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Adjacency matrices of zero-divisor graphs of integers modulo n

Matthew Young

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We study adjacency matrices of zero-divisor graphs of $\mathbb{Z}_n$ for various n. We find their determinant and rank for all n, develop a method for finding nonzero eigenvalues, and use it to find all eigenvalues for the case $n = p^3$, where p is a prime number. We also find upper and lower bounds for the largest eigenvalue for all n.

1. Introduction

Let R be a commutative ring with a unity. The notion of a *zero-divisor graph* of R was pioneered by Beck [1988]. It was later modified by Anderson and Livingston [1999] to be the following.

Definition 1.1. *The zero-divisor graph* $\Gamma(R)$ of the ring R is a graph with the set of vertices $V(R)$ being the set of zero-divisors of R and edges connecting two vertices $x, y \in R$ if and only if $x \cdot y = 0$.

To each (finite) graph Γ, one can associate the *adjacency matrix* $A(\Gamma)$ that is a square $|V(\Gamma)| \times |V(\Gamma)|$ matrix with entries $a_{ij} = 1$, if v_i is connected with v_j, and zero otherwise. In this paper we study the adjacency matrices of zero-divisor graphs $\Gamma_n = \Gamma(\mathbb{Z}_n)$ of rings $\mathbb{Z}_n$ of integers modulo n, where n is not prime. We note that the adjacency matrices of zero-divisor graphs of $\mathbb{Z}_p \times \mathbb{Z}_p$, $\mathbb{Z}_p[i]$, and $\mathbb{Z}_p[i] \times \mathbb{Z}_p[i]$, where p is a prime number and $i^2 = -1$, were studied in [Sharma et al. 2011].

2. Properties of adjacency matrices of Γ_n

Let $n = p_1^{t_1} \cdots p_s^{t_s}$, where $p_1, \dots, p_s$ are distinct primes. For any divisor d of n, we define $S(d) = \{k \in \mathbb{Z}_n \mid \gcd(k, n) = d\}$. If $d = p_1^{a_1} \cdots p_s^{a_s}$, we will also write $S(d) = S(a_1, \dots, a_s)$. We can easily compute the size of the sets $S(d)$.

Proposition 2.1. *For a divisor d of n, the cardinality of the set $S(d)$ is equal to $|S(d)| = \phi(n/d)$, where ϕ denotes Euler's totient function.*

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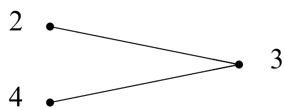
Proof. A positive integer m less than n is contained in the set $S(d)$ if and only if $\gcd(n,m) = d$, which happens if and only if $m = \hat{m}d$ and $\gcd(n/d, \hat{m}) = 1$. Thus, there is a one-to-one correspondence between the element m of $S(d)$ and integers $\hat{m}$, where $0 < \hat{m} < n/d$ and $\gcd(n/d, \hat{m}) = 1$. $\square$

Example 2.2. We illustrate Proposition 2.1 with the zero-divisor graph Γ_6 and its adjacency matrix:



$$A(\Gamma_6) = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad S(2) = \{2,4\}, \quad |S(2)| = \phi(6/2) = 2.$$

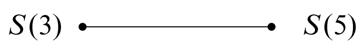
Theorem 2.3. *If* $n > 4$, *then* $\det A(\Gamma_n) = 0$.

Proof. Write $n = p_1^{t_1} p_2^{t_2} \cdots p_s^{t_s}$ as above. For $i = 1, \ldots, s$, if $p_i > 2$, then $|S(n/p_i)| = p_i - 1 > 1$. If a vertex v of Γ_n corresponding to some divisor d of n is adjacent to one of the elements of $S(n/p_i)$, then $d \cdot (n/p_i) = 0 \pmod n$. Thus the product of d with any multiple of n/p_i is also zero, and v is adjacent to every element of $S(n/p_i)$. So $A(\Gamma_n)$ will have repeated rows corresponding to each element of $S(n/p_i)$. Since $|S(n/p_i)| > 1$, we conclude that $\det A(\Gamma_n) = 0$.

