Chaotic Behavior and Equicontinuity of Iterates of an Interval Map

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Let f be a continuous function mapping an interval I into itself. For $n=1,2,\dots$, let $f^{n+1}=f\circ f^n$. Research in the social, biological and physical sciences often leads to a study of the sequence $\{f^n\}$ of iterates [M],[Y],[VSK]. Of importance to the researchers is the question, "if $|x-y|$ is small, will $|f^n(x)-f^n(y)|$ be small for all n?". For example, if x_0 denotes the actual initial population of a species of insect and y_0 the population as estimated by the researcher, one would hope that $|f^n(x_0)-f^n(y_0)|$, the error in estimating the population of the n^{th} generation, would be small if the initial estimate were good. In mathematical language, one would like the family $\{f^n\}$ to be equicontinuous.

Practical problems, however, don't usually lead to equicontinuity of $\{f^n\}$. In fact, one often finds chaotic behavior of various sorts. Theoretically, there could be almost certainly that chaos will arise. Bruckner and Hu [BH] recently established the following result.

Theorem. There exists a continuous function f mapping $[0,1]$ into itself and a set S of Lebesgue measure one such that for $x,y\in S(x\neq y)$,

$$\limsup_{n\to\infty}|f^n(x)-f^n(y)|=1,\quad \liminf_{n\to\infty}|f^n(x)-f^n(y)|=0.$$

Thus with this function f, one can be almost sure that both the true initial value x_0 and the estimate y_0 will be in S, and using $f^n(y_0)$ to predict $f^n(x_0)$ is of no value. The function f can be chosen arbitrarily close (uniformly) to the function $g(x)=4\alpha(1-x)$, a function in the logistic family often arising in practice [M],[P]. We should mention that in practice chaotic situations do arise, but situations in which satisfactory prediction is possible for x_0 and y_0 in some set large in measure or category also arise [P].

The purpose of this article is to determine conditions under which the ideal situation, equicontinuity of the family of iterates, does occur. We shall see that this happens only under exceptional circumstances.

Theorem 1. Let $f:I \to I$ be a continuous mapping of the unit interval into itself such that its fixed point set is a non-degenerate interval. Then a necessary and sufficient condition for $\{f^n\}$ to be equicontinuous is that for each x in I, $\{f^n(x)\}$ converges to a fixed point of f.

For a mapping of an interval I onto itself, there is a particularly simple necessary and sufficient condition for equicontinuity of the sequence $\{f^n\}$: namely that f^2 be the identity. That the condition is sufficient is obviois. The proof of the necessity depends on the following lemma.

Lemma. Let (X,d) be a compact metric space and $f:X \to X$ be a surjective mapping whose sequence of iterates $\{f^n\}$ is equicontinuous Then for any two points x an y in X, the two sequences of iterates $\{f^n(x)\}$ and $\{f^n(y)\}$ have convergent subsequences of the same indices $\{f^{n(i)}(x)\}$, $\{f^{n(i)}(y)\}$ such that $f^{n(i)}(x) \to x$, $f^{n(i)}(y) \to y$ as $i \to \infty$

Theorem 2. Let f be a continuous mapping of a compact interval I onto itself. Then a necessary and sufficient condition for $\{f^n\}$ to be equicontinuous is that f^2 be the identity mapping.

For mappings which are not necessarily surjective, a necessary and sufficient condition for $\{f^n\}$ to be equicontinuous is given by the following theorem.

Theorem 3. Let $f:I \to I$ be a continuous mapping. Then $\{f^n\}$ is equicontinuous if and only if $\bigcap_{n=1}^{\infty} f^n(I) = F_2$, where F_2 is the fixed point set of f^2.

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