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Variations of Hardy's Inequality

§1. Introduction. Let p and q be real numbers and $u(x)$, $v(x)$ non-negative extended real valued functions defined on $(0,\infty)$. In this paper, we are concerned with inequalities and their reverses of the form

$$(1.1) \qquad \left\{\int_0^\infty [Tf(x)u(x)]^q dx\right\}^{1/q} \le C\left\{\int_0^\infty [f(x)v(x)]^p dx\right\}^{1/p},$$

where C is a constant independent of f, and T is one of the averaging operators P_η, Q_η given by

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$$(1.2)\qquad (P_\eta f)(x) = x^{-\eta}\int_0^x f(t)\,dt; \quad (Q_\eta f)(x) = x^{-\eta}\int_x^\infty f(t)\,dt ,$$

η real, $x \in (o,\infty)$.

For $T = \begin{cases} P_o & \text{if } r>1 \\ Q_o & \text{if } r<1 \end{cases}$ Hardy [4, Theorems 330, 347] has shown that (1.1) holds if $p = q>1$ and $(u,v) = (x^{-r/p}, x^{1-r/p})$. And if in addition $0<p = q<1$ the reverse inequality of (1.1) holds with the same u and v. Using techniques involving the Euler-Lagrange differential equations, Beesack [2] established inequalities of the form (1.1) and their reverses for this T with $p = q$, $-\infty<p<\infty$, $p \neq 0,1$ and certain general pairs of weight functions (u,v). Artola, Talenti [7], Tomaselli [8] and Muckenhoupt [5] characterized the weights (u,v) for which (1.1) holds with $p=q$, $p\geq 1$. Recently, Bradley [3] solved this problem if $1\leq p\leq q<\infty$. It follows trivially from Bradley's result that the following holds:

<u>Theorem A</u>. If $T = P_\eta$, η real and $1\leq p\leq q<\infty$, then (1.1) holds, if and only if

$$(1.3)\qquad A \equiv \sup_{r>o}\left(\int_r^\infty \left(\frac{u(x)}{x^\eta}\right)^q dx\right)^{1/q}\left(\int_o^r v(x)^{-p'}\,dx\right)^{1/p'}$$

is finite. Here and throughout p and p' are related by $1/p+1/p' = 1$.

For the operator Q_η the result holds, if and only if

$$B \equiv \sup_{r>0} \left(\int_0^r \left(\frac{u(x)}{x^\eta} \right)^q dx \right)^{1/q} \left(\int_r^\infty v(x)^{-p'} dx \right)^{1/p'}$$

is finite.

In fact with a little more effort one can show that if $T = P_\eta$, $\eta \geq 0$ then

$$(1.4) \qquad A_1 \equiv \sup_{r>0} r^{-\eta} \left(\int_r^\infty u(x)^q dx \right)^{1/q} \left(\int_0^r v(x)^{-p'} dx \right)^{1/p'}$$

finite is both necessary and sufficient for (1.1) with $1 \leq p \leq q < \infty$. If $\eta < 0$, a strong type estimate (1.1) is not available. However, Andersen and Muckenhoupt [1] proved that (1.4) is necessary and sufficient for the weak type inequality

$$(1.5) \qquad \left\{ \int_{\{x: P_\eta f(x) > y\}} u(x)^q dx \right\}^{1/q} \leq C\, y^{-1} \left\{ \int_0^\infty [f(x) v(x)]^p dx \right\}^{1/p}$$

to hold.

It is the purpose of this paper to establish the appropriate results when $p<1$, $q<1$, $p\neq 0$, $q\neq 0$. As in [3] the main tool is the following well-known (see e.g. [6, Lemma 2.1]) result.

Lemma 1 [6]. Let $f(x)$, $g(x)$, $Z(x)$ be non-negative functions defined on (o,∞), where $Z(x)$ is increasing. If $k\geq 1$, then

$$(1.6)\qquad \int_0^\infty f(x)\left[\int_0^{Z(x)} g(y)\,dy\right]^k dx \leq \left\{\int_0^\infty g(y)\left[\int_{\zeta(y)}^\infty f(x)\,dx\right]^{1/k} dy\right\}^k$$

and

$$(1.7)\qquad \int_0^\infty f(x)\left[\int_{Z(x)}^\infty g(y)\,dy\right]^k dx \leq \left\{\int_0^\infty g(y)\left[\int_0^{\zeta(y)} f(x)\,dx\right]^{1/k} dy\right\}^k$$

where ζ is the inverse function of Z.

