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C. Alex Safsten and Logan C. Tatham

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A variational approach to a generalized elastica problem

C. Alex Safsten and Logan C. Tatham

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In this paper, we apply the calculus of variations to solve the elastica problem. We examine a more general elastica problem in which the material under consideration need not be uniformly rigid. Using, the Euler–Lagrange equations, we derive a system of nonlinear differential equations whose solutions are given by these generalized elastica curves. We consider certain simplifying cases in which we can solve the system of differential equations. Finally, we use novel numerical techniques to approach solutions to the problem in full generality.

1. Introduction

Historically, it has been of much interest to find the shape to which a material conforms when it is bent. This is known as the elastica problem, and has been studied by mathematicians including the Bernoullis, Euler, and Laplace [Euler 1786; Levien 2008]. Elastica problems date back to the thirteenth century mathematician Jordanus de Nemore, who mentioned them in his book, *De ratione ponderis* (see [Tartaglia 1565]). The problems were explored analytically by James Bernoulli in the seventeenth century, though he did not utilize variational methods, which is curious, as his brother Johann was instrumental in the development of the calculus of variations [Goldstine 1980].

Today, the elastica problem is still of great interest, as it has applications in many fields, including engineering, animation, and even industrial design. For example, at Brigham Young University, Professor David Morgan bends sheet metal to construct bowls. Professor Morgan employs a trial and error technique to find a shape which, when cut from a metal sheet, bends into an aesthetically pleasing bowl whose base is flush with the surface, as shown in Figure 1. Currently, Professor Morgan and other design professionals who utilize extreme deflections in their designs use a "guess and

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Figure 1. Left: An unfolded shape cut from sheet metal. Right: The metal shape folds to a bowl.

check" algorithm to determine how to cut and bend their material. An analytic expression for the bent shapes would facilitate the construction of industrial designers' art. Deflections are especially of interest to those in the field of engineering known as compliant mechanisms. Compliant mechanisms are devices which exhibit movement through bending the material of the device itself, as opposed to with hinges.

There are two ways of looking at the extreme deflection problem. First, given a material and boundary conditions representing the bending, what shape does it make? Second, given specific boundary conditions, can one vary the rigidity of the material to ensure it will bend to a specified shape. Euler, Bernoulli, and others answered the first of these questions in the case that the material is uniformly rigid, but the answer is the solution to a system of differential equations that is rather difficult to solve. We reproduce their results, and show how to find solutions to these differential equations in a few special cases, using a novel approach to fit boundary conditions. Furthermore, we present an algorithm capable of finding a more general class of solutions. We also explore the consequences for a material that is not uniformly rigid. Finally, we answer the second question and present a closed-form solution for a rigidity function that will make a material conform to the desired shape.

In this paper, we apply the calculus of variations and numerical methods for answering both of these questions. We analyze problems involving a "strip" of material, in which bending varies in direction, as opposed to a "sheet" of material, in which bending varies in several directions. We propose a simple model of the energy stored in a strip of flexible material and utilize this model to approach an analytic answer to these problems.

2. Preliminary

The calculus of variations. The calculus of variations is a theory of optimization. Optimization using variational calculus is analogous to optimization using differentiation in elementary calculus. However, rather than optimizing a function with respect to a single real variable, optimization methods in the calculus of variations seek to optimize a functional. A functional is a mapping $\mathscr{F} \to \mathbb{R}$, where $\mathscr{F}$ is a set of functions. Generally, we only consider functions that are n-times continuously differentiable on a closed interval; that is, $\mathscr{F} \subset C^n([a, b])$. The mapping usually takes the form

$$J = \int_a^b \mathcal{F}\big(x, f(x), f'(x), f''(x), \ldots, f^{(n)}(x)\big)\, dx, \tag{1}$$

where

- $f \in \mathscr{F}$,
- $\mathcal{F}$, when treated as a function only of x, is n-times differentiable,
- $\mathcal{F}$ is continuously differentiable in $f, f', \ldots, f^{(n)}$,
- J is the value to be optimized with respect to f.

Our primary assumption is that flexible materials in extreme deflection problems form shapes that minimize their stored energy. Therefore extreme deflection problems can be understood by letting f be a function whose plot in the xy-plane gives the shape of the material and

$$J = \int_a^b \mathcal{F}\big(x, f(x), \ldots\big)\, dx$$

be a functional that calculates the energy stored in the material from its shape. We endeavor to use the calculus of variations to find the function f that gives the minimum energy J. The following information on the calculus of variations comes from the excellent book [Goldstine 1980], which goes into greater depth for the interested reader.

It can be shown that if an extremum of the functional

$$J = \int_a^b \mathcal{F}\big(x, f(x), f'(x)\big)\, dx$$

occurs at f, then the following identity, known as the *Euler–Lagrange equation*, holds:

$$0 = \frac{\partial \mathcal{F}}{\partial f} - \frac{d}{dx}\frac{\partial \mathcal{F}}{\partial f'}. \tag{2}$$

For the more general functional

$$J = \int_a^b \mathcal{F}\big(x, f(x), f'(x), f''(x), \ldots, f^{(n)}(x)\big)\, dx,$$

the Euler–Lagrange equation extends as

$$0 = \frac{\partial \mathcal{F}}{\partial f} - \frac{d}{dx}\frac{\partial \mathcal{F}}{\partial f'} + \frac{d^2}{dx^2}\frac{\partial \mathcal{F}}{\partial f''} - \frac{d^3}{dx^3}\frac{\partial \mathcal{F}}{\partial f'''} + \cdots + (-1)^n \frac{d^n}{dx^n}\frac{\partial \mathcal{F}}{\partial f^{(n)}}. \tag{3}$$

