Real Analysis Exchange Vol. 10 (1984-85)

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Topologically Equivalent Measures in the Cantor Space II

1. Introduction

Consider pairs (X, μ) where X is a topological space and μ is a Borel measure in X. Two pairs (X, μ) and (Y, ν) are said to be isomorphic if there is a homeomorphism h from X onto Y such that $\mu(B) = \nu(h(B))$ for every Borel set B of X. If, in addition, X and Y are the same space, then μ and ν are said to be topologically equivalent measures in the space X. In [1] I began the study of such equivalences in the particular case that X is the Cantor space, and μ and ν are shift invariant product measures. In this article I propose to extend the study to other types of measure. However, in order to utilize an easily established result of strong geometrical flavor, the measures to be considered are not quite that different: they are of the form μf where f is a homeomorphism from the Cantor space to some product space and μ is still a shift invariant product measure in this other space.

2. C-pairs

By a C-pair is meant a pair (X, μ) where X is a space of the form

$$X = \prod_{n=1}^{\infty} S_n$$

with the product topology. Each factor S_n is finite and carries the discrete topology. μ is a Borel measure in X.

It is well known, of course, that whenever (X, μ) is a C-pair then X is homeomorphic to the Cantor space of infinite sequences of zeros and ones, and its topology is compatible with the metric d which, for any two points $x = (x_n)$ and $x' = (x_n')$, is given by the formula

$$d(x, x') = \sum_{n=1}^{\infty} (1/2)^n d_n(x_n, x_n').$$

By d_n is meant the metric on S_n which takes only the values 0 or 1. Furthermore, it can readily be seen that a countable basis for the topology of X consists of sets of the form

$$\langle i_1, i_2, \ldots, i_m \rangle = \{(x_n) \in X: x_j = i_j \text{ for } j = 1, 2, \ldots, m\}.$$

These sets are both open and closed and will be referred to as the special closed-open sets of X. They are obtained by fixing the first coordinate, the second coordinate, and so on, up to a finite number of coordinates.

Definition

Let t be an integer, $t \geq 2$, and let $p_1, p_2, \ldots, p_t$ be non-negative real numbers such that $p_1+p_2+\ldots+p_t = 1$. A C-pair (X, μ) is said to be of type $(t; p_1, p_2, \ldots, p_t)$ if the following two conditions hold:

(i) For each n, S_n is a set consisting of t elements which, for convenience, may be taken to be the integers 1, 2, ... , t.

(ii) μ is a shift invariant product measure $\mu = \prod_{n=1}^{\infty} \mu_n$ with $\mu_n(j) = p_j$ for all n and j.

Let (X, μ) be a C-pair of type $(t; p_1, p_2, \ldots, p_t)$. X can be expressed as a disjoint union

$$X = \bigcup_{j=1}^{t} \langle j \rangle,$$

where $\mu(\langle j \rangle) = p_j$ and $\text{diam}(\langle j \rangle) = 1/2$. Each one of the special closed-open sets $\langle i_1 \rangle$ can in turn be expressed as a disjoint union

$$\langle i_1 \rangle = \bigcup_{j=1}^{t} \langle i_1, j \rangle,$$

where $\mu(\langle i_1, j \rangle) = p_{i_1} p_j$ and $\text{diam}(\langle i_1, j \rangle) = 1/4$. By continuing this process a sequence of covers $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \ldots$ of X can be constructed with the following properties:

(2.1) $$\mathcal{U}_n = \{U_{i_1 i_2 \ldots i_n} : 1 \leqslant i_j \leqslant t, \quad 1 \leqslant j \leqslant n\}.$$

The members of $\mathcal{U}_n$ are mutually disjoint non-empty closed-open sets of diameter less than $1/n$.

