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Local Compactness and Porosity in Metric Spaces

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1. Introduction.

In recent years a number of articles concerning the notion of porosity of a linear set have appeared in the literature. This is due, in part, to the many questions in which this notion arises naturally. The interested reader may consult several recent issues of The Real Analysis Exchange, in particular the comprehensive articles by Bullen [2] and Thomson [4].

While linear porosity has been studied extensively, porosity in spaces more general than the line seems to have been ignored. The purpose of this article is to present another application of porosity, but in settings more general than the line. We offer a generalization of unilateral linear porosity to metric spaces and show that when the metric space A is a convex subset of a separable Banach space, porosity is intimately related to local compactness. More specifically, we show that when A is locally compact, every sphere in A contains nowhere dense compact sets that are not σ-porous, but when A is not locally compact, then every compact set is totally porous, that is, porous "in all directions." Thus, for such spaces, local compactness can be characterized in terms of the porosity of its compact subsets.

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We choose convex sets for the spaces we study for two reasons: firstly, many subspaces that arise naturally are convex, and some infinite dimensional convex sets are locally compact (e.g. the Lipschitz functions in $C[a,b]$ having a fixed coefficient M); secondly, convex sets (with more than one member) have sufficient structure to avoid the nuisances one must deal with when a space has "extraneous" parts such as isolated points.

In section 2 we provide the necessary definitions along with some discussion indicating why we chose those particular definitions. We also set forth some notation and some known results that we use repeatedly in the sequel.

Our main results appear in section 3.

2. Preliminaries.

We shall be concerned with the notion of porosity in metric spaces more general than R_1. In writing this preliminary section, we have three objectives:

i) to motivate the particular notion of porosity that we develop;

ii) to provide the definitions needed in our development; and

iii) to set forth the notation, conventions, and some known results that we shall use throughout the paper.

When dealing with porosity in R_1, one can deal with symmetric porosity or one can obtain a slightly more delicate development by distinguishing right porosity from left porosity. The latter may be more appropriate, for example, when one deals with unilateral generalized derivatives [1].

When one tries to generalize porosity even to R_2, one is immediately faced with some decisions: should one deal with symmetric porosity or should one try to obtain a more delicate version? If the latter, how does one do this? One can develop a notion of porosity "with respect to quadrants" or "with respect to spheres." The simplest and perhaps most flexible generalization of right or left porosity may be the following, which we give as a definition.

Definition 1. Let (X,ρ) be a metric space, B a subset of X and $x \in B$. Let S be a sphere in X such that x is in the boundary of S, i.e. $x \in \overline{S} - S$. Then B is said to be porous at x with respect to S if there exists a $\gamma > 0$ such that for every $\varepsilon > 0$ there exist spheres $S_1 \subset S_2 \subset S$ such that $x \in \overline{S}_2 - S_2$, $S_1 \cap B = \emptyset$ and $\varepsilon >$ diameter $S_1 \geq \gamma$ (diameter S_2).

If each $x \in B$ is porous with respect to some sphere, we say B is a porous set.

It is clear that a porous set must be nowhere dense. But even in R_2, a set B can be porous at a point x with respect to some sphere and yet not give much of a sense of porosity (for example if $B = S \setminus S'$ where S and S' are spheres, $S' \subseteq S$, S' tangent to S at x). We obtain a much stronger sense of porosity if we require porosity with respect to all spheres.

Definition 2. A set B that is porous at a point $x \in B$ with respect to every sphere containing x in its boundary is called totally porous at x. If B is totally porous at all of its points, we say that B is totally porous.

Thus, a totally porous set is very thin near its points.

Definition 3. A set B is nonporous if there exists $x \in B$ such that B fails to be porous at x.

Finally, a set B is called σ-porous if it is a countable union of porous sets.

While the preceding definitions make sense in any metric space, the theorems we obtain in section 3 below depend on avoiding certain nuisances. Since our objective is to relate porosity to local compactness, we week a setting that is reasonably general and at the same time avoids certain nuisances that would require many special considerations. We have chosen convex subsets of Banach spaces as our underlying metric spaces because such spaces seem to satisfy both requirements: convex subsets of Banach spaces are plentiful and, as we shall see, they provide enough structure to lead to our theorems without requiring many special assumptions.

Throughout section 3 we will make use of a theorem of Banach and Mazur [1 sec. 26 th. 1]:

Theorem A: Let $C = C[0,1]$ and let X be any separable Banach space. Then there exists a linear isometry ϕ of X onto a subspace of C.

Theorem A allows us to state our theorems for convex subsets of separable Banach spaces while presenting our proofs for the specific space C. This is so because the linearity of ϕ implies that convexity is preserved by ϕ. On the other hand, ϕ is an isometry so ϕ maps porous sets onto porous sets and spheres onto spheres.