If we are optimizing over a set of vector-valued functions

$$f(x) = \big(f_1(x), \dots, f_m(x)\big),$$

one can show that the Euler–Lagrange equation must be true for each component function. In our problem, we are examining shapes made by flexible material as functions mapping to $\mathbb{R}^2$. We find it easier to consider two independent functions $x = x(t)$ and $y = y(t)$. Our functional thus takes the form,

$$J = \int_a^b \mathcal{F}\big(t, x(t), y(t), x'(t), y'(t), \dots, x^{(n)}(t), y^{(n)}(t)\big)\,dt.$$

The Euler–Lagrange equation becomes a system of equations, where (3) is satisfied for both component functions; that is,

$$0 = \frac{\partial \mathcal{F}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{F}}{\partial x'} + \cdots + (-1)^n \frac{d^n}{dt^n}\frac{\partial \mathcal{F}}{\partial x^{(n)}},$$
$$0 = \frac{\partial \mathcal{F}}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{F}}{\partial y'} + \cdots + (-1)^n \frac{d^n}{dt^n}\frac{\partial \mathcal{F}}{\partial y^{(n)}}.$$

Note that the Euler–Lagrange equation is a necessary but not sufficient condition for an extremum, similar to solving $f'(x) = 0$ in elementary calculus optimization problems. The condition that gives sufficiency is known as the second variation, but it is a common practice to omit it in variational problems. It is very complex and in this paper, we will not need to address this problem, as it is obvious when solutions are minimizers. We also mention that while the Euler–Lagrange equation is satisfied by the extremum of the functional, it does not identify which local minima and maxima are global minima and maxima. This is similar to calculus in that solving $f'(x) = 0$ or $\nabla f(\boldsymbol{x}) = \mathbf{0}$ does not identify local extrema as global extrema. The Euler–Lagrange equation gives differential equations; extrema are found by solving these equations subject to boundary conditions.

Constrained optimization. One also may wish to minimize the functional subject to a constraint. For clarity, allow us to draw an analogy. In multivariable calculus, we may wish to optimize the function $f : \mathbb{R}^n \to \mathbb{R}$ subject to a constraint function $g(\boldsymbol{x}) = 0$. To do this, we solve the equations

$$\begin{aligned} \nabla f(\boldsymbol{x}) &= \lambda \nabla g(\boldsymbol{x}), \\ g(\boldsymbol{x}) &= 0, \end{aligned} \tag{4}$$

where λ is a constant.

This is known as the method of Lagrange multipliers. In the calculus of variations, one may wish to optimize the functional

$$J = \int_a^b \mathcal{F}\big(t, x(t), y(t), x'(t), y'(t)\big)\,dt \quad \text{subject to} \quad \Phi(x, y) = 0, \tag{5}$$

where $\Phi(x, y)$ is an ordinary function of $(x(t), y(t))$. To do this, we define

$$\mathcal{L} = \mathcal{F} - \lambda(t)\Phi$$

and then solve the system of equations

$$\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial x'} = 0, \quad \frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial y'} = 0, \quad \Phi(x, y) = 0. \tag{6}$$

So that the analogy becomes clear, note that solving (4) is equivalent to setting $L(\boldsymbol{x}) = f(\boldsymbol{x}) - \lambda g(\boldsymbol{x})$ and solving the unconstrained problem

$$\nabla L(\boldsymbol{x}) = \mathbf{0},$$
$$g(\boldsymbol{x}) = 0,$$

which is

$$\frac{\partial L}{\partial x} = 0, \quad \frac{\partial L}{\partial y} = 0, \quad g(\boldsymbol{x}) = 0.$$

There are other types of constrained optimization problems, such as the *isoparametric problem*

$$J = \int_a^b \mathcal{F}\big(x, f(x), f''(x)\big)\, dx \quad \text{subject to} \quad \int_a^b \sqrt{1 + (f')^2}\, dx = 1.$$

However, our analysis focuses on constrained problems having the same form as (5).

3. Results

Model. In our problem, the natural state of the system subject to an extreme deflection will be that for which stored energy is minimized relative to other curves that conform to the same boundary conditions. This invites the question: how does one measure the energy in a bent strip of material? The model that we propose is that potential energy stored in an infinitesimal section of deflected material is proportional to the curvature squared. This is the same model used for linear springs and thus it treats the material as if it were constructed from infinitesimal springs throughout. Our model then is

$$dU = \alpha(t)\kappa^2(t)\, ds, \tag{7}$$

where

- U is potential energy,
- $\alpha(t)$ is a positive, continuously differentiable spring coefficient,
- $\kappa(t)$ is the curvature,
- s is the arc length.

