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Wang Xie

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Shock profile for gas dynamics in thermal nonequilibrium

Wang Xie

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In this paper, we study the existence and the properties of a shock profile for a system of thermal nonequilibrium gas dynamics. We find a neat condition to ensure the existence of the shock profile. Moreover, we calculate the shock profile solution explicitly.

1. Introduction

The motion of a gas in local thermodynamic equilibrium is governed by the compressible Euler equations. In Lagrangian coordinates, the equations for one-dimensional flow read (see [Courant and Friedrichs 1948])

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ \left(e + \frac{1}{2}u^2\right)_t + (pu)_x = 0, \end{cases} \tag{1-1}$$

where v, u, p and e are, respectively, the specific volume, velocity, pressure and internal energy of the gas. For an ideal gas,

$$e = \frac{1}{\gamma - 1} pv, \tag{1-2}$$

where $\gamma > 1$ is the adiabatic constant. During rapid changes in the flow the internal energy e may lag behind the equilibrium value corresponding to the ambient pressure and density. The translational energy adjusts quickly, but the rotational and vibrational energy may take an order of magnitude longer. If we suppose that α of the degrees of freedom adjust instantaneously but a further α_f degrees of freedom take longer to relax, we may take (see [Whitham 1974])

$$e = \frac{\alpha}{2} pv + q, \tag{1-3}$$

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where q is the energy in the lagging degrees of freedom. In equilibrium, q would have the value

$$q_{\text{equil}} = \frac{\alpha_f}{2} pv. \tag{1-4}$$

A simple overall equation to represent the relaxation is (in Lagrangian coordinates)

$$q_t = -\frac{1}{\tau}\Big(q - \frac{\alpha_f}{2} pv\Big), \tag{1-5}$$

where $\tau > 0$ is the relaxation time. Therefore, in thermal nonequilibrium, we have the following system of equations to model the gas motion:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ \big(\frac{1}{2}\alpha pv + q + \frac{1}{2}u^2\big)_t + (pu)_x = 0, \\ q_t = -\frac{1}{\tau}1\big(q - \frac{1}{2}\alpha_f pv\big). \end{cases} \tag{1-6}$$

If the relaxation time τ is taken to be so short that $q = (\alpha_f/2)pv$ is an adequate approximation to the last equation in (1-6), we have the following equilibrium theory:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p_x = 0, \\ \big(\frac{1}{2}(\alpha + \alpha_f)pv + \frac{1}{2}u^2\big)_t + (pu)_x = 0. \end{cases} \tag{1-7}$$

The three characteristic speeds for (1-7) are

$$\lambda_1 = -\sqrt{\Big(1 + \frac{2}{\alpha + \alpha_f}\Big)\frac{p}{v}}, \quad \lambda_2 = 0, \quad \lambda_3 = \sqrt{\Big(1 + \frac{2}{\alpha + \alpha_f}\Big)\frac{p}{v}}.$$

For the system (1-7), the setup $((v_-, u_-, p_-), (v_+, u_+, p_+), \sigma)$ with two constant states (v_-, u_-, p_-) and (v_+, u_+, p_+) and speed σ is called a shock wave (see [Courant and Friedrichs 1948]) if the Rankine–Hugoniot conditions

$$\begin{cases} -\sigma(v_+ - v_-) = (u_+ - u_-), \\ \ \sigma(u_+ - u_-) = (p_+ - p_-), \\ \sigma\Big(\big(\frac{1}{2}(\alpha+\alpha_f)p_+v_+ + \frac{1}{2}u_+^2\big) - \big(\frac{1}{2}(\alpha+\alpha_f)p_-v_- + \frac{1}{2}u_-^2\big)\Big) = (p_+u_+ - p_-u_-) \end{cases} \tag{1-8}$$

hold, and some other entropy conditions hold, where v_-, v_+, p_-, p_+ are positive constants, u_- and u_+ are constants. A shock wave is called a 1-shock wave if

$$-\sqrt{\Big(1 + \frac{2}{\alpha + \alpha_f}\Big)\frac{p_-}{v_-}} > \sigma > -\sqrt{\Big(1 + \frac{2}{\alpha + \alpha_f}\Big)\frac{p_+}{v_+}}. \tag{1-9}$$