If $n{=}2^t$, we must have $t > 2$. Then $6 \in S(2)$, and $|S(2)| > 1$. So $A(\Gamma_n)$ will have repeated rows corresponding to 2 and 6, and $\det A(\Gamma_n) = 0$. $\square$

The sets $S(d)$ for all divisors d of a given integer n are an *equitable partition* of the set of vertices $V(\Gamma_n)$. That is, any two vertices in $S(d_i)$ have the same number of neighbors in $S(d_j)$ for all divisors d_i, d_j of n. This allows us to define a *projection graph* $\pi\Gamma_n$ as a graph with vertices $S(d)$ for all $d|n$ and edges connecting $S(d_i)$ with $S(d_j)$ if every element in $S(d_i)$ is connected with every element in $S(d_j)$ in Γ_n.

Example 2.4. The projection graph $\pi\Gamma_{15}$:



Proposition 2.5. *The number of vertices in the graph* $\pi\Gamma_n$, *where* $n = \prod_{i=1}^{s} p_i^{t_1}$, *is*

$$|V(\pi\Gamma_n)| = \prod_{i=1}^{s} (t_i + 1) - 2.$$

Proof. The vertices of $\pi\Gamma_n$ are the sets $S(d)$ that are in one-to-one correspondence with the divisors d of n. If the prime decomposition for n is $n = \prod_{i=1}^{s} p_i^{t_i}$ and the prime decomposition for a divisor d of n is $d = \prod_{i=1}^{s} p_i^{a_i}$, then we have $t_i + 1$

choices for the exponent a_i of p_i in d. The choice of all $a_i = 0$ leads to $d = 1$, and the choice of each $a_i = t_i$ leads to $d = n$, neither of which are proper divisors of n. So the number of proper divisors d is $\prod_{i=1}^{s}(t_i + 1) - 2$. ☐

Let $A(\pi(\Gamma_n))$ denote the adjacency matrix of $\pi(\Gamma_n)$. We will also consider the weighted adjacency matrix $\mathcal{A}(\pi(\Gamma_n))$, where $a_{ij} = |S(d_j)|$ whenever $S(d_i)$ is connected with $S(d_j)$. In the above example,

$$A(\pi(\Gamma_{15})) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \mathcal{A}(\pi(\Gamma_{15})) = \begin{bmatrix} 0 & 2 \\ 4 & 0 \end{bmatrix}.$$

The following theorem relates the ranks of the various adjacency matrices.

Theorem 2.6. *Let $n = \prod_{i=1}^{s} p_i^{t_i}$. Then,*

$$\operatorname{rank} A(\Gamma_n) = \operatorname{rank} A(\pi\Gamma_n) = \operatorname{rank} \mathcal{A}(\pi\Gamma_n) = \prod_{i=1}^{s}(t_i + 1) - 2.$$

Proof. Recall that vertices in $V(\pi\Gamma_n)$ correspond to sets $S(d)$, where $d|n$. Since each element of $S(d)$ contributes exactly the same row to the adjacency matrix $A(\Gamma_n)$, it follows that $\operatorname{rank} A(\Gamma_n) \leq |V(\pi\Gamma_n)|$. On the other hand, $\operatorname{rank} A(\pi\Gamma_n) \leq \operatorname{rank} A(\Gamma_n)$ since we just remove repeated rows and columns to get $A(\pi\Gamma_n)$ from $A(\Gamma_n)$. Obviously, $\operatorname{rank} A(\pi\Gamma_n) = \operatorname{rank} \mathcal{A}(\pi\Gamma_n)$. So it is enough to show $\operatorname{rank} A(\pi\Gamma_n) = |V(\pi\Gamma_n)|$.

Let $n = p_1^{t_1} p_2^{t_2} \cdots p_s^{t_s}$. Since the rank of the matrix does not change with permutations of rows, assume that the rows of $A(\pi\Gamma_n)$ correspond to the $S(d)$, with $d|n$, in the following order: $S(p_1)$, $S(n/p_1)$, $S(p_2)$, $S(n/p_2)$, $\ldots$, $S(p_s)$, $S(n/p_s)$, $S(p_i p_j)$ with i and j not necessarily distinct, $S(n/(p_i p_j))$ for all possible pairs of i and j, $S(p_i p_j p_k)$, $S(n/(p_i p_j p_k))$ for all possible triples i, j, k with i, j, k not necessarily distinct, etc.

We will compute the determinant of $A(\pi\Gamma_n)$ and, by showing that it is not zero, will prove that $|V(\pi\Gamma_n)|$ rows of $A(\pi\Gamma_n)$ are linearly independent.

