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Comments on a Functional Equation

In this note we study all solutions of the equations

$$(1)\qquad f(x+s) = af(x) \text{ and } f(x+t) = bf(x)$$
$$a>0,\ b>0, \text{ and } s/t \text{ irrational}$$

where a,b,s, and t are real numbers. In Rudin's book, Real and Complex Analysis Mcgraw-Hill (1974) p. 190, Problem 19, equation (1) appears with $a=b=1$. In this case, any Lebesgue measurable solution, f, must be Lebesgue equivalent (i.e. equal a.e.) to a constant. In the more general case ($a>0$ and $b>0$) investigated here we obtain the following results.

THEOREM. *If there is a nonzero solution of (1) which is positive at some point and bounded above on some interval, then* $a^t = b^s$.

Conversely,

THEOREM. *If* $a^t = b^s$, *then* $f(x) = ca^{x/s}$ *is a solution to (1) and furthermore, every Lebesgue measurable solution to (1) is equivalent to f.*

Finally, we show that:

THEOREM. *There is a solution to (1) which is not equivalent to a continuous function.*

The motivation for this study was to determine in what sense the function $f(x) = 2^x$ is characterized by the functional equation $f(x+1) = 2f(x)$. That is, what conditions can be appended to the functional equation $f(x+1) = 2f(x)$ to insure that any solution is equivalent to $f(x) = 2^x$.

As an initial remark, notice that if $f(0)=1$ and f satisfies (1) then for integers m and n, $f(ms) = a^m$ and $f(nt) = b^n$, so that for $y = ms+nt$,

$$f(x+y) = f(x)f(y).$$

That is, (1) reduces to a restricted Cauchy equation where the restriction is that one of x or y lies in the set

(2) $$D = \{ms+nt : m \text{ and } n \text{ are integers}\} .$$

This set D plays an important role in our investigation.

To better understand the the solutions to (1) we will investigate the general form of any possible solution.

First, let r_1 and r_2 be any two rational numbers. Then there exist unique integers m and n and a 4-tuple (p_1,q_1,p_2,q_2) such that

$$r_1s+r_2t = (ms+nt) + (p_1s/q_1 + p_2t/q_2), \text{ and}$$
$$(p_1,q_1,p_2,q_2) \in B = \{(p_1,q_1,p_2,q_2) : p_i \text{ and } q_i \text{ are nonnegative integers with } p_i \text{ relatively prime to } q_i \text{ if } p_i \neq 0.$$

Let H be a Hamel Basis for the reals which contains both s and t, and let C be the linear span of $\{s,t\}$. Let C' be the subspace spanned by $H-\{s,t\}$. If x is any real number, then x has a unique representation as

$$x = (r_1 s + r_2 t) + y$$

where r_1 and r_2 are rationals and $y \in C'$. Now, if F is any real valued function defined on $B \times C'$, then

$$f(x) = a^m b^n F(p_1, q_1, p_2, q_2, y) \tag{3}$$

satisfies (1), and it can easily be verified that all solutions to (1) have this form.

We now give two results which will be needed in our further analysis of the solutions of (1). The first result is well known.

LEMMA 1. *The set, D, defined in (2) is dense in the reals if and only if s/t is irrational.*

LEMMA 2. *Suppose that f is a solution of (1) and that f is bounded above on some interval. Then f is bounded above on every finite interval.*

Proof. Suppose that f is bounded above on some interval $J=[c,d]$, and suppose further that there is a compact interval I and $f(x_n) \to \infty$ for some sequence in I. As I is compact, we can assume that $x_n \to y$ for some y in I. Let I' be a neighborhood of y contained in I such that the length of I' is less than $(d-c)/4$. Let J' be a subinterval of J such that for any v,w in I, $J' + (v-w) \subset J$. Choose u_1 and u_2 in D such that $u_1 \in I'$ and $u_2 \in J'$; then for sufficiently large n, $x_n = x'_n + (u_1 - u_2)$ for some $x'_n \in J$.

But then,

$$f(x_n) = f(x'_n + u_1 + u_2) = a^m b^n f(x'_n)$$

and this contradicts the fact that $f(x_n)$.

We are now ready to prove our main results.

THEOREM 1. *If there is a nonzero solution of* (1) *which is positive at some point and bounded above on some interval, then* $a^t = b^s$

Proof. Suppose $f(y) > 0$ and define two supplementary functions as follows:

$$g(x) = a^{-x/s} f(x), \quad \text{and} \quad h(x) = b^{-x/t} f(x).$$

Then g and h are periodic with periods s and t respectively,and these functions are related by:

$$g(x) = a^{x/s} b^{-x/t} h(x)$$

Now if $a^t > b^s$, then

$$g(y \pm nt) = [a^{1/s} b^{-1/t}]^{y \pm nt} h(y)$$

which diverges as $n \to \infty$ since $h(y) = 0$. Hence, $a^t \leq b^s$. A symmetric argument using $h(x) = a^{-x/s} b^{x/t} g(x)$ shows that $b^s \leq a^t$.

THEOREM 2. *If* $a^t = b^s$ *then* $f(x) = ca^{x/s}$ *is a solution to* (1), *and furthermore, every Lebesgue measurable solution to* (1) *is equivalent to this function*, f.

Proof. It is easily verified that $f(x) = ca^{x/s}$ is a solution to (1) when $a^t = b^s$. Suppose $a^t = b^s$ and $g(x)$ is a Lebesgue measurable solution to (1). Let

$$G(x) = a^{-x/s}g(x)$$

Then, it is easy to show that G(x) is periodic with periods s and t. And, in fact, the set of periods of G contains the set D which is everywhere dense. This periodicity of G on D then insures that the set of points of density of each of the sets $E_a = \{x:G(x) > a\}$ is either of full measure or of measure zero. As G is measurable, this entails that G(x) is constant a.e. and the theorem is verified.