One can think of this as our definition of "flexible material". For parametric curves in the plane given by $\gamma(t) = (x(t), y(t))$, the curvature $\kappa(t)$ is given by

$$\kappa(t) = \frac{|x'(t)y''(t) - y'(t)x''(t)|}{(x'(t)^2 + y'(t)^2)^{3/2}}. \tag{8}$$

We will plot the shape that the deflected material makes in the xy-plane. We seek parametric equations of the form $x = x(t)$, $y = y(t)$, where $0 \le t \le 1$, whose plot represents the shape to which the deflected material conforms. We also introduce the arc-length parametrization constraint

$$(x')^2 + (y')^2 = 1 \quad \text{for all } t \in [0, 1]. \tag{9}$$

Observe that this greatly simplifies the expression for curvature in (8). Because any curve in the xy-plane with derivatives not both vanishing at the same point can be arc-length parametrized, this is not a restrictive assumption. Since integrating (7) gives the potential energy, which we seek to minimize, we are left to solve the problem

$$\min_{(x(t),y(t),\alpha(t))} \int_0^1 \alpha(t)\kappa^2(t)\,dt. \tag{10}$$

Now that we have a good understanding of the calculus of variations, we may apply it to (10). Specifically, we may apply the constrained Euler–Lagrange condition, equation (6), where[1]

$$\mathcal{F} = \alpha\kappa^2 \quad \text{and} \quad \Phi = (x')^2 + (y')^2 - 1 = 0.$$

This will give a system of differential equations, which are necessary conditions for their solutions to be minimizers of (10).

Proposition 3.1. *If $x(t)$ and $y(t)$ are solutions of the general problem*

$$\min_{(x(t),y(t),\alpha(t))} \int_0^1 \kappa^2\,dt,$$

then they satisfy

$$c_1 y' + \alpha' x'' y' + \alpha x''' y' = c_2 x' + \alpha' x' y'' + \alpha x' y'''$$

and

$$(x')^2 + (y')^2 = 1$$

on $[0, 1]$ *for some constants* c_1, c_2.

Proof. Since our curves are arc-length parametrized, curvature squared reduces to

$$\kappa^2 = (x'')^2 + (y'')^2.$$

[1] For convenience, we stop writing explicit dependence on t.

Thus, we may apply the constrained Euler–Lagrange condition, (6), where

$$\mathcal{F} = \alpha\kappa^2 = \alpha(x'')^2 + \alpha(y'')^2,$$
$$\Phi = (x')^2 + (y')^2 - 1 = 0.$$

Applying (6) to the x parameter gives

$$\begin{aligned} 0 &= \frac{d^2}{dt^2}\frac{\partial(\mathcal{F}-\lambda\Phi)}{\partial x''} - \frac{d}{dt}\frac{\partial(\mathcal{F}-\lambda\Phi)}{\partial x'} + \frac{\partial(\mathcal{F}-\lambda\Phi)}{\partial x} \\ &= \lambda' x' + \lambda x'' + \alpha'' x'' + 2\alpha' x''' + \alpha x^{(iv)}, \end{aligned}$$

which simplifies to

$$-(\lambda x')' = (\alpha' x'' + \alpha x''')'.$$

Integrating gives

$$-\lambda x' = c_1 + \alpha' x'' + \alpha x'''$$

for some constant c_1. A similar computation in y yields

$$-\lambda y' = c_2 + \alpha' y'' + \alpha y'''$$

for some constant c_2.

Thus we must solve

$$c_1 + \alpha' x'' + \alpha x''' = -\lambda x' \quad \text{and} \quad c_2 + \alpha' y'' + \alpha y'' = -\lambda y'.$$

Multiplying the first by y' and the second by x' allows us to eliminate λ:

$$c_1 y' + \alpha' x'' y' + \alpha x''' y' = -\lambda x' y' = c_2 x' + \alpha' x' y'' + \alpha x' y'''.$$

Recalling the constraint, we will solve the system

$$\begin{aligned} c_1 y' + \alpha' x'' y' + \alpha x''' y' &= c_2 x' + \alpha' x' y'' + \alpha x' y''', \\ (x')^2 + (y')^2 &= 1. \end{aligned} \tag{11}$$

This completes the proof. □

This is a very difficult system of differential equations to solve. Fortunately, we can simplify it significantly. Since $(x')^2(t) + (y')^2(t) = 1$ for all t, we can define a function $\theta(t)$ such that

$$x'(t) = \cos(\theta(t)) \quad \text{and} \quad y'(t) = \sin(\theta(t)).$$

Then

$$\begin{aligned} x''(t) &= -\sin(\theta(t))\theta'(t), \\ x'''(t) &= -\cos(\theta(t))(\theta')^2(t) - \sin(\theta(t))\theta''(t), \\ y''(t) &= \sin(\theta(t))\theta'(t), \\ y'''(t) &= -\sin(\theta(t))(\theta')^2(t) + \cos(\theta(t))\theta''(t). \end{aligned}$$

The interested reader may verify that when applying this substitution to (11), one arrives at the much simpler single differential equation

$$0 = c_2 \sin\theta - c_1 \cos\theta + (\alpha\theta')'. \tag{12}$$

This equation was first derived for $\alpha = 1$ by Gustav Kirchhoff in 1859. He observed that it is equivalent to the equation of motion for a pendulum, which is the basis for a multitude of analogies from elastica curves to pendulum dynamics [Love 1906].

Equation (12) contains two parameters, c_1 and c_2, and the rigidity function $\alpha(t)$, each of which is completely arbitrary (with the stipulation that α is positive and continuously differentiable). If c_1 and c_2 are both zero, (12) is linear, so we call solutions associated with this case the linear solutions. We can also simplify (12) by assuming a constant rigidity function such that $\alpha'(t) = 0$. We call solutions in this case constant rigidity solutions. We examine solutions in all four cases.