The first row corresponding to $S(p_1)$ has 1 in the second column corresponding to $S(n/p_1)$ and the rest of the entries are 0. Expand the determinant of $A(\pi\Gamma_n)$ along the first row to get

$$\det A(\pi\Gamma_n) = -\det A_{1,2},$$

where $-\det A_{1,2}$ is the cofactor of the $(1, 2)$ entry of $A(\pi\Gamma_n)$. Note that the first column of $A_{1,2}$ has 1 in the first row and the rest of the entries are 0. Expand $\det A_{1,2}$ along the first column to get $\det A_{1,2} = \det A^{(2)}$, where $A^{(2)}$ is a matrix obtained from $A(\pi\Gamma_n)$ by deleting the first two rows and columns. So, we can conclude that $\det A(\pi\Gamma_n) = -\det A^{(2)}$.

We repeat this procedure for all $S(p_i)$ and $S(n/p_i)$ to get

$$\det A(\pi\Gamma_n) = (-1)^s \det A^{(2s)},$$

where $A^{(2s)}$ is obtained from $A(\pi\Gamma_n)$ by deleting the first $2s$ rows and columns.

Now consider $S(p_i p_j)$. In $\pi\Gamma_n$, the vertex corresponding to $S(p_i p_j)$ is adjacent to vertices of $S(n/p_i)$, $S(n/p_j)$ and $S(n/(p_i p_j))$. However, in the matrix $A^{(2s)}$, the row corresponding to $S(p_i p_j)$ will have only one 1 in the column corresponding to $S(n/(p_i p_j))$ since the columns corresponding to $S(n/p_i)$ and $S(n/p_j)$ were deleted. So we can repeat the procedure of expanding the determinant along the rows and columns corresponding to $S(p_i)$ to expand the determinant along the rows and then the columns corresponding to $S(n/(p_i p_j))$.

Then continue to $S(n/(p_i p_j p_k))$ in a similar fashion. In the end we will be left either with a 2×2 matrix of determinant -1, or a 1×1 matrix of determinant 1. So $\det A(\pi\Gamma_n) = (-1)^m$, where m is the number of distinct divisors d of n such that $d < \sqrt{n}$. It follows that $\operatorname{rank} A(\pi\Gamma_n) = |V(\pi\Gamma_n)|$. The result follows from Proposition 2.5. □

Corollary 2.7. *We have* $\det A(\pi\Gamma_n) = (-1)^m$, *where* $m = \lfloor(\operatorname{rank} A)/2\rfloor$.

Proof. This follows from the proof of Theorem 2.6. □

Corollary 2.8. *We have* $\det \mathcal{A}(\pi\Gamma_n) = (-1)^m \prod_{d|n} |S(d)|$, *where* $m = \lfloor(\operatorname{rank} \mathcal{A})/2\rfloor$.

Proof. This result follows from the previous corollary and the fact that $\mathcal{A}(\pi\Gamma_n)$ is obtained from $A(\pi\Gamma_n)$ by multiplying the j-th column by $|S(d_j)|$. □

Corollary 2.9. *The multiplicity of the eigenvalue* 0 *of* $A(\Gamma_n)$ *is*

$$n - \phi(n) - \prod_{i=1}^{s}(t_i + 1) + 2.$$

Proof. The multiplicity of the eigenvalue 0 of $A(\Gamma_n)$ is $|V(\Gamma_n)| - |V(\pi(\Gamma_n))|$. The number of vertices of Γ_n is the number of positive integers less than n and not relatively prime to n, which is $n - \phi(n)$. The number of vertices of $\pi\Gamma_n$ is

$$\prod_{i=1}^{s}(t_i + 1) - 2.$$ □

Part of the following result is known, but we will include it for the convenience of the reader.

Proposition 2.10. *A nonzero* $\lambda \in \mathbb{R}$ *is an eigenvalue of* $A(\Gamma_n)$ *if and only if it is an eigenvalue of* $\mathcal{A}(\pi(\Gamma_n))$.

Thus, to find all the nonzero eigenvalues of $A(\Gamma_n)$, it is enough to find all the eigenvalues of $\mathcal{A}(\pi(\Gamma_n))$. The proposition makes it especially easy when n has few factors. When $n = pq$ is a product of distinct primes, Γ_n is a bipartite graph. It is known (and easy to see) that the nonzero eigenvalues of $\mathcal{A}(\Gamma_{pq})$ are $\pm\sqrt{(p-1)(q-1)}$ with multiplicity 1. When $n = p^2$, the matrix $\mathcal{A}(\Gamma_n)$ is a 1×1 matrix with $p-1$ as a sole entry and hence the eigenvalue. So we consider the following two examples.