A shock wave is called a 3-shock wave if

$$\sqrt{\left(1+\frac{2}{\alpha+\alpha_f}\right)\frac{p_-}{v_-}} > \sigma > \sqrt{\left(1+\frac{2}{\alpha+\alpha_f}\right)\frac{p_+}{v_+}}. \tag{1-10}$$

In this paper, we consider a 3-shock wave, because a 1-shock wave can be handled by the same method. For a 3-shock wave, it follows from (1-8) and (1-10) that

$$v_- < v_+, \quad u_- > u_+, \quad p_- > p_+. \tag{1-11}$$

A shock profile for the 3-shock wave $((v_-, u_-, p_-), (v_+, u_+, p_+), \sigma)$ is a traveling-wave solution for system (1-6) of the form $(v, u, p, q)((x-\sigma t)/\tau)$ satisfying

$$(v, u, p, q)(\pm\infty) = (v_\pm, u_\pm, p_\pm, (\alpha_f/2)p_\pm v_\pm). \tag{1-12}$$

So we have

$$\begin{cases} -\sigma v' - u' = 0, \\ -\sigma u' + p' = 0, \\ -\sigma\left(\frac{1}{2}\alpha p v + q + \frac{1}{2}u^2\right)' + (pu)' = 0, \\ -\sigma q' = -\left(q - \frac{1}{2}\alpha_f p v\right), \end{cases} \tag{1-13}$$

where $' = d/d\xi$ and $\xi = (x-\sigma t)/\tau$.

In this paper, we are interested in the existence and properties of the shock profile. For a general hyperbolic system with relaxation, the existence of the shock profile has been proved in [Yong and Zumbrun 2000] by using the center manifold method with the assumption that the shock strength is sufficiently small. In this paper, we find the sufficient and necessary condition, which is

$$\frac{p_-}{p_+} < 1 + \frac{2\alpha_f}{\alpha(1+\alpha+\alpha_f)},$$

to ensure the existence of the shock profile. Moreover, we can calculate the shock profile solution in some explicit details. This is in sharp contrast to the abstract construction in [Yong and Zumbrun 2000]. Before we state our theorem, we introduce some notation. Let

$$\begin{aligned} m &= \sigma v_- + u_- = \sigma v_+ + u_+, \\ P &= -\sigma u_- + p_- = -\sigma u_+ + p_+, \\ Q &= -\sigma\left(\tfrac{1}{2}\alpha + \alpha_f p_- v_- + \tfrac{1}{2}u_-^2\right) + p_- u_- \\ &= -\sigma\left(\tfrac{1}{2}\alpha + \alpha_f p_+ v_+ + \tfrac{1}{2}u_+^2\right) + p_+ u_+, \\ f(v) &= -\sigma^2(1+\alpha)v + \left(1+\tfrac{1}{2}\alpha\right)(\sigma^2 v_- + p_-) \\ &= -\sigma^2(1+\alpha)v + \left(1+\tfrac{1}{2}\alpha\right)(\sigma^2 v_+ + p_+). \end{aligned} \tag{1-14}$$

Theorem. *Suppose the two constant states* (v_-, u_-, p_-), (v_-, u_-, p_-) *and the speed* σ *satisfy the Rankine–Hugoniot conditions* (1-8) *and the Lax shock condition* (1-10).

(1) *If*

$$\frac{p_-}{p_+} < 1 + \frac{2\alpha_f}{\alpha(1+\alpha+\alpha_f)}, \tag{1-15}$$

then there exists a solution to the problem (1-13) *and* (1-12).