Throughout, the function f is assumed non-negative.

§2. Main Results. In the theorems below our weight functions satisfy certain conditions akin to A, B, and A_1. Let η be real and let

$$(2.1) \qquad K_\eta(r) = r^{-\eta} \left(\int_0^r u(t)^q dt \right)^{1/q} \left(\int_0^r v(t)^{-p'} dt \right)^{1/p'} .$$

Assume that $\inf_{r>0} K_\eta(r) = K_\eta > 0$.

Similarly write

$$(2.2) \qquad J_\eta(r) = r^{-\eta} \left(\int_r^\infty u(t)^q dt \right)^{1/q} \left(\int_r^\infty v(t)^{-p'} dt \right)^{1/p'} ,$$

and assume that $\inf_{r>0} J_\eta(r) = J_\eta > 0$.

<u>Theorem 1</u>. Let $p \le q \le 0$.

i) If $\eta \ge 0$ and (u,v) satisfies (2.1) with $K_\eta(r)$ non-decreasing, then

$$(2.3) \qquad \left\{ \int_0^\infty [f(t)v(t)]^p dt \right\}^{1/p} \le C \left\{ \int_0^\infty [(P_\eta f)(x)u(x)]^q dx \right\}^{1/q} .$$

ii) If $\eta \le 0$ and (u,v) satisfies (2.2) with $J_\eta(r)$ non-increasing, then (2.3) holds with P_η replaced by Q_η .

Proof. i) Let

$$h(t) = \left[\int_0^t v(s)^{-p'} ds\right]^{1/(pp')} .$$

Then by Hölder's inequality ([4, Thm. 189])

$$(P_\eta f)(x)^q \le x^{-\eta q} H(x)^q \left[\int_0^x [f(t)v(t)h(t)]^p dt\right]^{q/p} ,$$

where

$$H(x) = \left[\int_0^x [v(t)h(t)]^{-p'} dt\right]^{1/p'} .$$

Now, multiplying by $u(x)^q$ and integrating we obtain by (1.6) with $k = q/p$

$$I \equiv \int_0^\infty [(P_\eta f)(x)u(x)]^q dx \le \int_0^\infty \left[\frac{u(x)H(x)}{x^\eta}\right]^q \left[\int_0^x [f(t)v(t)h(t)]^p dt\right]^{q/p} dx$$

$$\le \left\{\int_0^\infty [f(t)v(t)h(t)]^p \left[\int_t^\infty \left[\frac{u(x)H(x)}{x^\eta}\right]^q dx\right]^{p/q} dt\right\}^{q/p} .$$

By (2.1)

$$H(x)^q = \left\{\int_0^x v(t)^{-p'}\left[\int_0^t v(s)^{-p'}ds\right]^{-1/p} dt\right\}^{q/p'}$$

$$= \left\{p'\left[\int_0^t v(s)^{-p'}ds\right]^{1/p'}\Bigg|_0^x\right\}^{q/p'} = (p')^{q/p'}\left[\int_0^x v(s)^{-p'}ds\right]^{q/(p'p')}$$

$$= (p')^{q/p'}K_\eta(x)^{q/p'}x^{\eta q/p'}\left[\int_0^x u(s)^q ds\right]^{-1/p'} ,$$

so that by the assumption $K_\eta(x)$ is non-decreasing and $K_\eta(x) \geq K_\eta > 0$,

$$I \leq (p')^{q/p'}\left\{\int_0^\infty [f(t)v(t)h(t)]^p\left[\int_t^\infty \frac{u(x)^q K_\eta(x)^{q/p'}}{x^{\eta q(1-1/p')}}\left(\int_0^x u(s)^q ds\right)^{-1/p'}dx\right]^{p/q}dt\right\}^{q/p}$$

$$\leq (p')^{q/p'}\left\{\int_0^\infty [f(t)v(t)h(t)]^p t^{-\eta}K_\eta(t)^{p/p'}\left[p\left(\int_0^x u(s)^q ds\right)^{1/p}\Bigg|_t^\infty\right]^{p/q}dt\right\}^{q/p}$$

$$\leq (p')^{q/p'}(-p)\left\{\int_0^\infty [f(t)v(t)h(t)]^p t^{-\eta}K_\eta(t)^{p/p'}K_\eta(t)t^\eta\left(\int_0^t v(s)^{-p'}ds\right)^{-1/p'}dt\right\}^{q/p}$$

$$\leq (p')^{q/p'}(-p)K_\eta^q\left\{\int_0^\infty [f(t)v(t)]^p dt\right\}^{q/p} ,$$

which is the result with $C = 1/\left[(p')^{1/p'}(-p)^{1/q}K_\eta\right]$.

