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Jaromír Uher, Leninova 56, 160 00 Praha 6, Czechoslovakia

Symmetrically Differentiable Functions are Differentiable Almost Everywhere

In this note we show that any function f defined on the real line R and symmetrically semicontinuous at almost every point of a measurable set $E \subset R$ is differentiable at almost every point of E at which it possesses a symmetric derivative, possibly infinite. Since the existence of the symmetric derivative at a point implies symmetric semicontinuity at that point, we get as a corollary that a function possessing a symmetric derivative almost everywhere is measurable. This solves a well-known problem, which was, according to [1], posed by W. Sierpiński in 1928.[1)]

Recall that the upper symmetric derivative of a function f at $x \in R$ is $\overline{f}^s(x) = \limsup_{h \to 0} \frac{f(x+h) - f(x-h)}{2h}$ and that the lower symmetric derivative is

$$\underline{f}^s(x) = \liminf_{h \to 0} \frac{f(x+h) - f(x-h)}{2h} .$$

If $\overline{f}^s(x) = \underline{f}^s(x)$, the common value, finite or infinite, is called the symmetric derivative of f at x. A function f defined on R is said to be upper (lower) symmetrically

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1) Editorial Note: The paper referred to is, W. Sierpiński, <u>Sur une hypothèse de M. Mazurkiewicz</u>, Fund. Math., 11(1928), 148-150. The question asked there is a weaker form of the measurability question which has been answered by Szpilrajn and Preiss.

semicontinuous at x, if

$$\limsup_{h\to 0_+} f(x+h) - f(x-h) \le 0$$

$$(\liminf_{h\to 0_+} f(x+h) - f(x-h) \ge 0)\ ,$$

and it is said to be symmetrically semicontinuous at x, if it is upper or lower symmetrically semicontinuous at x. We shall also use the usual notations $\overline{D}f(x)$, $\underline{D}f(x)$, $\overline{D}^+f(x)$, $\overline{D}^-f(x)$, $\underline{D}^+f(x)$, and $\underline{D}^-f(x)$ for ordinary, one-sided, upper, and lower derivatives. If M is a subset of the real line, we denote by $|M|$ its outer Lebesgue measure,

$$D(M) = \left\{x \in R\ ;\ \lim_{h\to 0_+} \frac{|M\cap(x-h,x+h)|}{2h} = 1\right\}\ ,$$

$$D_+(M) = \left\{x \in R\ ;\ \lim_{h\to 0_+} \frac{|M\cap(x,x+h)|}{h} = 1\right\}\ ,\ \text{and}$$

$$D_-(M) = \left\{x \in R\ ;\ \lim_{h\to 0_+} \frac{|M\cap(x-h,x)|}{h} = 1\right\}\ .$$

We remark that even if M is not measurable, $D(M)$, $D_+(M)$ and $D_-(M)$ are measurable.

The results mentioned in the beginning will be easy consequences of the following lemma.

Lemma. Assume that f is a real-valued function defined on the real line, E is a measurable subset of the real line and $K \in R$ such that (i) $E \subset D(E)$,

(ii) $E \subset D\{z \in R\ ;\ \underline{f}^s(z) > K\}$, and

(iii) f is symmetrically semicontinuous at each point of E.

Then $\underline{D}f(z) \geq K$ at almost every point $z \in E$.

__Proof.__ Let

$$F = \{z \in R \; ; \; \underline{f}^s(z) > K\}$$

and

$$F_n = \left\{ z \in F \; ; \; |h| < \frac{1}{n} \Rightarrow \frac{f(z+h) - f(z-h)}{2h} > K \right\} .$$

Clearly $F = \bigcup_{n=1}^{\infty} F_n$ and $|D(F) - \bigcup_{n=1}^{\infty} D(F_n)| = 0$.

The main part of the proof of the lemma will be accomplished by showing that $\underline{D}^+ f(z) \geq K$ whenever $z \in D(F_n) \cap E$ for some natural number n. To prove this assertion, we assume $z = 0$ and we choose $\Delta \in (0, \frac{1}{n})$ such that for every $a \in (0, \Delta)$

$$\frac{|(-a, a) \cap F_n|}{2a} > (1 - \frac{1}{64}) \quad \text{and} \quad \frac{|(-a, a) \cap E|}{2a} > (1 - \frac{1}{64}) .$$

First we prove that

(*) whenever $x \in (0, \Delta)$, one may find a measurable set $B \subset (\frac{3}{4}x, x)$ with $|B| > \frac{3}{16}x$ and with $f(y) - f(0) \geq Ky$ for each $y \in B$.