Linear, constant rigidity solutions. The easiest case is the linear, constant rigidity case. In this case, $c_1 = c_2 = \alpha'(t) = 0$. Equation (12) reduces to $\theta''(t) = 0$, the solutions to which are

$$\theta(t) = c_3 t + c_4. \tag{13}$$

If $c_3 = 0$, we get

$$\begin{aligned} x(t) &= \cos(c_4)t + c_5, \\ y(t) &= \sin(c_4)t + c_6. \end{aligned} \tag{14}$$

If $c_3 \neq 0$, then

$$\begin{aligned} x(t) &= \frac{1}{c_3}\sin(c_3 t + c_4) + c_5, \\ y(t) &= -\frac{1}{c_3}\cos(c_3 t + c_4) + c_6. \end{aligned} \tag{15}$$

Thus, after fitting boundary conditions by identifying c_3, c_4, c_5, and c_6, we find that linear constant rigidity solutions are either line segments as in (14), or arcs of circles as in (15).

Linear, nonconstant rigidity solutions. Also relatively easy to solve is the linear, nonconstant rigidity case. Here, we have $c_1 = c_2 = 0$. Equation (12) reduces to

$$\alpha(t)\theta'(t) = c_3.$$

To have physical meaning, $\alpha(t)$ must be nonnegative. If it is also nonvanishing, we know that $c_3/\alpha(t)$ is continuous, so it is integrable. Thus, by direct integration, we can solve for $\theta(t)$:

$$\theta(t) = c_4 + \int_0^t \frac{c_3}{\alpha(t')}\,dt'. \tag{16}$$

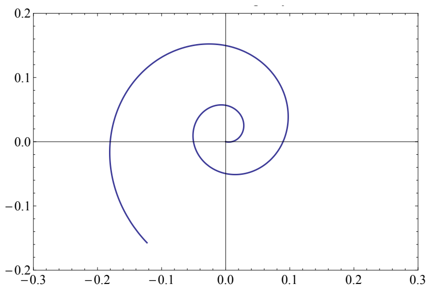

Figure 2. A solution to the linear, nonconstant rigidity case. The rigidity, in this case, is lower at the end near the origin. In regions of lower rigidity, curvature requires less energy, resulting in a spiral which grows tighter with lower rigidity.

This is as far as we can go without specifying $\alpha(t)$. As an example, suppose $\alpha(t) = at+b$, that describes a material which is more rigid on one end than the other. Then,

$$\theta(t) = c_4 + \frac{c_3}{a}\ln(at+b),$$

and

$$\begin{aligned} x'(t) &= \cos\left(c_4 + \frac{c_3}{a}\ln(at+b)\right), \\ y'(t) &= \sin\left(c_4 + \frac{c_3}{a}\ln(at+b)\right). \end{aligned} \tag{17}$$

By integrating once more, we arrive at

$$\begin{aligned} x(t) &= \frac{(at+b)\left(a\sin\left(\frac{c_3\log(at+b)}{a}+c_4\right) - c_3\cos\left(\frac{c_3\log(at+b)}{a}+c_4\right)\right)}{a^2+c_3^2} + c_5, \\ y(t) &= \frac{(at+b)\left(c_3\sin\left(\frac{c_3\log(at+b)}{a}+c_4\right) + a\cos\left(\frac{c_3\log(at+b)}{a}+c_4\right)\right)}{a^2+c_3^2} + c_6. \end{aligned} \tag{18}$$

We have provided a plot of this solution in Figure 2 for $a=1$, $b=0.1$, and $c_3=c_4=5$. The rigidity is lower at the end near the origin. Since that cost of curvature in terms of energy is lower in regions of less rigidity, the spiral is tighter there.

Nonlinear, constant rigidity solutions. In the case of nonlinear, constant rigidity solutions, c_1 and c_2 are not both zero, and $\alpha(t)$ is a constant function. This is perhaps the most interesting case because it is possible to find exact solutions that are far more general than any linear solutions. We first show how to determine solutions by using special functions, and then show how to apply boundary conditions to these solutions. Finally, we present some numerical techniques for approximating solutions when applying boundary conditions proves too difficult. In this section, we suppose $\alpha = 1$.

Jacobi elliptic and amplitude functions. The Jacobi elliptic functions are a class of special functions, sn, cn, and dn (along with others we will not discuss here)[Prasolov and Solovyev 1997]. They are often written as functions of the variable u with respect to a parameter k as $\operatorname{sn}(u,k)$, $\operatorname{cn}(u,k)$, and $\operatorname{dn}(u,k)$. The Jacobi amplitude function, usually given by $\operatorname{am}(u,k)$, is related to the Jacobi elliptic functions by

$$\operatorname{sn}(u,k) = \sin(\operatorname{am}(u,k)),$$
$$\operatorname{cn}(u,k) = \cos(\operatorname{am}(u,k)),$$
$$\operatorname{dn}(u,k) = \frac{d}{du}\operatorname{am}(u,k) = \sqrt{1-k^2\sin^2(\operatorname{am}(u,k))}.$$

It is easy to take a second derivative to demonstrate

$$\frac{d^2}{du^2}\operatorname{am}(u,k) = -\frac{k^2}{2}\sin(2\operatorname{am}(u,k)).$$

These functions and their properties will be useful in finding an exact solution to (11).

