'_E = 0$ on P and so that G and E satisfy E_1 (take φ to be the modulus of continuity for G). Such a function G cannot be LG since by the Baire category theorem a LG function must satisfy a Lipschitz condition on (a dense subset of) a nonempty portion of P.

One may suspect that if in the definition of E_2 a stronger requirement is made on φ (such as $\varphi'(0) = 0$ or $\varphi \equiv 0$) that it may be possible to obtain the conclusions (3) and/or (4). This is not the case. In fact, we have constructed examples which show that the requirement $\varphi \equiv 0$ is not enough to get either (3) or (4). Since it is impossible to get these conclusions by only strengthening the requirement on φ, we will now add restrictions to the proximity of the points x_i and y_i.

(3.6) Definition: The function F and the path system E satisfy the condition E_3 if there is a function φ with $\varphi'(0) = 0$ so that all the requirements for E_2 are met and in addition:

(iv) $$|(y_1 - x_1) - (y_2 - x_2)| \leq \varphi(y-x).$$

(3.7) Theorem: If F and E satisfy E_3 and F is E differentiable, then $F'_E \in B_1$.

Condition E_3 does not suffice, however, for conclusion (4), even with $\varphi \equiv 0$. An example can be constructed along the following lines.

Let P_0 be a perfect subset of a Hamel basis containing a rational, say $P_0 \subset [0,1]$. For each non zero integer let $P_n = P_0 + \frac{1}{n}$. Define F on $\bigcup_{n=-\infty}^{\infty} P_n$ by $F(x) = \frac{1}{n}$, $x \in P_n$ $(n \neq 0)$ and $F(x) = 0$ on P_0. It is easy to verify that $\bigcup_{n=-\infty}^{\infty} P_n$ is closed and F is continuous on this set. Extend F to a continuous function on $\mathbb{R}$, $\hat{F}$ differentiable on $\mathbb{R} - \bigcup_{n=-\infty}^{\infty} P_n$. We choose $E_x = E_x^*$ in an obvious manner for $x \notin P_0$. For $x \in P_0$, take $E_x = P_0$ and $E_x^* = \{x\} \cup \{x + \frac{1}{n} : n \neq 0\}$. One verifies easily that $F_E'(x) = 0$ and $F_{E^*}'(x) = 1$ for $x \in P_0$ so conclusion (4) fails. One can also verify that E satisfies the EIC on P_0 while E^* satisfies E_3 with $\varphi \equiv 0$. (Actually, E^* satisfies a slight variant of E_3 in that the intervals adjacent to $[x,y]$ in the definition of E_3 have length $2(y-x)$ rather than $y - x$. This distinction offers no essential difficulty -- we chose our example for simplicity of presentation. See [C] for a complete analysis).

We obtain a condition that does suffice for conclusion (4) by replacing the requirement (iv) by the stronger condition (iv')

(iv') $\quad |y_1 - x_1| \leqslant \varphi(y-x)$ and $|y_2 - x_2| \leqslant \varphi(y-x)$.

(3.8) Definition: The function F and the path system E satisfy the condition E_4 if the requirements for E_3 are met with (iv') replacing (iv).

We thus have an apparent improvement of (4) in Theorem 0. Theorem 3.9 shows, however, that there is no real improvement.

3.9. Theorem: If F and E satisfy E_4 and F is E differentiable, then there is a system E^* satisfying the EIC so that F is E^* differentiable and $F_E'(x) = F_{E^*}'(x)$ for all x.

Thus, one can view E_4 as a pre-EIC condition; the system E can be "improved" to a system E^* satisfying EIC without altering F_E'.

Remark: A number of results in the literature can be stated in terms of "improvement of the path system." For example suppose F is E differentiable with $d(E_x, x) = 1$ for all x; that is, F is approximately differentiable. It then follows that F is differentiable at each point of some dense, open set G, and $F'_{ap} = F'$ on G. Thus, for the path system E^* defined by

$$E^*_x = \begin{cases} E_x, & x \notin G \\ \mathbb{R} & x \in G \end{cases}$$

we have

$F'_E = F'_{E^*}$, but E^* is a much "fuller" system than E.

A similar statement is valid for a continuous function F that is differentiable with respect to a nonporous system E [BT]. (That system may, for example, consist of sequential paths). If F is a Lipschitz function, then F is actually differentiable, so we can take $E^*_x = \mathbb{R}$ for all x.

Suppose now that F and E satisfy E_4. By Theorem 3.9, we can replace E with a system E^* satisfying the EIC. It follows that F is LG and thus approximately differentiable a.e. with $F'_E = F'_{ap}$ a.e. (because of conclusion (4)). In fact, by (3.5), and [BOT], F is actually differentiable a.e. on a dense-open set.

The condition E_4 may appear artificial, but it does provide a useful test for improvement to EIC. For example, one can show that if F is in Lipschitz class β $(\beta > \frac{1}{2})$ and is E differentiable with respect to s system E satisfying, for all $x \in \mathbb{R}$,

$$E_x \cap \left[x + \frac{1}{n+1}, x + \frac{1}{n}\right] \neq \emptyset$$

and

$$E_x \cap \left[x - \frac{1}{n}, x - \frac{1}{n+1}\right] \neq \emptyset,$$

then F and E satisfy E_4.

It follows that E can be improved to a system E^* satisfying EIC, and all the discussion above applies.

For $\beta = 1$, we obtain much stronger results, of course.

This example also illustrates a way in which non-porosity conditions can substitute for the EIC. It would be of interest fo find general conditions relating conditions of non-porosity type on E to moduli of continuity on F that permit improvement of E to E^* satisfying the EIC.

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