__Proof of (*).__ Let $x \in (0, \Delta)$. We put $A = (F_n + \frac{3}{4}x) \cap E \cap (\frac{3}{4}x, x)$ and note that $|A| > \frac{x}{4} - \frac{5}{128}x$.

Let

$E^+ = \{z \in E \; ; \; f$ is lower symmetrically semicontinuous at $z\}$,

$E^- = \{z \in E \; ; \; f$ is upper symmetrically semicontinuous at $z\}$,

$A^+_{m,k} = \{z \in A \cap E^+ ; \; 0 < h < \frac{1}{m} \Rightarrow f(z+h) - f(z-h) > -\frac{1}{k}\}$, and

$A^-_{m,k} = \{z \in A \cap E^- ; \; 0 < h < \frac{1}{m} \Rightarrow f(z-h) - f(z+h) > -\frac{1}{k}\}$.

Let

$$A^* = \bigcap_{k=1}^{\infty} [\bigcup_{m=1}^{\infty} (D(A^+_{m,k}) \cup D(A^-_{m,k}))].$$

We put

$$B = A^* \cap [\tfrac{2}{3}D(F_n) + \tfrac{x}{2}] \cap [\tfrac{1}{2}E + \tfrac{3}{4}x] \cap (\tfrac{3}{4}x,\ x) \ .$$

Since A^* is a measurable subset of $(\frac{3}{4}x,\ x)$ with

$$|A^*| = |A| > \frac{x}{4} - \frac{5}{128}x \ ,$$

since

$$|(\tfrac{2}{3}D(F_n) + \tfrac{x}{2}) \cap (\tfrac{3}{4}x,\ x)| > \frac{x}{4} - \frac{1}{64}x \ ,$$

and since

$$|(\tfrac{1}{2}E + \tfrac{3}{4}x) \cap (\tfrac{3}{4}x,\ x)| > \frac{x}{4} - \frac{1}{128}x \ ,$$

$$|B| > \frac{x}{4} - \frac{1}{16}x \ .$$

To prove the last part of the statement (*) let $y \in B$. Then

(1) $2(y - \frac{3}{4}x) + \frac{3}{4}x - \frac{y}{2} = \frac{3}{2}(y - \frac{x}{2}) \in D(F_n)$, since $B \subset \frac{2}{3}D(F_n) + \frac{x}{2}$

and

(2) $2(y - \frac{3}{4}x) \in E$, since $B \subset \frac{1}{2}E + \frac{3}{4}x$.

Let

$$C^1 = (F_n - \tfrac{3}{4}x + \tfrac{y}{2}) \cap E \cap (0,\ \tfrac{x}{2}) \ .$$

Then

(3) $2(y - \frac{3}{4}x) \in D(C^1)$ according to (1) and (2) .

Let $\epsilon \in (0, \frac{1}{16}x)$ be an arbitrary positive number. Let $C^{1+} = C^1 \cap E^+$ and $C^{1-} = C^1 \cap E^-$, and choose, for each $t \in C^{1+}$ (resp. $t \in C^{1-}$), a $\delta^+(t) \in (0, \epsilon)$ (resp. a $\delta^-(t) \in (0, \epsilon)$) such that $f(t + h) - f(t - h) > - \epsilon$ for each $h \in (0, \delta^+(t))$ (resp. $f(t - h) - f(t + h) > - \epsilon$ for each $h \in (0, \delta^-(t))$.

Let

$$C^2 = \bigcup_{t \in C^{1+}} (t-\delta^+(t), t] \cup \bigcup_{t \in C^{1-}} [t, t+\delta^-(t)) .$$

Then (3) and the definition of C^2 imply

(4) $2(y - \frac{3}{4}x) \in D(C^2)$ and C^2 is measurable.

