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A Property of Borel Measurable Functions and Extendable Functions

The results announced here will involve real-valued functions defined on a closed interval. We begin by defining some of the types of functions that are involved.

Definition. A function $f:X \rightarrow Y$ is said to be

(1) a connectivity function provided that if C is a connected subset of X, then the graph of f restricted to C is connected in $X \times Y$;

(2) a Darboux function provided that if C is a connected subset of X, then $f(C)$ is connected in Y; and

(3) an extendable function provided that there exists a connectivity function $g:X \times I \rightarrow Y$ such that $f(x) = g(x,0)$ for each x in X where $I = [0,1]$.

Since a projection map is continuous, a connectivity function is also a Darboux function. From Stallings paper,[21], it follows that if f is extendable, then f is a connectivity function.

Definition. A function $f:[a,b] \rightarrow$ Reals is said to have

(4) property B provided that for each pair of open intervals (p,q) and E if $(p,q) \cap f^{-1}(E)$ is uncountable, then there exists a perfect set P such that $P \subset (p,q) \cap f^{-1}(E)$ and

(5) a <u>perfect road</u> at $x \in [a,b]$ if there exists a perfect set P containing x as a bilateral point of accumulation such that $f|P$ is continuous at x. If x is an endpoint, then the bilateral condition is replaced with a unilateral condition.

Clearly, Borel measurable functions have property B. Also if a function has a perfect road at each point, then it must have property B. Thus every extendable function must have property B,[9]. However, there exist connectivity functions that do not have property B,[7]. Thus it follows that within the class of all functions $[a,b] \to \text{Reals}$, only the following implications hold.

$$\begin{tikzcd}
\text{Extendable} \arrow[r] \arrow[dr] \arrow[dd] & \text{Connectivity} \arrow[r] & \text{Darboux} \\
& \text{Perfect Road} \arrow[dl] & \\
\text{Property B} & &
\end{tikzcd}$$

However we have the following theorem.

Theorem. If $f:[a,b] \to \text{Reals}$ is a Darboux function having property B, then f has a perfect road at each point.

In [2], Brown, Humke, and Laczkovich gave an improvement of a proof of an implication (connectivity + Baire class 1 $\longrightarrow$ perfect road) in theorem 1.1, Chapter II, of Bruckner's book, [4], by noting that with a slight modification it follows that for Borel measurable functions, if f is Darboux, then f has a perfect road at each point. The theorem above improves that improvement to a class of functions (functions having property B) that contains 2^c many. This follows because there are 2^c many extendable functions,[11].

Finally, using a category argument we can construct the following example,[11].

Example. There exist extendable functions $f_1, f_2 : [0,1] \longrightarrow$ Reals such that $f_1 + f_2$ does not have property B.

This example shows that there exist two extendable functions whose sum is not extendable. This leads to the following questions.

Questions. (1) Is the composition of two extendable functions extendable? (2) Is the uniform limit of a sequence of extendable functions extendable? (3) Is any function the pointwise limit of a sequence of extendable functions? (4) Is any function the sum of two extendable functions?

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