(2.2) For fixed $i_1, i_2, \ldots, i_n$,

$$U_{i_1 i_2 \ldots i_n} = \bigcup_{j=1}^{t} \{U_{i_1 i_2 \ldots i_n j} \in \mathcal{U}_{n+1}\}.$$

(2.3) $$\mu(U_{i_1 i_2 \ldots i_n}) = p_{i_1} p_{i_2} \ldots p_{i_n}.$$

<u>Theorem 2.1</u>

Let X be a complete metric space and μ a Borel measure in X. If there exist an integer $t \geqslant 2$, non-negative real numbers $p_1, p_2, \ldots, p_t$ with $p_1+p_2+\ldots+p_t=1$, and a sequence of covers of X with properties (2.1), (2.2) and (2.3), then (X, μ) is isomorphic to a C-pair of type $(t; p_1, p_2, \ldots, p_t)$.

Proof

Put $Y = \prod_{n=1}^{\infty} S_n$, where $S_n = \{1, 2, \ldots, t\}$, and let $\nu = \prod_{n=1}^{\infty} \nu_n$ be the product measure in Y given by $\nu_n(j) = p_j$ for all n and j. Since X is complete and $\lim_{n\to\infty}(\text{diam } U_{i_1 i_2 \ldots i_n}) = 0$, for every $y = (i_1, i_2, i_3, \ldots)$ in Y the intersection $U_{i_1} \cap U_{i_1 i_2} \cap U_{i_1 i_2 i_3} \cap \ldots$ consists of a single point x_y of X. Thus, a function h from Y to X can be properly defined by setting $h(y) = x_y$ for each y in Y. h is an onto function on account of the fact that the families $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \ldots$ are covers of X; and it is one-one because each cover $\mathcal{U}_n$ consists of disjoint sets. Clearly, $h(\langle i_1, i_2, \ldots, i_n\rangle) = U_{i_1 i_2 \ldots i_n}$ and $h^{-1}(U_{i_1 i_2 \ldots i_n}) = \langle i_1, i_2, \ldots, i_n\rangle$. Since the family of all sets $U_{i_1 i_2 \ldots i_n}$ is a basis for the topology of X, and the family of special closed-open sets is a basis for the topology of Y, both functions h and h^{-1} are continuous. Finally, for $U = \langle i_1, i_2, \ldots, i_n\rangle$, it is the case that $\nu(U) = p_{i_1} p_{i_2} \ldots p_{i_n} = \mu(U_{i_1 i_2 \ldots i_n}) = \mu(h(U))$. But every open set of Y is a disjoint union of countably many special closed-open sets. Hence, the equation $\nu(U) = \mu(h(U))$ holds for every open set U of Y, and, consequently, for every Borel set as well.

The purely topological content of Theorem 2.1, namely, that every compact metric space is the continuous image of the Cantor space, is a well known result. Theorem 2.1 was inspired by the proof that A. H. Schoenfeld [2] has supplied for this well known result.

It is obvious to observe that the converse statement of Theorem 2.1 is nearly true. Indeed, suppose that (X, μ) is isomorphic to a C-pair (Y, ν) through a homeomorphism h form Y to X. Since Y has a sequence of covers with properties (2.1), (2.2) and (2.3), h carries these covers to covers of X with the same properties, except that there is no guarantee that a set of diameter less than 1/n in Y is carried by h to a set of diameter also less than 1/n in X.