(ii) The proof is similar to that of part (i) except now we define $h(t) = \left[\int_t^\infty \nu(s)^{-p'} ds\right]^{1/(pp')}$. We omit the details.

<u>Corollary 1</u>. If $p \le q < 0$, $\eta \ge 0$, $\alpha < 0$ and $\beta = \alpha - \eta$, then

$$(2.4) \qquad \int_0^\infty (P_\eta f)(x)^q x^{\alpha q - 1} dx \le C \left\{\int_0^\infty (f(t)t)^p t^{\beta p - 1} dt\right\}^{q/p},$$

where

$$C = |p| / \left[|\alpha q| \, |\beta|^{q/p'}\right].$$

If $\eta \le 0$ and $\alpha \ge 0$, (2.4) holds with P_η replaced by Q_η.

<u>Proof</u>. Let $u(t) = t^{\alpha - 1/q}$, $\nu(t) = t^{\beta + 1/p'}$. Then by (2.1) (respectively (2.2))

$$K_\eta = 1/\left[|\alpha q|^{1/q} (p')^{1/p'} |\beta|^{1/p'}\right] = J_\eta$$

and the result follows with $C = (p')^{q/p'}(-p)K_\eta^q$ (respectively $C = (p')^{q/p'}(-p)J_\eta^q$).

If $\eta = 0$ and $r = 1 - \alpha p$, then the constant $C = \left[\frac{|p|}{|r-1|}\right]^p$ is best possible as was shown by Beesack [2].

Corollary 2. Suppose $p \le q < 0$ and (u,v) satisfies

$$\left(\int_0^r \left(\frac{u(x)}{x}\right)^q dx\right)^{1/q} \left(\int_0^r v(x)^{-p'} dx\right)^{1/p'} = K > 0$$

for all $r > 0$. If f is non-decreasing then

$$\left\{\int_0^\infty (v(x) f(x))^p dx\right\}^{1/p} \le C \left\{\int_0^\infty (u(x) f(x))^q dx\right\}^{1/q} .$$

Observe that $(P_0 f)(x) \le x f(x)$ and hence the result follows from Theorem 1.

We now wish to consider certain weak type estimates for P_η with $\eta \le 0$ and for Q_η with $\eta \ge 0$. First note that if T is an (linear) operator and $u(E) = \int_E u(x)^q dx$, $E \subset (o,\infty)$, $q<0$, then a simple change of variable shows that for $\lambda > 0$

$$\left\{\int_0^\infty |Tf(x) u(x)|^q dx\right\}^{1/q} = \left\{\int_0^\infty u\left(\left\{x : |Tf(x)|^q > y\right\}\right) dy\right\}^{1/q}$$

$$= \left\{\int_0^\infty u\left(\left\{x : |Tf(x)| < y^{1/q}\right\}\right) dy\right\}^{1/q}$$

$$= \left\{-q \int_0^\infty t^{q-1} u\left(\left\{x : |Tf(x)| < t\right\}\right) dt\right\}^{1/q}$$

$$\leq \left\{ -q \int_{\lambda}^{\infty} t^{q-1} \mu\left(\left\{x : |Tf(x)| < t\right\}\right) dt \right\}^{1/q}$$

$$\leq \lambda \left\{ \int_{\{x:|Tf(x)|<\lambda\}} u(x)^q \, dx \right\}^{1/q} .$$

Theorem 1 yields therefore the weak type inequality

$$\left\{ \int_0^{\infty} [f(t)v(t)]^p dt \right\}^{1/p} \leq C\lambda \left\{ \int_{\{x:|Tf(x)|<\lambda\}} u(x)^q \, dx \right\}^{1/q} ,$$

where T is P_η $(\eta \geq 0)$, respectively, Q_η, $(\eta \leq 0)$.

Corresponding to the weak type results of [1] we prove the following Theorem.