In order to use Jacobi functions to solve (12), we will apply some transformations. We define $R=\sqrt{c_1^2+c_2^2}$ and $\phi = -\arctan(c_1/c_2)$ so that (12) can be written as

$$\theta''(t) = -R\sin(\theta(t)+\phi).$$

Next, we make the transformation

$$\tau(t) = \tfrac{1}{2}(\theta(t)+\phi),$$

which gives

$$\tau''(t) = -\frac{R}{2}\sin(2\tau(t)). \tag{19}$$

Proposition 3.2. *Given R and ϕ, the function*

$$\tau(t) = \operatorname{am}\left(c_3t+c_4, \frac{\sqrt{R}}{c_3}\right) \tag{20}$$

is a solution to (19), where c_3 and c_4 are constants.

Proof. Let $u = c_3t+c_4$. Then letting $\tau(t)$ be defined as in (20), we have

$$\tau''(t) = \frac{d^2\tau}{dt^2} = \frac{d^2\tau}{du^2}\left(\frac{du}{dt}\right)^2 = -c_3^2\frac{R}{2c_3^2}\sin 2\tau(t) = -\frac{R}{2}\sin(2\tau(t)). \qquad \square$$

Working backwards through our substitutions, we can see that solutions for $x'(t)$ and $y'(t)$ are

$$\begin{aligned} x'(t) &= \cos\left(2\operatorname{am}\left(c_3(t+c_4), \frac{\sqrt{R}}{c_3}\right) - \phi\right),\\ y'(t) &= \sin\left(2\operatorname{am}\left(c_3(t+c_4), \frac{\sqrt{R}}{c_3}\right) - \phi\right). \end{aligned} \tag{21}$$

Remark 3.3. Using trigonometric identities, it is not hard to also express these in terms of sn and cn. However, the expressions are very long and not very enlightening, so we do not present them here.

Though it may be possible to find closed-form expressions for $x(t)$ and $y(t)$, we will not find it necessary to do so. We simply define the solutions as

$$\begin{aligned} x(t) &= c_5 + \int_0^t \cos\left(2\,\mathrm{am}\left(c_3(t'+c_4), \frac{\sqrt{R}}{c_3}\right) - \phi\right) dt', \\ y(t) &= c_6 + \int_0^t \sin\left(2\,\mathrm{am}\left(c_3(t'+c_4), \frac{\sqrt{R}}{c_3}\right) - \phi\right) dt'. \end{aligned} \tag{22}$$

Fitting boundary conditions. Now we have found the solutions, but it can still be very difficult to fit boundary conditions. At this point, we find it easiest to apply computational techniques to find a solution. For example, one of the simplest nontrivial solutions we may wish to find is the shape formed when one bends a strip of paper holding both ends together, as if to fold the paper in half, but not creasing the paper. We call this a "teardrop" shape. Such a shape has the following boundary conditions:

$$x(0) = 0, \quad y(0) = 0, \quad x'(0) = 1. \tag{23}$$

We also apply natural symmetries of the problem,

$$\begin{aligned} x(t) &= x(1-t), \\ y(t) &= -y(1-t), \end{aligned} \tag{24}$$

from which we obtain boundary conditions for $x(1)$, $y(1)$, and $x'(1)$. Furthermore, the $y'(0)$ and $y'(1)$ boundary conditions are determined by the arc-length constraint. We have six free variables. Four of these are apparent in (21), namely R, ϕ, c_3, and c_4. The other two are constants of integration that come from integrating (21), which we call c_5 and c_6. We will usually set both of these to zero so that one end of our solution is at the origin. In the following claim, we argue that $\phi = 0$.

Claim 3.4. *In the equations*

$$\begin{aligned} x'''y' + c_1 y' &= x'y''' + c_2 x', \\ (x')^2 + (y')^2 &= 1, \end{aligned}$$

with symmetries (24), *we have* $c_1 = 0$.

Proof. This will use some facts of even and odd functions. First, the product of any two functions with the same parity is even and the product of two functions with opposite parity is odd. Furthermore, the sets of even and odd functions are both closed under addition and scalar multiplication.

Now, we note that odd order derivatives of $x(t)$ and $y(t)$ are odd and even (respectively) about $t = \frac{1}{2}$. One can write (11) with $\alpha = 1$ as

$$c_1 y' = x'y''' - x'''y' + c_2 x'.$$

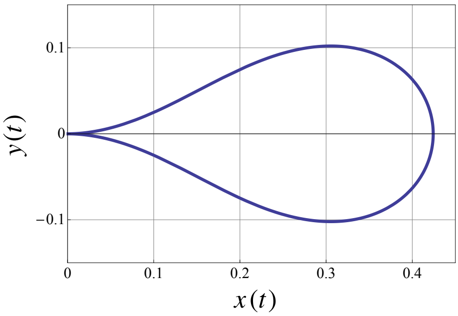

Figure 3. A solution to the teardrop problem, in which the ends of a strip of paper are pinched together.

On one hand, since y' is even, the left-hand side is even. On the other hand, since x', x''' are odd and y', y''' are even, all terms on the right side are odd. Because odd functions are closed under subtraction, the right-hand side is odd. Thus, $c_1 y$ is both even and odd, meaning

$$c_1 y'(t) \equiv 0 \quad \text{for all } t.$$

Since $y(t)$ is not a constant function, one may conclude that $c_1 = 0$. □

Since $c_1 = 0$, and $\phi = -\arctan(c_1/c_2)$, we have $\phi = 0$.