3. Consequences of Theorem 2.1

A C-pair of a given type may well be isomorphic to a C-pair of a different type. In particular instances this can be established by appealing to Theorem 2.1. For example, the pair $(X = \prod_{n=1}^{\infty} \{1, 2\}_n, \mu)$ of type (2; 1/2, 1/2) and the pair $(Y = \prod_{n=1}^{\infty} \{1, 2, 3\}_n, \nu)$ of type (3; 1/2, 1/4, 1/4) are isomorphic. To prove this a sequence of covers $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3, \ldots$ of X will be constructed satisfying properties (2.1), (2.2) and (2.3) with $t = 3$, $p_1 = 1/2$, $p_2 = p_3 = 1/4$. Let the first cover $\mathcal{U}_1$ consist of the special closed-open sets $U_1 = \langle 1\rangle$, $U_2 = \langle 2, 1\rangle$ and $U_3 = \langle 2, 2\rangle$. These sets are mutually disjoint, of diameter less than 1, and $\mu(U_1) = 1/2$, $\mu(U_2) = \mu(U_3) = 1/4$. In general, if covers $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n$ have been constructed satisfying properties (2.1), (2.2) and (2.3), then the cover $\mathcal{U}_{n+1}$ is constructed as follows. Let $U_{i_1 i_2 \ldots i_n} = \langle j_1, j_2, \ldots, j_s\rangle$ be any of the special closed-open sets in $\mathcal{U}_n$. Put $U_{i_1 i_2 \ldots i_n 1} = \langle j_1, j_2, \ldots, j_s, 1\rangle$, $U_{i_1 i_2 \ldots i_n 2} = \langle j_1, j_2, \ldots, j_s, 2, 1\rangle$ and $U_{i_1 i_2 \ldots i_s 3} = \langle j_1, j_2, \ldots, j_s, 2, 2\rangle$. The cover $\mathcal{U}_{n+1}$ is defined to consist of the special closed-open sets $U_{i_1 i_2 \ldots i_{n+1}}$. These sets are clearly disjoint and of diameter less than 1/n+1. Property (2.1) is satisfied, and the same is easily seen to be true of properties (2.2) and (2.3). The function h from Y to X defined, as in the proof of Theorem 2.1, using the covers $\mathcal{U}_1, \mathcal{U}_2, \ldots$ establishes an isomorphism between (X, μ) and (Y, ν). In fact, it is even possible to describe the action of h on the points $(i_1, i_2, \ldots)$ of Y. Put f(1) = 1; f(2) = 2, 1; f(3) = 2, 2. Then

$h(i_1, i_2, \ldots) = (f(i_1), f(i_2), \ldots)$.

Theorem 3.1

Let (X, μ) be a C-pair of type $(s; q_1, q_2, \ldots, q_s)$. In order for (X, μ) to be isomorphic to a C-pair of type $(t; p_1, p_2, \ldots, p_t)$ it is sufficient that there exist disjoint special closed-open sets U_1, $U_2, \ldots, U_t$ in X such that $X = U_1 \cup U_2 \cup \ldots \cup U_t$ and $\mu(U_j) = p_j$ for all j.

Proof

Write $U_j = \langle j_1, j_2, \ldots, j_{n(j)} \rangle$ for $j = 1, 2, \ldots, t$. Then $p_j = \mu(U_j) = q_{j_1} q_{j_2} \ldots q_{j_{n(j)}}$. Let $\mathcal{U}_1$ be the cover of X consisting of the special closed-open sets $U_1, U_2, \ldots, U_t$. Suppose that covers $\mathcal{U}_1, \mathcal{U}_2, \ldots, \mathcal{U}_n$ of X have been constructed satisfying properties (2.1), (2.2) and (2.3), each cover consisting of special closed-open sets. Let $U_{i_1 i_2 \ldots i_n}$ be any member of $\mathcal{U}_n$, say $U_{i_1 i_2 \ldots i_n} = \langle k_1, k_2, \ldots, k_r \rangle$. For fixed j, $1 \leqslant j \leqslant t$, put $U_{i_1 i_2 \ldots i_n j} = \langle k_1, k_2, \ldots, k_r, j_1, j_2, \ldots, j_{n(j)} \rangle$, and define the cover $\mathcal{U}_{n+1}$ to consist of the sets $U_{i_1 i_2 \ldots i_{n+1}}$. These sets are disjoint special closed-open sets of diameter less than $1/n+1$. Properties (2.1) and (2.2) are satisfied, and since $\mu(U_{i_1 i_2 \ldots i_n j}) = q_{k_1} q_{k_2} \ldots q_{k_r} q_{j_1} q_{j_2} \ldots q_{j_{n(j)}} = \mu(U_{i_1 i_2 \ldots i_n}) p_j = p_{i_1} p_{i_2} \ldots p_{i_n} p_j$, property (2.3) is satisfied as well. Thus, it is possible to construct a sequence of covers of X with the requirements of Theorem 2.1. Hence, (X, μ) is isomorphic to a C-pair of type $(t; p_1, p_2, \ldots, p_t)$.