Let k and m be natural numbers such that $\frac{1}{k} < \epsilon$ and $y \in D(A^+_{m,k}) \cup D(A^-_{m,k})$. Choose $\beta \in (0, \min(\frac{1}{m}, \epsilon))$ such that

$$(y - \beta, y + \beta) \subset (\tfrac{3}{4}x, x) .$$

Put

$$C^3 = \left[((y-\beta, y) \cap A^+_{m,k}) \cup ((y, y+\beta) \cap A^-_{m,k}) \right] - \tfrac{3}{4}x$$

and

$$C^4 = (2C^3) \cap C^2 .$$

If $y \in D(A^+_{m,k})$, then $2(y - \frac{3}{4}x) \in D_-(2C^3)$ which, together with (4), implies that one may find a point $\tau \in C^4 \cap 2((y-\beta, y) - \frac{3}{4}x)$.

Since $\mathcal{T} \epsilon 2C^3$, there is $u \epsilon (y-\beta, y) \cap A^+_{m,k}$ such that $\mathcal{T} = 2(u-\frac{3}{4}x)$. Then

(5) $f(y) - f(2u - y) > -\epsilon$ according to the definition of $A^+_{m,k}$, since $u \epsilon A^+_{m,k}$, $\epsilon > \frac{1}{k}$ and $0 < y - u < \beta < \frac{1}{m}$. In addition,

(6) $u - \frac{3}{4}x \epsilon F_n$ according to the definition of A, since $u \epsilon A^+_{m,k} \subset A$.

If $y \epsilon D(A^-_{m,k}) - D(A^+_{m,k})$, then $2(y - \frac{3}{4}x) \epsilon D_+(2C^3)$ which, together with (4), implies that one may find a point $\mathcal{T} \epsilon C^4 \cap 2((y, y+\beta) - \frac{3}{4}x)$. Since $\mathcal{T} \epsilon 2C^3$, there is $u \epsilon (y, y+\beta) \cap A^-_{m,k}$ such that

$$\mathcal{T} = 2(u - \frac{3}{4}x) .$$

Then

(5′) $f(y) - f(2u - y) > -\epsilon$ according to the definition of $A^-_{m,k}$, since $u \epsilon A^-_{m,k}$, $\epsilon > \frac{1}{k}$ and $0 < u - y < \beta < \frac{1}{m}$. In addition,

(6′) $u - \frac{3}{4}x \epsilon F_n$ according to the definition of A, since $u \epsilon A^-_{m,k} \subset A$.

Finally, we use $\mathcal{T} \epsilon C^2$ to choose $t \epsilon C^1$ such that $\mathcal{T} \epsilon [t, t + \delta^-(t)]$ or $\mathcal{T} \epsilon (t - \delta^+(t), t]$. We also note that

(7) $t + \frac{3}{4}x - \frac{y}{2} \epsilon F_n$ according to the definition of C^1.

Now we are ready to estimate

$$f(y) - f(0) = [f(y) - f(2u-y)] + [f(2u-y) - f(2t - 2(u - \frac{3}{4}x))] +$$
$$+ [f(2t - 2(u - \frac{3}{4}x)) - f(2(u - \frac{3}{4}x))] + [f(2(u - \frac{3}{4}x)) - f(0)] .$$

The first term is greater than $-\epsilon$ according to (5) and (5′). To estimate the second term we first note that since $t \in C^1 \subset (0, \frac{x}{2})$ and since $y \in B \subset (\frac{3}{4}x, x)$, $y - t \in (\frac{x}{4}, x)$. Then since $|u-y| < \beta < \epsilon$ and since

$$|t - 2(u - \tfrac{3}{4}x)| = |t - \mathcal{T}| < \epsilon, \quad |(2u - y - 2t + 2(u - \tfrac{3}{4}x)) - (y-t)|$$

$$\leq 2|u-y| + |t - 2(u - \tfrac{3}{4}x)| < 3\epsilon < \frac{x}{4} .$$

It follows that

$$0 < 2u - y - 2t + 2(u - \tfrac{3}{4}x) < \frac{2}{n} .$$

Therefore (7) implies

$$f(2u - y) - f(2t - 2(u - \tfrac{3}{4}x)) > K(2u - y - 2t + 2(u - \tfrac{3}{4}x)) .$$