We now present our process for finding a solution to the ‘teardrop problem”.

(1) With $c_1 = 0$, we know $\phi = 0$, and $R = c_2$. We make a guess as to the value of c_2. After several trials, we find that $c_2 = 137$ gives reasonable results. We will later change the value of R to be exact.

(2) The functions given by (21) are periodic, and represent the derivatives of x and y. Two conditions that must be met are $x(1) = x(0)$ and $y(1) = y(0)$. We find the period T such that $x(t) = x(t \pm T)$ and $y(t) = y(t \pm T)$. We find a value for c_3 such that $\int_0^T y'(t)\,dt = 0$ via numeric integration. (The numeric integrator is not exact, but is correct to five decimal places).

(3) The constant c_4 merely shifts $x'(t)$ and $y'(t)$ in time. We use a root finder to find a value for c_4 such that $x'(0) = 1$ and $y'(0) = 0$.

(4) We finally fit all the boundary conditions by rescaling the parameter c_3 so that the period is 1. We do this by making the substitution $c_3 \to c_3/T$ and $R \to R/T^2$ so that the value of $\sqrt{R}/c_3$ is unchanged.

(5) Finally, we plot the result, as shown in Figure 3.

Through a similar process to the steps above, but with different boundary conditions, we arrive at a solution for the “bump problem”, where the ends of a flexible material on a flat surface are pushed in, keeping the ends flat. This solution is shown in Figure 4. Table 1 gives the values of the various constants for both solutions. We

Curve	Bound. cond. (goal)	Bound. cond. (actual)	Parameter values
	$x(0) = 0$	$x(0) = 0$	
	$x'(0) = 1$	$x'(0) = 1$	$R = 54.2257$
	$x(1) = 0$	$x(1) = 0$	$\phi = 0$
teardrop,	$x'(1) = -1$	$x(1) = -1$	$c_3 = 5.38408$
Figure 3	$y(0) = 0$	$y(0) = 0$	$c_4 = 0.174661$
	$y'(0) = 0$	$y'(0) = 0$	$c_5 = 0$
	$y(1) = 0$	$y(1) = -2.676 \times 10^{-5}$	$c_6 = 0$
	$y'(1) = 0$	$y'(1) = 0$	
	$x(0) = 0$	$x(0) = 0$	
	$x'(0) = 1$	$x'(0) = 1$	$R = 209.804$
	$x(1) = .7822$	$x(1) = .7822$	$\phi = 0$
bump,	$x'(1) = 1$	$x(1) = 1$	$c_3 = 3.06405$
Figure 4	$y(0) = 0$	$y(0) = 0$	$c_4 = 0$
	$y'(0) = 0$	$y'(0) = 0$	$c_5 = 0$
	$y(1) = 0$	$y(1) = 0$	$c_6 = 0$
	$y'(1) = 0$	$y'(1) = 0$	

Table 1. The constants derived in order to force (22) to meet the specified boundary conditions.

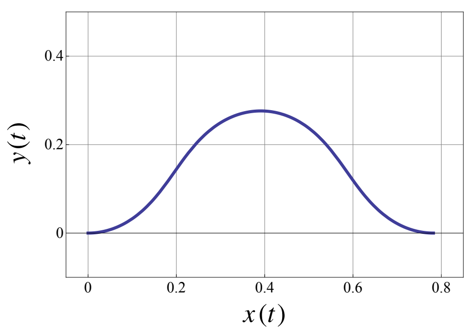

Figure 4. A solution to the bump problem, in which the ends of a strip of a flexible material are pushed together along a flat surface.

do not assert that the teardrop or bump solutions we found here match boundary conditions exactly, nor do we insist that these are unique. We merely emphasize that they are exact solutions to (11) which *nearly* match specified boundary conditions.

These problems lend themselves to analytic solutions, but others are not so easy. To solve more difficult problems, we found standard numeric differential equation solving techniques to be fruitless. But, in harmony with the variational nature of this problem, there is an optimization technique we can apply: evolutionary algorithms.

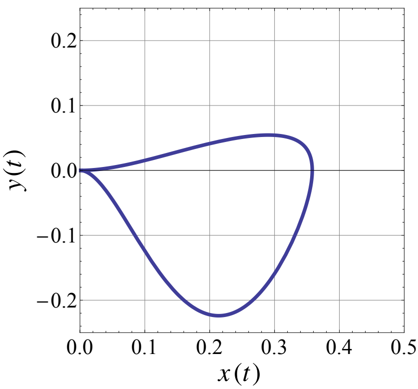

Figure 5. A curve generated by random-coefficient polynomials with boundary conditions enforced.

Evolutionary algorithms. As the name suggests, evolutionary algorithms utilize the same principles as evolutionary biology for selective optimization. The purpose of our evolutionary algorithm is to find a polynomial approximation to a shape that satisfies given boundary conditions and minimizes stored energy.

We assume that the solutions to (11) are analytic, and therefore have a Taylor series expansion. Then, we generate a fixed number N of (uniformly distributed) random-coefficient polynomial pairs, called a *generation*. Each pair forms a parametric curve that satisfies our boundary conditions and has length 1. (See Figure 5). We do not, however, require the resulting parametric curve to have constant arc-length. This is because the arc-length constraint is impossible to enforce with nonlinear polynomials, and the image generated by a pair of polynomials can also be traced out by a constant arc-length function.