Example 2.11. Consider Γ_{p^3}. The matrix $\mathcal{A}(\Gamma_{p^3})$ takes the form

$$\mathcal{A} = \begin{bmatrix} 0 & p-1 \\ p(p-1) & p-1 \end{bmatrix}.$$

Its characteristic polynomial is $p(\lambda) = \lambda^2 - (p-1)\lambda - p(p-1)^2$, and the eigenvalues are $\lambda_{1,2} = \frac{1}{2}(p-1)(1 \pm \sqrt{1+4p})$.

Example 2.12. Consider Γ_{p^2q}, where p and q are distinct primes. In this case,

$$\mathcal{A}(\Gamma_{p^2q}) = \begin{bmatrix} 0 & p-1 & 0 & 0 \\ (p-1)(q-1) & p-1 & q-1 & 0 \\ 0 & p-1 & 0 & p^2-p \\ 0 & 0 & q-1 & 0 \end{bmatrix}.$$

The characteristic polynomial of this matrix is

$$p(\lambda) = \lambda^4 - (p-1)\lambda^3 - 2p(p-1)(q-1)\lambda^2 + p^2(p-1)(q-1)\lambda + p(p-1)^2(q-1)^3.$$

The roots can be found by the formulas for the roots of the fourth degree polynomial, but are too cumbersome to include here.

3. Estimates on eigenvalues

Since the increase in the number of factors of n leads to a rapid increase of the size of the adjacency matrix and the degree of the characteristic polynomial, one can use some known results to approximate the nonzero eigenvalues of $A(\Gamma_n)$. Since $A(\Gamma_n)$ is symmetric, all its eigenvalues are real numbers. We will number them from largest to smallest $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$.

The *degree* of a vertex of a graph is the number of the edges incident to this vertex. Given a (finite) graph Γ, let $\operatorname{maxdeg}(\Gamma) = \max\{\deg v \mid v \in V(\Gamma)\}$ and

$$\operatorname{avedeg}(\Gamma) = \frac{\sum_{v \in V(\Gamma)} \deg v}{|V(\Gamma)|}.$$

It is known (see [Brouwer and Haemers 2012, Proposition 3.1.2]) that

$$\operatorname{avedeg}(\Gamma) \leq \lambda_1 \leq \operatorname{maxdeg}(\Gamma).$$

We next compute $\operatorname{maxdeg}(\Gamma_n)$ and $\operatorname{avedeg}(\Gamma_n)$.

Proposition 3.1. *Let $n = p_1^{t_1} \cdots p_s^{t_s}$ with $p_1 < p_2 < \cdots < p_s$. Then*

$$\operatorname{maxdeg}(\Gamma_n) = n/p_1 - 1.$$

Proof. For a divisor d of n, denote the corresponding vertex in Γ_n by v_d. The vertex v_d is connected to a vertex v_c corresponding to a divisor c of n by an edge in Γ_n if and only if $dc \equiv 0 \pmod{n}$. This happens if and only if c is a multiple of n/d (and is less than n). There are $d-1$ such multiples, and so $\deg(v_d) = d-1$. Then the vertex with the largest degree will correspond to the largest divisor of n, which is n/p_1. Since for any $d|n$, the degrees of all vertices in $S(d)$ are the same and the sets $S(d)$ partition $V(\Gamma_n)$, $\operatorname{maxdeg}(\Gamma_n) = n/p_1 - 1$. □

Proposition 3.2. *The average degree of the graph Γ_n is*

$$\operatorname{avedeg}(\Gamma_n) = \frac{\sum_{d|n, d\neq n} \phi(n/d)(d-1)}{n - \phi(n) - 1}.$$

Proof. To compute the average degree, we take the degree $d-1$ of a vertex in $S(d)$, multiply by the cardinality $\phi(n/d)$ of the set $S(d)$, sum these products over all proper divisors d of n, and divide by the total number of vertices $n - \phi(n) - 1$. □

The estimate of the eigenvalue λ_1 using the average degree of the graph is inconvenient to use. So we use the results on interlacing and on bipartite subgraphs of Γ_n for alternative estimates that are easier to use. Let Γ be any graph and Δ an *induced subgraph*, that is, a subgraph obtained from Γ by deleting some vertices and all edges incident to the deleted vertices. The following result is known (see [Brouwer and Haemers 2012, Proposition 3.2.1] or [Godsil and Royle 2001, Theorem 9.1.1]). Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$ be eigenvalues of $A(\Gamma)$ and $\theta_1 \geq \cdots \geq \theta_l$ be the eigenvalues of $A(\Delta)$; then $\lambda_i \geq \theta_i \geq \lambda_{k-l+i}$ for $i = 1, 2, \ldots, l$. With the interlacing result in mind, we prove the following proposition.