<u>Theorem 2</u>. a) If $\eta < 0$, and (u,v) satisfies (2.1), then for $p<0$, $q<0$

$$\int_{\{x:P_\eta f(x)<\lambda\}} u(x)^q \, dx \leq K_\eta^q \, \lambda^{-q} \left\{ \int_0^{\infty} [f(t)v(t)]^p dt \right\}^{q/p} . \tag{2.5}$$

b) If $\eta > 0$ and (u,v) satisfies (2.2), then for $p \leq q < 0$

$$\int\limits_{\{x:Q_\eta f(x)<\lambda\}} u(x)^q\,dx \le J_\eta^q\,\lambda^{-q}\left\{\int_0^\infty [f(t)v(t)]^p dt\right\}^{q/p} .$$

Proof (a). Since $\eta<0$, $P_\eta f(x)$ is non-decreasing, so that $\{x:(P_\eta f)(x)>\lambda\} = (o,r)$, where r is the smallest number satisfying

$$r^{-\eta}\int_0^r f(x)dx = \lambda .$$

Therefore by (2.1) and Hölder's inequality

$$\int\limits_{\{x:(P_\eta f)(x)<\lambda\}} u(x)^q dx = \int_0^r u(x)^q dx$$

$$= K_\eta(r)^q\, r^{\eta q}\left\{\int_0^r v(x)^{-p'}dx\right\}^{-q/p'}$$

$$= K_\eta(r)^q\left\{\lambda^{-1}\int_0^r f(x)dx\right\}^q\left\{\int_0^r v(x)^{-p'}dx\right\}^{-q/p'}$$

$$\le K_\eta(r)^q\,\lambda^{-q}\left\{\int_0^r [f(x)v(x)]^p dx\right\}^{q/p}$$

from which the first part of the theorem follows.

To prove b) note that $(Q_\eta f)(x) = (P_{-\eta}g)(\frac{1}{x})$, $\eta>0$ where $g(t) = t^{-2}f(1/t)$. Now as in part a)

$$\{x:(Q_\eta f)(x)<\lambda\} = \{x:(P_{-\eta}g)(1/x)<\lambda\} = \{t^{-1}:(P_{-\eta}g)(t)<\lambda\}$$

$$= (r^{-1},\infty), \quad \text{where } r \text{ is the smallest}$$

number satisfying

$$r^\eta \int_0^r t^{-2}f(1/t)\,dt = r^\eta \int_{1/r}^\infty f(x)\,dx = \lambda \qquad (x = \tfrac{1}{t}).$$

Therefore,

$$\int_{\{x:(Q_\eta f)(x)<\lambda\}} u(x)^q dx = \int_{1/r}^\infty u(x)^q dx$$

$$= \left[J_\eta(1/r)r^{-\eta}\left(\int_{1/r}^\infty v(t)^{-p'}dt\right)^{-1/p'}\right]^q$$

$$\le J_\eta^q\,\lambda^{-q}\left(\int_{1/r}^\infty f(x)\,dx\right)^q\left(\int_{1/r}^\infty v(x)^{-p'}dx\right)^{-q/p'}$$

$$\le J_\eta^q\,\lambda^{-q}\left(\int_0^\infty [v(x)f(x)]^p dx\right)^{q/p}$$

by Hölder's inequality.

Note that unlike Theorem 1 no monotonicity assumption for $K_\eta(r)$ or $J_\eta(r)$ is required in this result. It is therefore possible to give a converse of Theorem 2, part a) if $0 < \int_0^r v(t)^{-p'}dt < \infty$ and in case of part b) if $0 < \int_r^\infty v(t)^{-p'}dt < \infty$.

<u>Theorem 3</u>. If $p<0$, $q<0$ and $\eta<0$, then (2.5) (with K_η replaced by C) implies (2.1) and $C \le K_\eta$.

<u>Proof</u>. Since $\{x:(P_\eta f)(x) < \lambda\} = (0,r)$ where r is the smallest number satisfying $r^{-\eta}\int_0^r f(x)dx = \lambda$, then with $f(x) = v(x)^{-p'}$ on $(0,r]$ and zero for $x>r$ in (2.5) yields

$$\int_0^r u(x)^q dx \le C^q\lambda^{-q}\left\{\int_0^r v(t)^{p-pp'}dt\right\}^{q/p}$$

$$= C^q\left\{r^{-\eta}\int_0^r v(t)^{-p'}dt\right\}^{-q}\left\{\int_0^r v(t)^{-p'}dt\right\}^{q/p}$$

$$= C^q\, r^{\eta q}\left\{\int_0^r v(t)^{-p'}dt\right\}^{-q/p'} .$$

Hence

$$r^{-\eta}\left\{\int_0^r u(x)^q dx\right\}^{1/q}\left\{\int_0^r v(t)^{-p'}dt\right\}^{1/p'} \geq C$$

and the result follows.