Let A be a convex subset of C, let $f \in A$ and let $\varepsilon > 0$. By the sphere $S(f,\varepsilon)$ we mean as usual the set $\{g \in A : \|g - f\| < \varepsilon\}$. We mention that there exist nowhere dense perfect subsets of R_1 which are non σ-porous. It sould be clear that such sets can be chosen of arbitrarily small diameter. (See Tkadlec [5] for a general method of constructing such sets.) .

Throughout this paper R_n will denote Euclidean n-space, I_0 will denote the unit interval [0,1], and C will be $C[0,1]$.

3. Porosity and local compactness.

We turn now to a development of our main results. We first show that if some closed sphere in a convex subset of a separable Banach space is compact, then all such closed spheres are. We begin with a lemma that will be useful in the proof of this result.

Lemma 1. Let A and B be subsets of C with B compact, $f \in A$, $g \in B$, $0 < \alpha < 1$, $h_\alpha = \alpha f + (1-\alpha)g$, $x,y \in I_0$ and ω a modulus of continuity for B. Then

$$|h_\alpha(y) - h_\alpha(x)| \geq \alpha|f(y) - f(x)| - \omega(|y-x|) \tag{1}$$

and

$$\|h_\alpha - g\| = \alpha\|f-g\|. \tag{2}$$

Proof. We have

$$\begin{aligned}
&|h_\alpha(y) - h_\alpha(x)| \\
&= |\alpha f(y) + (1-\alpha)g(y) - \alpha f(x) - (1-\alpha)g(x)| \\
&= |\alpha[f(y) - f(x)] + (1-\alpha)[g(y) - g(x)]| \\
&\geq \alpha|f(y) - f(x)| - (1-\alpha)\omega(|y-x|) \\
&\geq \alpha|f(y) - f(x)| - \omega(|y-x|)
\end{aligned}$$

establishing (1).

Furthermore, $|h_\alpha(x) - g(x)| = |\alpha f(x) - \alpha g(x)| = \alpha|f(x) - g(x)|$ for all x, i.e. $\|h_\alpha - g\| = \alpha\|f-g\|$ which is (2).

Proposition 1. Let A be a closed convex subset of a separable Banach space X. If A contains a compact sphere, then all closed spheres in A are compact.

Proof. By the Banach-Mazur Theorem quoted in section 2, we may assume without loss of generality that $X = C$. Thus a sphere $S(f,\varepsilon)$ is a set of the form: $\{g \in A : \|g - f\| < \varepsilon\}$ where $f \in A$. Suppose that $\overline{S}$ is not compact. Let S_1 be any sphere in A. We show $\overline{S_1}$ is not compact. Let g be the center of S_1, and N be so large that $S(g,N) \supset \overline{S}$. Let $\alpha = \frac{\varepsilon}{2N}$ where ε is the radius of S_1. It follows from lemma 1 (2) that $h_\alpha = \alpha k + (1-\alpha)g \in S_1$ for all $k \in S$. Suppose $\overline{S_1}$ is compact. Let ω be a modulus of continuity for $\overline{S_1}$. From lemma 1 (1) we see that for $f \in S$ and $h_\alpha = \alpha f + (1-\alpha)g$,

$$|h_\alpha(y) - h_\alpha(x)| \geq \alpha|f(y) - f(x)| - \omega(|y-x|).$$

But $\overline{S}$ is bounded and not compact. Therefore S is not equicontinuous. Thus there exists $\beta > 0$ such that to every $\delta > 0$ there corresponds an $f \in S$ and $x,y \in I_0$, $|x-y| < \delta$, with $|f(x) - f(y)| \geq \beta$. By choosing δ so that $\omega(\delta) \leq \frac{\alpha\beta}{6}$, we see that for some $x,y \in I_0$ with $|x-y| < \delta$, we have

$$|h_\alpha(y) - h_\alpha(x)| \geq \alpha\beta - \frac{\alpha\beta}{6} = \frac{5}{6}\alpha\beta > \frac{1}{6}\alpha\beta \geq \omega(\delta).$$

Since $h_\alpha \in S_1$, this violates the assumption that ω is a modulus of continuity for S_1.

We have shown that if A has a non-compact closed sphere, then all closed spheres in A are non-compact.

Theorem 1 below establishes our first connection between local compactness and porosity.

Theorem 1. Let X be a separable Banach space and A a closed non-locally compact convex subset of X, with more than one element. Let B be any compact subset of A. Then B is totally porous with respect to A.