(2) *If*

$$\frac{p_-}{p_+} \geq 1 + \frac{2\alpha_f}{\alpha(1+\alpha+\alpha_f)}, \tag{1-16}$$

the problem (1-13) *and* (1-12) *does not admit a smooth solution.*

(3) *In case* (1), *that is, if* (1-15) *holds, the solution of the problem* (1-13) *and* (1-12) *satisfying* $v(0) = v_0$ *for some constant* v_0 *satisfying* $v_- < v_0 < v_+$ *is given by*

$$\begin{aligned} 2f(v_+)\big(\ln(v_+-v)-\ln(v_+-v_0)\big)-2f(v_-)\big(\ln(v-v_-)-\ln(v_0-v_-)\big) \\ = -\sigma\xi(1+\alpha+\alpha_f)(v_+-v_-), \end{aligned} \tag{1-17}$$

$$u(\xi) = m - \sigma v(\xi), \quad p(\xi) = m\sigma + P - \sigma^2 v(\xi) \tag{1-18}$$

for $-\infty < \xi < +\infty$. *For this solution, we have*

$$v'(\xi) > 0, \quad u'(\xi) < 0, \quad p'(\xi) < 0 \tag{1-19}$$

for $-\infty < \xi < +\infty$, *and*

$$C_1 v'(\xi) \leq \exp\left(-\frac{1+\alpha+\alpha_f}{2f(v_+)}\sigma(v_+-v_-)\xi\right) \leq C_2(v_+ - v(\xi)) \tag{1-20}$$

for $\xi > 0$, *and*

$$C_3 v'(\xi) \leq \exp\left(\frac{1+\alpha+\alpha_f}{2f(v_-)}\sigma(v_+-v_-)\xi\right) \leq C_4(v(\xi) - v_-) \tag{1-21}$$

for $\xi < 0$, *where* C_i $(i = 1, 2, 3, 4)$ *are some positive constants. For* $u(\xi)$ *and* $p(\xi)$, *we have similar estimates.*

2. Proofs

To prove our theorem, we start by integrating (1-13) to get

$$\begin{cases} \sigma v + u = m, \\ -\sigma u + p = P, \\ -\sigma\left(\frac{1}{2}\alpha p v + q + \frac{1}{2}u^2\right) + pu = Q, \end{cases} \tag{2-1}$$

where m, P and Q are given by (1-14). By the third equation of (2-1), we have

$$q=-\left(\frac{\alpha}{2}pv+\frac{u^2}{2}\right)+\frac{pu-Q}{\sigma}. \tag{2-2}$$

Substituting (2-2) into the fourth equation of (1-13), using (2-1) and (2-2), we get

$$f(v)\frac{dv}{d\xi}=\frac{1}{\sigma}\left(\frac{\alpha+\alpha_f}{2}pv+\frac{u^2}{2}-\frac{pu-Q}{\sigma}\right), \tag{2-3}$$

where $f(v)$ is given by (1-14). So

$$f(v_-)=v_-\left(-\frac{\alpha}{2}\sigma^2+\left(1+\frac{\alpha}{2}\right)\frac{p_-}{v_-}\right). \tag{2-4}$$

In view of (1-10), we have

$$\begin{aligned}-\frac{\alpha}{2}\sigma^2+\left(1+\frac{\alpha}{2}\right)\frac{p_-}{v_-}&>-\frac{\alpha}{2}\left(1+\frac{2}{\alpha+\alpha_f}\right)\frac{p_-}{v_-}+\left(1+\frac{\alpha}{2}\right)\frac{p_-}{v_-}\\&=\left(1-\frac{\alpha}{\alpha+\alpha_f}\right)\frac{p_-}{v_-}>0.\end{aligned} \tag{2-5}$$

Therefore

$$f(v_-)>0.$$

By (1-14), we get

$$f(v_+)=v_+\big(-\tfrac{1}{2}\alpha\sigma^2+\big(1+\tfrac{1}{2}\alpha\big)p_+/v_+\big). \tag{2-6}$$

Let

$$\bar{v}=\frac{1+\frac{1}{2}\alpha}{1+\alpha}\,\frac{\sigma^2v_++p_+}{\sigma^2}. \tag{2-7}$$

Then

$$f(\bar{v})=0. \tag{2-8}$$

So, if

$$v_+<\bar{v}, \tag{2-9}$$

then, because f is a decreasing function,

$$f(v_+)>0. \tag{2-10}$$

In the next lemma, we will give a neat condition to ensure (2-10).