Proposition 3.3. *Let λ_1 be the largest eigenvalue of $\mathcal{A}(\Gamma_n)$:*

(1) *If n is a product containing two or more distinct primes, then $\lambda_1 \geq \sqrt{\phi(n)}$.*

(2) *If $n = p^t$, then $\lambda_1 \geq p^{\lceil t/2 \rceil} - 1$.*

Proof. Let $n = p_1^{t_1} p_2^{t_2} \cdots p_s^{t_s}$. Consider the bipartite subgraph of Γ_n induced by the vertices in the sets $S(p_1^{t_1})$ and $S(n/p_1^{t_1})$. The largest eigenvalue μ_1 of this subgraph is

$$\mu_1 = \sqrt{|S(p_1^{t_1})| \left| S\left(\frac{n}{p_1^{t_1}}\right) \right|} = \sqrt{\phi\left(\frac{n}{p_1^{t_1}}\right) \phi(p_1^{t_1})} = \sqrt{\phi(n)}.$$

The interlacing results give $\mu_1 \leq \lambda_1$.

To prove the second statement, notice that all vertices of $S(p^i)$ are connected to all the vertices of $S(p^j)$ whenever $i+j \geq t$, $1 \leq i, j \leq t-1$. In the case t is even, Γ_n contains a complete subgraph induced by vertices in the sets $S(p^{t/2}), \ldots, S(p^{t-1})$. The largest eigenvalue μ_1 of this complete subgraph equals the number of vertices in the subgraph, which we compute next. By Proposition 2.1, $|S(d)| = \phi(n/d)$, so the number of vertices in the complete subgraph will be $\phi(p)+\phi(p^2)+\cdots+\phi(p^{t/2})$. This can be expressed as $(p-1)+p(p-1)+p^2(p-1)+\cdots+p^{t/2-1}(p-1)$, which after summing the arithmetic progression becomes $p^{t/2}-1=\mu_1$. In the case of t odd, Γ_n contains a complete subgraph induced by vertices in the sets $S(p^{\lceil t/2\rceil}), \ldots, S(p^{k-1})$. Using Proposition 2.1 and summing up the number of vertices as in case of t even gives $p^{\lfloor t/2\rfloor} = \mu_1$. □

We can use the above estimates on the largest eigenvalue λ_1 of $A(\Gamma_n)$ to prove that there are only finitely many graphs with small eigenvalues.

Theorem 3.4. *For any positive integer k, there exists only a finite number of integers n such that all the eigenvalues of $A(\Gamma_n)$ are less or equal than k.*

Proof. We will use the estimates μ_1 on λ_1 obtained in Proposition 3.3. If, for a given $k>0$, we have $\lambda_1 \leq k$, then $\mu_1 \leq k$. We will show $\mu_1 \leq k$ is only possible for a finite number of integers n. Suppose $n = p^t$. Then $p^{t/2}-1$ must be less than or equal to k. Thus $t \leq 2\log_p(k+1)$, and there are only finitely many such positive integers.

Now suppose n is divisible by at least two distinct primes, say $n = p_1^{t_1}\cdots p_s^{t_s}$. The Euler function on n can be computed as

$$\phi(n) = \phi(p_1^{t_1})\cdots\phi(p_s^{t_s}) = p_1^{t_1-1}(p_1-1)\cdots p_s^{t_s-1}(p_s-1).$$

Since there are only finitely many primes less than $k+1$, and only a finite number of possible exponents $t_1,\ldots,t_s$ that satisfy $0 < t_1,\ldots,t_s \leq 2\log_2 k$, there are only finitely many positive integers n such that $\phi(n) \leq k^2$. □