Proof. In view of the Banach-Mazur Theorem, we may assume $X = C$. Let $g \in B$ and we show B is totally porous at g. Let S be a sphere having g in its boundary and let u be the center of S. Without loss of generality, assume S has diameter less than 1. Let $\varepsilon > 0$. By Proposition 1, $\overline{S}$ is not compact. Using the notation of Proposition 1, we let $\beta > 0$ be the constant used in the proof of Proposition 1. We construct spheres S_1 and S_2 meeting the requirements of porosity of B with respect to S.

From lemma 1 (2) we see that a suitable convex combination of u with g gives rise to a function $v \in S$ such that $||v-g|| < \frac{\varepsilon}{2}$. One easily verifies that the sphere $S_2 = S(v,||v-g||)$ is contained in S, g is in its boundary, and diameter $S_2 < \varepsilon$. (Note v is on the "line segment" determined by u and g.)

We now find a sphere $S_1 \subset S_2$ such that $S_1 \cap B = \emptyset$, and diameter $S_1 > \frac{\beta}{12}$ (diameter S_2). Let ω be a modulus of continuity for B. As in the proof of Proposition 1 and using its notation, we find that for $\alpha = \frac{||v-g||}{2}$ and $\delta = \omega^{-1}\left(\frac{\alpha\beta}{6}\right)$, there exist points $x,y \in I_0$ and $h \in S_2$ such that $|x-y| < \delta$, $|h(x) - h(y)| > 5\omega(\delta)$ and $||h-v|| < \alpha$.

Let $S_1 = S\big(h,\omega(\delta)\big)$. Then $S_1 \subset S_2$ since $\omega(\delta) = \frac{\alpha\beta}{6} < \alpha$. To see that $S_1 \cap B = \emptyset$, we need only observe that if $f \in A \cap S_1$, then $|f(x) - f(y)| > 2\omega(\delta)$, so $f \notin B$. Finally, diameter $S_1 = \frac{\alpha\beta}{3}$, diameter $S_2 = 2||v-g|| = 4\alpha$. Thus, since $||v-g|| < \frac{\varepsilon}{2}$, $\varepsilon >$ diameter S_1 $= \frac{\beta}{12}$(diameter S_2). Since β depended only on S (and not on S_1 or S_2), we have shown that B is porous at g with respect to S. Since S is an arbitrary sphere containing g in its boundary, B is totally porous.

It is now natural to ask whether all nowhere dense subsets of A are totally porous if A is a convex non-locally compact subset of a separable Banach space X. If so, then all first category convex subsets of X will be σ-porous. In particular, subspaces of $C[a,b]$ (such as the space of functions that satisfy a Lipschitz condition) that are first category would automatically be σ-porous and σ-porosity would give no additional information about the size of a first category set. That is not the case, however, as Theorem 2, below, shows.

The proof of Theorem 2 involves a step that needs explanation. In the setting of that theorem we must show that under certain conditions convex sets of functions from a set A intersect coordinate lines in full intervals. Without those conditions this property is not in general true. Consider for a moment the set $A \subset C$ consisting of those functions f of the form $f(x) = ax$, $-1 \leq a \leq 1$. Let $0 < \delta < 1$, and consider $V = \{f \in A : \|f\| < \delta\}$. This neighborhood of the zero function relative to the set A has radius δ, yet for $y \neq 0$, there is no $v \in V$ for which $v(0) = y$. A less extreme case would be one in which there is $x_0 \in I$, $\delta > 0$ and $h \in A$ such that the projection of $S(h,\delta)$ onto the x_0 coordinate did not fill up the interval centered at $h(x_0)$ and having length 2δ. To deal with the difficulty this type of situation may create in the proof of Theorem 2, we begin with a simple calculation.

Lemma 2. Let A be a convex subset of C, $\{v,h,g\} \subset A$, $\delta > 0$, $\varepsilon > 0$. Suppose $x_0 \in I_0$, $h(x_0) + \varepsilon < g(x_0)$ and $S(v,\delta) \subset S(h,\varepsilon)$ and let

$$\gamma_1 = \frac{\|g-h\| + \varepsilon}{g(x_0) - h(x_0) - \varepsilon}.$$

Then for each $y \in [v(x_0), v(x_0) + \frac{\delta}{\gamma_1}]$ there exists a function $w \in S(v,\delta) \cap A$ with $w(x_0) = y$.

If the function g above is replaced by $f \in A$ with $f(x_0) < h(x_0) - \varepsilon$, then there is a constant γ_2 such that for each $y \in [v(x_0) - \frac{\delta}{\gamma_2}, v(x_0)]$, there exists $w \in S(v,\delta) \cap A$ with $w(x_0) = y$.

Proof. Since $g(x_0) > h(x_0) + \varepsilon > v(x_0) + \delta$, we have $\|g-v\| > \delta$ and so we may choose α such that $0 < \alpha < 1$ and $\|\alpha v + (1-\alpha)g - v\| = \delta$.