Lemma. (1) *If* $\dfrac{p_-}{p_+}<1+\dfrac{2\alpha_f}{\alpha(1+\alpha+\alpha_f)}$, *then* $v_+<\bar{v}$ *and thus* $f(v_+)>0$.

(2) *If* $\dfrac{p_-}{p_+}=1+\dfrac{2\alpha_f}{\alpha(1+\alpha+\alpha_f)}$, *then* $v_+=\bar{v}$ *and* $f(v_+)=0$.

(3) *If* $\dfrac{p_-}{p_+} > 1 + \dfrac{2\alpha_f}{\alpha(1+\alpha+\alpha_f)}$, *then* $v_+ > \bar{v}$ *and* $f(v_+) < 0$.

Proof. First, we use (1-8) to show that

$$\tfrac{1}{2}(\alpha+\alpha_f)(p_+v_+ - p_-v_-) = (v_- - v_+)\tfrac{1}{2}(p_+ + p_-). \tag{2-11}$$

In fact, by the third equation of (1-8), we have

$$\begin{aligned}\tfrac{1}{2}(\alpha+\alpha_f)(p_+v_+ - p_-v_-) &= \frac{1}{\sigma}(p_+u_+ - p_-u_-) - \tfrac{1}{2}(u_+^2 - u_-^2)\\ &= \frac{1}{\sigma}(p_+u_+ - p_-u_-) - \tfrac{1}{2}(u_+ + u_-)(u_+ - u_-).\end{aligned} \tag{2-12}$$

By the second equation of (1-8), we have $(u_+ - u_-) = (1/\sigma)(p_+ - p_-)$. This, together with (2-12), implies that

$$\begin{aligned}\tfrac{1}{2}(\alpha+\alpha_f)(p_+v_+ - p_-v_-) &= \frac{1}{\sigma}\big(p_+u_+ - p_-u_- - \tfrac{1}{2}(p_+ - p_-)(u_+ + u_-)\big)\\ &= \frac{1}{\sigma}\big(\tfrac{1}{2}p_+u_+ - \tfrac{1}{2}p_-u_- - \tfrac{1}{2}p_+u_- + \tfrac{1}{2}p_-u_+\big)\\ &= \frac{1}{\sigma}(u_+ - u_-)(p_+ + p_-).\end{aligned} \tag{2-13}$$

This proves (2-11). Dividing by p_-v_+ both sides of (2-11), we get

$$\frac{\alpha+\alpha_f}{2}\left(\frac{p_+}{p_-} - \frac{v_-}{v_+}\right) = \left(\frac{v_-}{v_+} - 1\right)\frac{(p_+/p_-)+1}{2}.$$

We solve for v_-/v_+ from this to get

$$\frac{v_-}{v_+} = \frac{(\alpha+\alpha_f)(p_+/p_-) + (p_+/p_-) + 1}{(\alpha+\alpha_f) + (p_+/p_-) + 1}. \tag{2-14}$$

It is easy to verify that $v_+ < \bar{v}$ is equivalent to

$$\sigma^2 < \left(1 + \frac{2}{\alpha}\right)\frac{p_+}{v_+}. \tag{2-15}$$

From the first and second equations of (1-8), we know that

$$\sigma^2 = \frac{p_- - p_+}{v_+ - v_-}. \tag{2-16}$$

So $v_+ < \bar{v}$ is equivalent to

$$\frac{p_- - p_+}{v_+ - v_-} < \left(1 + \frac{2}{\alpha}\right)\frac{p_+}{v_+}. \tag{2-17}$$

Now we use (2-14) to show (2-17) if (1-15) is true.