A corresponding result holds for $Q_\eta f$.

§3. The Case $0<q\leq p<1$.

Theorem 4. Suppose $0<q\leq p<1$.

a) If $\eta\leq 0$ and (u,v) satisfies (2.2) with $J_\eta(r)$ non-increasing, then

$$(3.1)\qquad \left\{\int_0^\infty [f(t)v(t)]^p dt\right\}^{1/p} \leq C\left\{\int_0^\infty [(P_\eta f)(x)u(x)]^q dx\right\}^{1/q}$$

where $C = 1/[(-p')^{1/p'}p^{1/q}J_\eta]$.

b) If $\eta\geq 0$ and (u,v) satisfies (2.1) with $K_\eta(r)$ non-decreasing, then (3.1) holds with P_η replaced by Q_η and J_η by K_η in the constant C .

Proof. a). Let $h(t) = \left\{\int_t^\infty v(s)^{-p'}ds\right\}^{1/(p'p)}$. Then by

Hölder's inequality

$$(P_\eta f)(x)^q \geq x^{-\eta q} H(x)^q \left[\int_0^x [f(t)v(t)h(t)]^p dt\right]^{q/p} ,$$

where

$$H(x)^q = \left\{\int_0^x [v(t)h(t)]^{-p'} dt\right\}^{q/p'} .$$

Multiplying by $u(x)^q$ and integrating we obtain by (1.7) with $k = p/q$

$$I \equiv \int_0^\infty [u(x)(P_\eta f)(x)]^q dx \geq \int_0^\infty \left[\frac{u(x)H(x)}{x^\eta}\right]^q \left[\int_0^x [f(t)v(t)h(t)]^p dt\right]^{q/p} dx$$

$$\geq \left\{\int_0^\infty [f(t)v(t)h(t)]^p \left[\int_t^\infty \left[\frac{u(x)H(x)}{x^\eta}\right]^q dx\right]^{p/q} dt\right\}^{q/p} .$$

But by (2.2)

$$H(x)^q = \left\{\int_0^x v(t)^{-p'} \left(\int_t^x v(s)^{-p'} ds\right)^{-1/p} dt\right\}^{q/p'}$$

$$= \left\{-p' \left(\int_t^\infty v(s)^{-p'} ds\right)^{1/p'} \Bigg|_0^x\right\}^{q/p'}$$

$$\geq (-p')^{q/p'} \left(\int_x^\infty v(s)^{-p'} ds \right)^{q/(p'p')}$$

$$= (J_\eta(x) x^\eta)^{q/p'} (-p')^{q/p'} \left(\int_x^\infty u(s)^q ds \right)^{-1/p'}$$

and substituting we obtain

$$I \geq (-p')^{q/p'} \left\{ \int_0^\infty [f(t)v(t)h(t)]^p \left[\int_t^\infty \frac{u(x)^q J_\eta(x)^{q/p'}}{x^{\eta q(1-1/p')}} \left(\int_x^\infty u(s)^q ds \right)^{-1/p'} dx \right]^{p/q} dt \right\}^{q/p}$$

$$\geq (-p')^{q/p'} \left\{ \int_0^\infty [f(t)v(t)h(t)]^p t^{-\eta} J_\eta(t)^{p/p'} \left[\int_t^\infty u(x)^q \left(\int_x^\infty u(s)^q ds \right)^{-1/p'} dx \right]^{p/q} dt \right\}^{q/p}$$

Integrating the inner integral and applying again (2.2) we obtain

$$I \geq (-p')^{q/p'} p \left\{ \int_0^\infty [f(t)v(t)h(t)]^p t^{-\eta} J_\eta(t)^{p-1} \left(\int_t^\infty u(s)^q ds \right)^{1/q} dt \right\}^{q/p}$$

$$= (-p')^{q/p'} p \left\{ \int_0^\infty [f(t)v(t)h(t)]^p J_\eta(t)^p h(t)^{-p} dt \right\}^{q/p}$$

$$\geq (-p')^{q/p'} p\, J_n^q \left\{ \int_0^\infty [f(t)v(t)]^p dt \right\}^{q/p}$$

from which a) follows.