Now

$$\frac{g(x_0) - v(x_0)}{g(x_0) - h(x_0) - \varepsilon} \geq 1 \geq \frac{\|g-v\|}{\|g-h\| + \varepsilon}.$$

Thus

$$g(x_0) - v(x_0) \geq \frac{\|g-v\|}{\gamma_1}.$$

So

$$(1-\alpha)[g(x_0) - v(x_0)] \geq \frac{(1-\alpha)\|g-v\|}{\gamma_1}.$$

But

$$(1-\alpha)[g(x_0) - v(x_0)] = \alpha v(x_0) + (1-\alpha)g(x_0) - v(x_0)$$

and

$$(1-\alpha)\|g-v\| = \delta$$

so we get

$$\alpha v(x_0) + (1-\alpha)g(x_0) \geq v(x_0) + \frac{\delta}{\gamma_1} .$$

The result follows by taking convex combinations of v and $\alpha v + (1-\alpha)g$.

The existence of γ_2 is obtained by an analogous argument.

Theorem 2. Let X be a separable Banach space and $A \subset X$ be convex with more than one element. If $S \subset A$ is a sphere, then there exists $B \subset S$ with B closed nowhere dense and non σ-porous in A.

Proof. Again because of the Banach-Mazur Theorem of section 2, we may assume $X = C$. Let f and g be in S with $f \neq g$ and without loss of generality let x_0 be a point where $f(x_0) < g(x_0)$. By a theorem of Tkadlec [5] there is a set $E \subset (f(x_0), g(x_0))$ with E closed, nowhere dense and non σ-porous. Let $B = \{s \in S : s(x_0) \in E\}$. Then B is nowhere dense and closed in A. Now suppose $B = \bigcup_{k=1}^{\infty} B_k$. We must show some B_k is non-porous at some point $h \in B_k$.

Let $E_k = \{y \in E : s(x_0) = y$ for some $s \in B_k\}$. Then $E = \bigcup_{k=1}^{\infty} E_k$ since S is convex. Thus there exists k such that E_k is non-porous at some point $z \in E_k$. By a suitable convex combination of f and g we obtain a function $h \in S$ with $h(x_0) = z$. We will show B_k is non-porous at h. Choose ε so that

$$f(x_0) < h(x_0) - \varepsilon < h(x_0) + \varepsilon < g(x_0).$$

Let γ_1 and γ_2 be as in Lemma 2 and set $\gamma = \max\{\gamma_1,\gamma_2\}$. Suppose $\eta > 0$ is given. Since E_k is non-porous at $h(x_0)$, we can find ε_1 such that $0 < \varepsilon_1 < \varepsilon$ and $I \cap E_k \neq \emptyset$ for all intervals $I \subset (h(x_0) - \varepsilon_1,\ h(x_0) + \varepsilon_1)$ for which $|I| \geq \eta\varepsilon_1$. Now suppose $M \subset A$ is a sphere with h in the boundary of M and diam $M < \varepsilon_1$ and $v \in M$. If $S(v,\delta) \subset M$, then

$$(v(x_0) - \delta,\ v(x_0) + \delta) \subset (h(x) - \varepsilon_1,\ h(x) + \varepsilon_1).$$

So if $\delta \geq \gamma\eta\varepsilon_1$, then $\frac{\delta}{\gamma} \geq \eta\varepsilon_1$ so there exists $y \in E_k \cap (v(x_0) - \frac{\delta}{\gamma},\ v(x_0) + \frac{\delta}{\gamma})$. But then by lemma 2, there exists $w \in A \cap S(v,\delta)$ with $w(x_0) = y$. So $w \in B_k$ and $S(v,\delta) \cap B_k \neq \emptyset$. Thus we see that if S is a sphere, $S \subset M$ and diam $S \geq 2\gamma\eta\varepsilon_1$, then $S \cap B_k \neq \emptyset$. Theorem 2 now follows from the observation that M is an arbitrary sphere having h in its boundary and of diameter $< \varepsilon_1$ and that γ depends on f, g, h and ε, but not on ε_1.

Corollary 1. Let X be a separable Banach space and $A \subset X$ be locally compact, convex with more than one element. Then every closed sphere in A contains a compact nowhere dense subset B that is non σ-porous in A.

Proof. Since A is locally compact, the closure of each sphere S is compact by Proposition 1. The set B in Theorem 2 is a closed subset of S and is therefore compact.

We summarize the results of this section with Theorem 3. Its proof entails no more than piecing together Proposition 1, Theorem 1 and Corollary 1.

Theorem 3. Let X be a separable Banach space. Let A be a convex subset of X having more than one element. There exists a nonporous compact subset B of A if and only if A is locally compact. When A is locally compact, we can choose B nowhere dense and non σ-porous. When A is not locally compact, every compact subset of A is totally porous.

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