REMARK 1. It is easy to construct a discontinuous solution of (1), as follows:

$$f(x) = \begin{cases} a^m b^n \text{ for } x = ms+nt \text{ in } D \\ 0 \text{ if } x \notin D \end{cases}$$

Then f is continuous nowhere, but, f is measurable since f = 0 a.e.

REMARK 2. Consider equations (1) with the assumption that s/t is rational. If f is a nonzero solution of this new system of equations, then

$$a^t = b^s, \text{ and}$$
$$f(x) = a^{x/s}G(x)$$

where G(x) is a periodic function of fundamental period p and p has the property that p divides every integral linear combination of s and t. To see that this is the case, one can use an argument analogous to that used in THEOREM 2. above; the exception is that although G has periods s and t, the set of all periods is not dense.

Now in what follows we will construct a solution of (1) which is not equivalent to a continuous function. Actually our aim is obtain such a solution which is also Lebesgue measurable, but, unfortunately, we are unable to determine the measurability of our example. In the case of the Cauchy equation, any solution that is different from the exponential equation is automatically not equivalent to a continuous function. For our equation (1), however, this question is more difficult.

THEOREM 3. <u>There is a solution of</u> (1) <u>which is not equivalent to a continuous function</u>.

Proof. For this example we need to define the following sets.

$$E = \{r_1 s + r_2 t + y : y \in C', y > 0, r_1 \text{ and } r_2 \text{ are rational}, r_1 \geq 0\}$$

$$F = \{r_1 s + r_2 t + y : y \in C', y > 0, r_1 \text{ and } r_2 \text{ are rational}, r_1 \leq 0\}$$

$$G = \{r_1 s + r_2 t + y : y \in C', y > 0, r_1 \text{ and } r_2 \text{ are rational}, r_1 \leq 0\}$$

$$J = \{r_1 s + r_2 t + y : y \in C', y < 0, r_1 \text{ and } r_2 \text{ are rational}, r_1 \geq 0\}$$

$$U = \{r_1 s + r_2 t : r_1 \text{ and } r_2 \text{ are rational}\}$$

Notice that the union of these sets is the reals, and that $E=-F$, $G=-J$, and $m(U)=0$ where $m()$ denotes Lebesgue measure. In what follows, $m^*(\cdot)$ and $m_*()$ denote Lebesgue outer and inner measure respectively. First observe that C' is not Lebesgue measurable and indeed C' has zero inner measure and full outer measure in every interval. Without loss of generality we assume $b=1$. Now let $x=r_1 s+r_2 t+y$ where $y \in C'$ and r_1 and r_2 are rationals. To get our example, we substitute in (3) for F the function which is 1 for $y>0$ and 0 for $y<0$. This implies that in (3), the function $f(x)$ is at least 1 either on the set E or G depending on whether $a \geq 1$ or $a \leq 1$; also,

$f(x)=0$ on $F \cup J$. Suppose that there is a continuous function g such that

$$m^*(\{x:f(x)\neq g(x)\}) \neq 0.$$

Then since the outer measure of both of the sets $E \cup G$ and $F \cup J$ is positive in every interval, g must assume the value of 0 as well as values in excess of 1 in every interval. This, of course, contradicts the continuity of g, and f has the desired properties.

REMARK 3. Suppose $s=s(a)$ is a function whose domain is the positive reals, and that f is a function which satisfies the functional equation

$$f(x+s(a)) = af(x) \tag{4}$$

for every real number x, and every positive real a. Notice that for any nonzero solution of (4), the function s must necessarily be 1-1, and consequently, the set $\{s(a):a>0\}$ is uncountable. Hence, there are numbers a and b such that $s(a)/s(b)$ is irrational. This means that a nonzero solution of (4) is also a solution of (1) for suitable s and t. Consequently, if there is a solution of (4) which is positive at some point and bounded above on some interval, then the ratios $s(a)/s(b)$ must be $\ln a/\ln b$. This means that there is a nonzero constant k such that for each positive a, $s(a) = k\ln a$.

REMARK 4. The results of this paper do have some generalizations in the context of metric topological groups, and in this remark we give one of them. Let G and H be metric topological groups, $a,b \in H$, $s,t \in G$, and $f:G \rightarrow H$ be a mapping such that

$$f(xs) = af(x) \quad \text{and} \quad f(xt) = bf(x) \tag{5}$$

for each x in G. Suppose that $\{s,t\}$ generates G in the sense that the set of all words in s, s^{-1}, t, t^{-1} is dense in G. Then, any nontrivial f satisfying (9) has the following property.

THEOREM. Suppose that $f(U)$ is sequentially compact for some open set U in G. Then, $f(K)$ is sequentially compact whenever K is.

Proof. Let K be a sequentially compact set in G. If $f(K)$ is not sequentially compact, then there exist $x_n \rightarrow y$ in K such that the sequence $\{f(x_n)\}$ has no convergent subsequence. Since G is a topological group, there exists an open set V and a neighborhood $U(y)$ such that $U(y)U(y)^{-1}V \subset U$. Let u,v be two words in s and t such that $u \in U(y)$ and $v \in V$. Now, for any $x \in U(y)$, $x=(xu^{-1})v(v`u-1u) \in Uv^{-1}u$. Since f satisfies (5) and $f(U)$ is sequentially compact, it is clear that $f(Uv^{-1}u)$ is sequentially compact. But for large n, $x_n \in U(y)$ and $f(x_n) \in f(U(y)) \subset f(Uv^{-1}u)$, which implies that $f(x_n)$ has a convergent subsequence. This is a contradiction and the proof follows.

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