We then arrange the N members of the generation, ordering the members by their stored energy calculated via numeric integration of (7). The worst 90% (meaning those with the highest energy) are then removed from the population. Only the most fit members of a generation remain. Here, "most fit" is deemed to mean those with the least stored energy. We introduce a mutation factor by perturbing the coefficients of the remaining polynomials by multiplying them by a randomly determined factor. Finally, we "breed" these polynomials to generate a new generation of N curves. The breeding process consists of randomly selecting two distinct curves and constructing a new polynomial by taking a randomly weighted average of every pair of coefficients. We also constrain the new curve to satisfy the specified boundary conditions and normalize its arc-length.

The members of the new generation have characteristics of their predecessors. Since only the lowest-energy curves are selected from each generation, generally the average energy per curve from each generation is no greater than that of the previous generation. Thus, repeating the process for many generations gives us increasingly

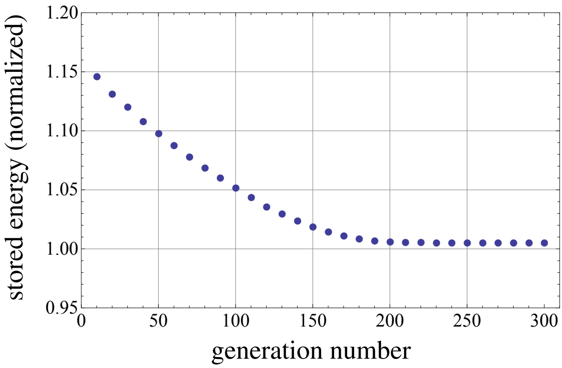

Figure 6. The energy of the best curve in the population for each generation. The scale of the energies is normalized so that the energy of the exact solution is 1.

accurate approximations to the lowest-energy curve conforming to our specifications. Figure 6 shows how the energy of the best curve in the population decreases with each generation. While the evolutionary algorithm does not provide an exact solution to our differential equation, it does provide a polynomial approximation to the image of a curve that does solve the differential equation. As an example, we have applied the evolutionary algorithm to the teardrop problem, and obtained the approximation in Figure 7 after more than 1000 generations. We can calculate the stored energy in this approximation by integrating (7) and compare it to the stored energy in the exact solution, as presented in Table 2.

The polynomials to obtain the best approximation curve are given by

$$x(t) = \sum_{n=0}^{6} a_n t^n, \qquad y(t) = \sum_{n=0}^{9} b_n t^n.$$

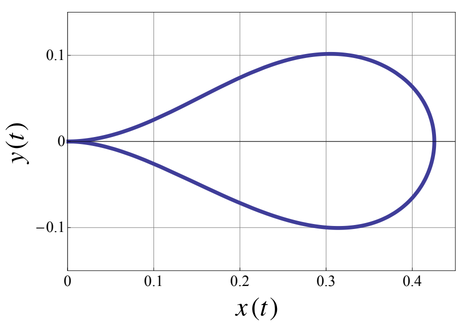

Figure 7. An approximate solution to the teardrop problem.

Curve	Figure	Stored energy (normalized)
random curve	5	2.2781
best approximation	7	1.0054
exact solution	3	1

Table 2. The total stored energy of the exact solution is slightly lower than the stored energy of the best approximation. We also include the energy of the random curve to show how close the approximation is.

The coefficients a_i and b_i are

$$a_0 = 0, \quad a_1 = 1.34525, \quad a_2 = 0.184519, \quad a_3 = -2.8019,$$
$$a_4 = -0.568997, \quad a_5 = 4.0875, \quad a_6 = -2.24638,$$

$$b_0 = 0, \quad b_1 = 0, \quad b_2 = -6.64108, \quad b_3 = 28.9681,$$
$$b_4 = -36.8535, \quad b_5 = 3.26174, \quad b_6 = 1.8281, \quad b_7 = 60.7485,$$
$$b_8 = -83.0815, \quad b_9 = 31.7697.$$

Nonlinear, nonconstant rigidity solutions. While we can find nonlinear solutions in the constant rigidity case, we have been unable to find exact solutions in the nonconstant rigidity case. Nevertheless, we can find some interesting numerical solutions. The evolutionary algorithm abandons the arc-length constraint, while the rigidity function relies on arc-length parametrization. To use the evolutionary algorithm, therefore, requires a reparametrization of $\alpha(t)$, which is a hard problem in general. Alternatively, we could calculate the value of $\alpha(t)$ for every iteration of the numeric integrator, but such a process is computationally slow. Therefore, we are content with solving (12) subject to initial conditions, with no guarantee as to the resulting boundary conditions. We provide two examples. Figure 8 shows a nonconstant rigidity function that takes the form of an inverted Gaussian curve.

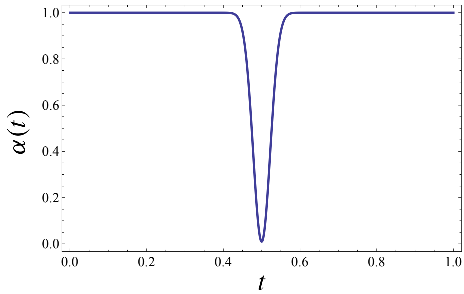

Figure 8. A rigidity function $\alpha(t)$ designed to model a crease in the material.