Part b) follows analogously, only now one defines

$$h(t) = \left[\int_0^t v(s)^{-p'} ds \right]^{1/(pp')}$$

and uses (1.6). The details are omitted.

<u>Corollary 3</u> [4, Thm. 347]. If $T = \begin{cases} P_0 & \text{if } r > 1 \\ Q_0 & \text{if } r < 1 \end{cases}$ and $0<p<1$, then

$$\int_0^\infty x^{-r} Tf(x)^p dx \geq \left(\frac{p}{|r-1|} \right)^p \int_0^\infty x^{-r} (xf(x))^p dx .$$

The constant in question is best possible.

This Corollary follows from Theorem 4 with $u(t) = t^{-r/p}$ and $v(t) = t^{1-r/p}$, $q = p$.

In the final theorem we assume that (u,v) satisfies

$$\text{(3.2)} \qquad \inf_{r>0} r^{-\eta}\left(\int_r^\infty u(x)^q dx\right)^{1/q} \left(\int_0^r v(x)^{-p'} dx\right)^{1/p'} \equiv A_2 > 0 .$$

<u>Theorem 5</u>. Suppose $0<q\le p<1$ and $\eta\le 0$. Then

$$\left\{\int_{\{x:P_\eta f(x)>1\}} [(P_\eta f)(x)]^q \ln[(P_\eta f)(x)] u(x)^q dx\right\}^{1/q} \ge A_2 q^{-1/q}\left\{\int_0^\infty [f(x)v(x)]^p dx\right\}^{1/p}$$

<u>Proof</u>. Since $P_\eta f$ is increasing and continuous

$$\int_{\{x:(P_\eta f)(x)>\lambda\}} [(P_\eta f)(x)u(x)]^q dx = \int_r^\infty [(P_\eta f)(x)u(x)]^q dx \ge [(P_\eta f)(r)]^q \int_r^\infty u(x)^q dx ,$$

where r is the largest number satisfying

$$(P_\eta f)(r) = r^{-\eta}\int_0^r f(x)dx = \lambda .$$

Now by (3.2) and Hölder's inequality

$$\int_{\{x:(P_\eta f)(x)>\lambda\}} [(P_\eta f)(x)u(x)]^q dx \geq A_2^q [P_\eta f)(r)]^q r^{\eta q} \left[\int_0^r v(x)^{-p'}dx\right]^{-q/p'}$$

$$= A_2^q [(P_\eta f)(r)]^q \lambda^{-q} \left[\int_0^r f(x)dx\right]^q \left[\int_0^r v(x)^{-p'}dx\right]^{-q/p'}$$

$$\geq A_2^q [(P_\eta f)(r)]^q \lambda^{-q} \left[\int_0^r [f(x)v(x)]^p dx\right]^{q/p}$$

$$\geq A_2^q \lambda^{-q} \left\{\int_0^r [(P_\eta f)(x)f(x)v(x)]^p dx\right\}^{q/p},$$

since $P_\eta f$ is increasing. But since $(P_\eta f)^{-1}(\lambda) = r$, we obtain on multiplying by λ^{-1} and integrating

$$\int_{\{x:(P_\eta f)(x)>1\}} [(P_\eta f)(x)]^q \ln[(P_\eta f)(x)]u(x)^q dx = \int_1^\infty \lambda^{-1}\left[\int_{\{x:(P_\eta f)(x)>\lambda\}} [(P_\eta f)(x)u(x)]^q dx\right]d\lambda$$

$$\geq A_2^q \int_1^\infty \lambda^{-q-1}\left[\int_0^{(P_\eta f)^{-1}(\lambda)} [(P_\eta f)(x)f(x)v(x)]^p dx\right]^{q/p} d\lambda$$

$$\geq A_2^q \left\{ \int_0^\infty [(P_\eta f)(x) f(x) \nu(x)]^p \left[\int_{(P_\eta f)(x)}^\infty \lambda^{-q-1} d\lambda \right]^{p/q} dx \right\}^{q/p}$$

$$= A_2^q \left\{ \int_0^\infty [(P_\eta f)(x) f(x) \nu(x)]^p \left[\frac{(P_\eta f)(x)^{-q}}{q} \right]^{p/q} dx \right\}^{q/p}$$

$$= A_2^q\, q^{-1} \left\{ \int_0^\infty [f(x) \nu(x)]^p dx \right\}^{q/p}$$

where we used inequality (1.7).

A corresponding result for negative p and q can be obtained from Theorem 2.

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