Example 3.5. We will find all positive integers n such that all the eigenvalues of $A(\Gamma_n)$ are less than or equal to $k=2$. Suppose $n=p^t$. If t is even, then we must have $p^{t/2}-1 \leq 2$. The only such possibilities are $p=2, t=2$ and $p=3, t=2$. If t is odd, we must have $p^{\lfloor t/2\rfloor} \leq 2$. This happens only if $p=2, t=3$. If n is divisible by at least two distinct primes, we must have $\phi(n) \leq 4$, and computations show that this is satisfied only for $n = 6, 10, 12$. For $n = 4, 6, 8, 9, 10, 12$, we compute the eigenvalues of $A(\Gamma_n)$ and see that for all above n except $n=12$, the eigenvalues are less or equal than 2. Thus, the adjacency matrices of Γ_4, Γ_6, Γ_8, Γ_9 and Γ_{10} have all their eigenvalues less or equal than 2.

Example 3.6. For $k=3$, the graphs all of whose eigenvalues are less than or equal to 3, in addition to the graphs of Example 3.5, are Γ_{12}, Γ_{14} and Γ_{15}. For $k=4$, the new graphs (in addition to the ones with eigenvalues less or equal to 3) are Γ_{16}, Γ_{21}, Γ_{22}, Γ_{25}, Γ_{26} and Γ_{34}.

As we were considering the above examples, we noticed that for adjacency matrices of even rank, precisely half of the nonzero eigenvalues were positive and half negative. For adjacency matrices of odd rank, we always had one more positive eigenvalue than negative. We investigate this further.

The independence number $\alpha(\Gamma)$ of a graph Γ is the size of the largest set of pairwise nonadjacent vertices. Let r denote the number of eigenvalues of a (weighted) adjacency matrix $A(\Gamma)$ of a graph Γ, $r_+(A)$ the number of positive eigenvalues, and $r_-(A)$ the number of negative eigenvalues. It is known (see [Brouwer and Haemers 2012, Theorem 3.5.4]) that $\alpha(\Gamma) \le r - r_+(A)$ and $\alpha(\Gamma) \le r - r_-(A)$. We use this fact to show the following result.

Theorem 3.7. *Suppose the rank of $A(\Gamma_n)$ is r. Then $A(\Gamma_n)$ has $\lceil r/2 \rceil$ positive eigenvalues and $\lfloor r/2 \rfloor$ negative eigenvalues.*

Proof. We first note that it is enough to prove the theorem for $\mathcal{A}(\pi\Gamma_n)$ since $A(\Gamma_n)$ and $\mathcal{A}(\pi\Gamma_n)$ have the same nonzero eigenvalues. We start by computing the independence number of $\pi\Gamma_n$. Recall that the vertices of $\pi\Gamma_n$ are the sets $S(d)$, for all divisors d of n, and $S(d_1)$ is connected to $S(d_2)$ by an edge if n divides the product $d_1 d_2$. So for all $d \le \sqrt{n}$, the vertices $S(d)$ are pairwise not connected. It is easy to see that all the vertices of $\pi\Gamma_n$ can be split into pairs $S(d)$ and $S(n/d)$, with $S(\sqrt{n})$ without a pair if $\sqrt{n}$ is a divisor of n. So the set of $S(d)$ with $d|n$ and $d < \sqrt{n}$ is the maximal nonadjacent set of cardinality $\lfloor r/2 \rfloor$, where r denotes the number of vertices of $\pi\Gamma_n$. The number of vertices of $\pi\Gamma_n$ is equal to the rank of $\mathcal{A}(\pi\Gamma_n)$ by Theorem 2.6. So the independence number $\alpha(\pi\Gamma_n)$ is equal to $\lfloor r/2 \rfloor$.

For r even, we have $r/2 \le r - r_+(\mathcal{A})$ and $r/2 \le r - r_-(\mathcal{A})$, which implies the statement of the theorem. For r odd, it remains to show that we have one more positive eigenvalue than negative. By Corollary 2.8, we know that the sign of $\det \mathcal{A}(\pi\Gamma_n)$ is given by $(-1)^{\lfloor r/2 \rfloor}$. On the other hand, $\det \mathcal{A}(\pi\Gamma_n)$ is equal to the product of eigenvalues. So the parity of the number of negative eigenvalues must determine the sign of $(-1)^{\lfloor r/2 \rfloor}$. Since $\lfloor r/2 \rfloor \le r - r_-(\mathcal{A})$ and $\lfloor r/2 \rfloor \le r - r_+(\mathcal{A})$, we must have $r_-(\mathcal{A}) = \lfloor r/2 \rfloor$. □

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