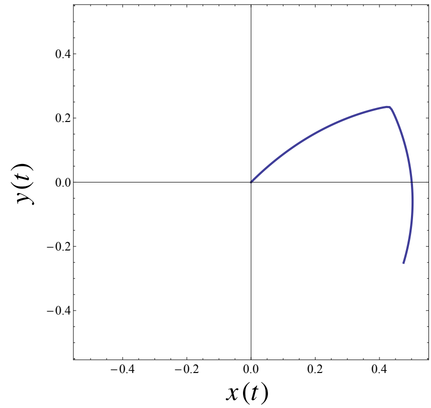

Figure 9. A solution to (12) with respect to the rigidity function shown in Figure 8.

This is meant to model a material with a crease. The resulting solution is given in Figure 9. Figure 10 shows a nonconstant rigidity function that takes the form of a hyperbolic tangent curve. This is meant to model a material that is loose on one end and stiff on the other, with a very sharp transition (imagine a piece of rubber attached to a steel bar). The resulting solution is given in Figure 11.

Finding rigidity. We now consider the second question that was posed in the introduction: given specific boundary conditions, can one vary the rigidity of the material to ensure it will bend to a specified shape? We may rewrite (12) as

$$(\alpha\theta')' = c_1 \cos\theta - c_2 \sin\theta = c_1 x' + c_2 y',$$

where c_1 and c_2 may be chosen freely (as long as they are both nonzero). We may integrate and solve for α as

$$\alpha = \frac{1}{\theta'}(c_1 x - c_2 y). \tag{25}$$

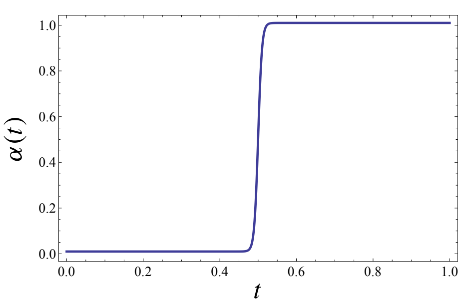

Figure 10. A rigidity function $\alpha(t)$ deigned to model a material with one loose end and one stiff end.

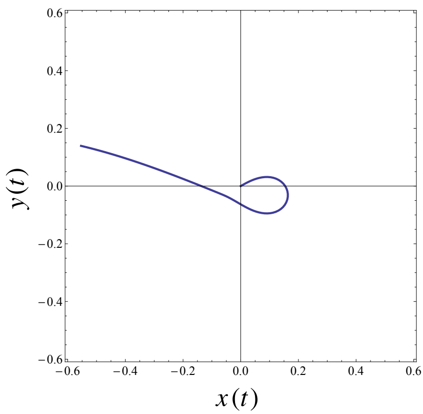

Figure 11. A solution to (12) with respect to the rigidity function shown in Figure 10.

Recall that $\theta(t)$ is defined to be the function satisfying

$$x'(t) = \cos(\theta(t)) \quad \text{and} \quad y'(t) = \sin(\theta(t)).$$

But taking derivatives of these equations yields

$$x'' = -\theta' \sin\theta = -\theta' y' \quad \text{and} \quad y'' = \theta' \cos\theta = \theta' x'$$

so that [2]

$$\theta' = -\frac{x''}{y'} = \frac{y''}{x'}.$$

Substituting this back into (25) gives the nice identity[3]

$$\alpha(t) = \frac{x'(t)}{y''(t)}\big(c_1 x(t) - c_2 y(t)\big) = -\frac{y'(t)}{x''(t)}\big(c_1 x(t) - c_2 y(t)\big). \tag{26}$$

Thus, if one wants a material with shape given by $(x(t), y(t))$, equation (26) gives the variable rigidity function that will make the material conform to that shape.[4] This method, of course, may not work if $x''(t) = 0$ or $y''(t) = 0$ at a point. A careful choice of c_1 and c_2 may mitigate this problem in some cases.

4. Conclusion

In this paper, we have explored analytic and numeric methods for solving the extreme deflection problem on a strip of flexible material. We used the calculus

[2] There is consistency between this equation and differentiating the arc-length constraint $(x')^2 + (y')^2 = 1$.

[3] This may be recast into several forms, but these are the simplest.

[4] Note that this equation is only valid if $\alpha(t) > 0$.

of variations to derive a system of differential equations (11), which has solutions in several forms. In particular, we confirmed results from Euler and others by deriving (21). We provided a technique for applying boundary conditions to these solutions. Often, fitting boundary conditions is difficult, so we have also explored numeric methods for approximating solutions. We also solved for what rigidity function a material would need in order for it to conform to a given shape.

Next steps. There is still much research that can be done on the elastica problem. Currently, we are only able to solve (11) subject to boundary conditions to numerical accuracy. Is there a way to find an exact solution for general boundary conditions? Can we prove that these solutions are unique?

One could also explore the consequences of varying the bending in another direction. That is, rather than examining a strip of flexible material bent in one direction, one could examine a sheet bent in two directions. The curvature for these surfaces is much more complicated, but the applications to solutions for this problem are more abundant.

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alex.safsten@gmail.com *Mathematics Department, Brigham Young University, 295 TMCB, Provo, UT 84602, United States*

logantatham@gmail.com *Mathematics Department, Brigham Young University, 295 TMCB, Provo, UT 84602, United States*

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