To illustrate theorem 3.1 consider a C-pair (X, μ) of type $(2; 1/3, 2/3)$. Put $U_1 = \langle 1 \rangle$, $U_2 = \langle 2, 1 \rangle$, $U_3 = \langle 2, 2 \rangle$. X is the disjoint union of these special sets and since $\mu(U_1) = 1/3$, $\mu(U_2) = 2/9$, $\mu(U_3) = 4/9$, (X, μ) is isomorphic to a C-pair of type (3; 1/3, 2/9, 4/9). On the other hand, put $V_1 = \langle 1, 1\rangle$, $V_2 = \langle 1, 2\rangle$, $V_3 = \langle 2\rangle$. This time $\mu(V_1) = 1/9$, $\mu(V_2) = 2/9$, $\mu(V_3) = 2/3$. Thus, (X, μ) is also isomorphic to a C-pair of type (3; 1/9, 2/9, 2/3). Therefore, two C-pairs of types (3; 1/3, 2/9, 4/9) and (3; 1/9, 2/9, 2/3) are always isomorphic.

Note that the condition in Theorem 3.1 that the closed-open sets U_j be special cannot be dropped. Indeed, let (X, μ) be a C-pair of type (4; 1/4, 1/4, 1/4, 1/4). Put $U_1 = \langle 1\rangle$ and $U_2 = \langle 2\rangle \cup \langle 3\rangle \cup \langle 4\rangle$. Then X is the disjoint union of U_1 and U_2, and $\mu(U_1) = 1/4$, $\mu(U_2) = 3/4$. However, (X, μ) cannot be isomorphic to a C-pair of type (2; 1/4, 3/4), for, indeed, by an easy application of Theorem 3.1, (X, μ) can be seen to be isomorphic to a C-pair of type (2; 1/2, 1/2). But, by Theorem 3.3 of [1], two C-pairs of types (2; 1/4, 3/4) and (2; 1/2, 1/2) cannot be isomorphic, and this in spite of the fact that both measures take values on closed-open sets which are dyadic rationals.

Theorem 3.2

If the requirement that the closed-open sets U_j be special is dropped, then the condition for a C-pair to be isomorphic to a C-pair of a given type as stated in Theorem 3.1 is necessary.

Proof

Let $h: X \to Y$ establish an isomorphism between (X, μ) and a C-pair (Y, ν) of type $(t; p_1, p_2, \ldots, p_t)$. Put $V_j = \langle j\rangle$ for $j = 1, 2, \ldots, t$. These are mutually disjoint closed-open sets of Y such that $Y = V_1 \cup V_2 \cup \ldots \cup V_t$ and $\nu(V_j) = p_j$. The sets $U_j = h^{-1}(V_j)$ are mutually disjoint closed-open sets of X such that $X = U_1 \cup U_2 \cup \ldots \cup U_t$ and $\mu(U_j) = \nu(V_j) = p_j$. Observe that while the sets V_j are special, the same is not necessarily true of the sets U_j.

Let (X, μ) and (Y, ν) be two C-pairs. Denote by C the Cantor space of infinite sequences of zeros and ones, and let f and g be homeomorphisms from C to X and Y, respectively. Put $\mu_1(B) = \mu(f(B))$ and $\nu_1(B) = \nu(g(B))$ for every Borel set B of C. Clearly, μ_1 and ν_1 are Borel measures which are topologically equivalent in C if and only if the C-pairs (X, μ) and (Y, ν) are isomorphic. Thus, the results of Theorem 3.1 and Theorem 3.2 can be carried to measures in C of the form μf where f is a homeomorphism from C to some product space X and μ is a shift invariant product measure in X.

The author is grateful to Professor Robert Zink for his comments and suggestions .

References

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Received May 21, 1984