The third term is not less than $-\epsilon$ since $t \in C^1$ and $2(u - \frac{3}{4}x) = \mathcal{T}$ belongs to $(t - \delta^+(t), t]$ or $[t, t + \delta^-(t))$. To estimate the last term, we use (6) or (6′) to show that $u - \frac{3}{4}x \in F_n$ which, together with $u - \frac{3}{4}x \in (0, \frac{x}{4})$ gives

$$f(2(u - \tfrac{3}{4}x)) - f(0) > K2(u - \tfrac{3}{4}x) .$$

Hence

$$f(y) - f(0) > Ky - 2\epsilon - |K| \cdot |2(t - 2(u - \tfrac{3}{4}x))| - |K| \cdot |2(y - u)| >$$

$$> Ky - (2 + 4|K|) \cdot \epsilon .$$

Since $\epsilon \in (0, \frac{x}{16})$ is arbitrary, $f(y) - f(0) \geq Ky$, which finishes the proof of (*).

Next we prove that $\underline{D}^+ f(0) \geq K$ by showing that $f(x) - f(0) \geq Kx$ for each $x \in (0, \Delta)$.

Let $C^5 = F_n \cap (\frac{7}{8}x, x)$, and $C = 2C^5 - x$.

Then $C \subset (\frac{3}{4}x, x)$ and $|C| = 2|C^5| > \frac{x}{4} - \frac{1}{16}x$.

Let B be a measurable set with the properties described in (*).

Then $|C \cap B| > \frac{x}{4} - \frac{x}{8} = \frac{x}{8} > 0$, hence there is $v \in C^5$ such that $2v - x \in B$.

From (*) we see that $f(2v - x) - f(0) \geq K(2v - x)$.

Since $v \in F_n$ and $v \in (\frac{7}{8}x, x)$, $f(x) - f(2v - x) > K2(x - v)$.

Hence $f(x) - f(0) > Kx$ for $x \in (0, \Delta)$ and thus $\underline{D}^+ f(0) \geq K$.

Therefore $\underline{D}^+ f(z) \geq K$ for each $z \in \bigcup_{n=1}^{\infty} D(F_n) \cap E$, hence $\underline{D}^+ f \geq K$ almost everywhere in E.

Using this statement for the function $\varphi(x) = -f(-x)$, we see that $\underline{D}^+ \varphi \geq K$ almost everywhere in $-E$, hence $\underline{D}^- f \geq K$ almost everywhere in E, which finishes the proof of the lemma.

<u>Theorem 1</u>. Let f be a real-valued function defined on the real line. Then f is differentiable at almost every point of the set

$$D\{x \in R;\ \overline{f}^s(x) < +\infty \text{ or } \underline{f}^s(x) > -\infty\} -$$
$$- D\{x \in R;\ f \text{ is not symmetrically semicontinuous at } x\}.$$

Proof.

Let $E_n^+ = \{x \in R;\ \underline{f}^s(x) > -n\}$, $E_n^- = \{x \in R;\ \overline{f}^s(x) < n\}$ and

$A = \{x \in R;$ f is not symmetrically semicontinuous at $x\}$.

Using the preceding lemma, we see that $\underline{D}f > -\infty$ at almost every point of each of the sets $D(E_n^+) - (A \cup D(A))$, and that $\overline{D}f < +\infty$ at almost every point of each of the sets $D(E_n^-) - (A \cup D(A))$.

From [2], Theorem 3, p. 171, we deduce that f is differentiable at almost every point of

$$\bigcup_{n=1}^{\infty}[D(E_n^+) \cup D(E_n^-)] - (A \cup D(A)),$$

hence it is differentiable at almost every point of the set

$$D(\bigcup_{n=1}^{\infty}(E_n^+ \cup E_n^-)) - D(A) .$$

Using that the existence of the symmetric derivative implies symmetric semicontinuity, we get the following corollaries.

Corollary 1. If a function f has the symmetric derivative at almost every point of a measurable set $E \subset R$, then f is differentiable almost everywhere in E.

Corollary 2. If a function f has the symmetric derivative almost everywhere in R, then f is measurable.

References

[1] L. Larson: On the symmetric derivative, Real Anal. Exchange, 6/1980-81/, 235-241

[2] S. Saks: Théorie de l'intégrale, Warszawa 1933

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