By (2-14), we have

$$
\begin{aligned}
\frac{p_- - p_+}{v_+ - v_-} &= \frac{(p_-/p_+) - 1}{1 - (v_-/v_+)}(p_+/v_+) \\
&= \frac{(p_-/p_+) - 1}{1 - \dfrac{(\alpha+\alpha_f)(p_+/p_-) + (p_+/p_-) + 1}{(\alpha+\alpha_f) + (p_+/p_-) + 1}}\left(\frac{p_+}{v_+}\right) \\
&= \frac{(p_-/p_+) - 1}{(\alpha+\alpha_f)(1 - (p_+/p_-))}\big[(\alpha+\alpha_f) + (p_+/p_-) + 1\big]\left(\frac{p_+}{v_+}\right) \\
&= \frac{(p_-/p_+) - 1}{1 - (p_+/p_-)}\left(1 + \frac{(p_+/p_-) + 1}{\alpha+\alpha_f}\right)\left(\frac{p_+}{v_+}\right) \\
&= \frac{(p_-/p_+)(p_-/p_+ - 1)}{(p_-/p_+) - 1}\left(1 + \frac{(p_+/p_-) + 1}{\alpha+\alpha_f}\right)\left(\frac{p_+}{v_+}\right) \\
&= \left(\frac{p_-}{p_+}\right)\left(1 + \frac{(p_+/p_-) + 1}{\alpha+\alpha_f}\right)\left(\frac{p_+}{v_+}\right) \\
&= \left(\frac{p_-}{p_+} + \frac{1}{\alpha+\alpha_f}\left(1 + \frac{p_-}{p_+}\right)\right)\left(\frac{p_+}{v_+}\right) \\
&= \left(\left(1 + \frac{1}{\alpha+\alpha_f}\right)\left(\frac{p_-}{p_+}\right) + \frac{1}{\alpha+\alpha_f}\right)\left(\frac{p_+}{v_+}\right).
\end{aligned}
\tag{2-18}
$$

So, if (1-15) holds, then we have

$$
\begin{aligned}
&\left(1 + \frac{1}{\alpha+\alpha_f}\right)\frac{p_-}{p_+} + \frac{1}{\alpha+\alpha_f} \\
&\qquad < \left(1 + \frac{1}{\alpha+\alpha_f}\right)\left(1 + \frac{2\alpha_f}{\alpha(1+\alpha+\alpha_f)}\right) + \frac{1}{\alpha+\alpha_f} \\
&\qquad = 1 + \frac{1}{\alpha+\alpha_f}\left(2 + \frac{2\alpha_f}{\alpha(1+\alpha+\alpha_f)}\right) + \frac{2\alpha_f}{\alpha(1+\alpha+\alpha_f)} \\
&\qquad = 1 + \frac{1}{\alpha+\alpha_f}\left(\frac{2\alpha(1+\alpha+\alpha_f) + 2\alpha_f}{\alpha(1+\alpha+\alpha_f)}\right) + \frac{2\alpha_f}{\alpha(1+\alpha+\alpha_f)} \\
&\qquad = 1 + \frac{2}{\alpha+\alpha_f}\,\frac{(\alpha+\alpha_f)(1+\alpha)}{\alpha(1+\alpha+\alpha_f)} + \frac{2\alpha_f}{\alpha(1+\alpha+\alpha_f)} \\
&\qquad = 1 + \frac{2(1+\alpha)}{\alpha(1+\alpha+\alpha_f)} + \frac{2\alpha_f}{\alpha(1+\alpha+\alpha_f)} \\
&\qquad = 1 + \frac{2}{\alpha}.
\end{aligned}
\tag{2-19}
$$

Inequality (2-17) follows from (2-18) and (2-19). This proves item (1) of the Lemma. Items (2) and (3) follow from the same arguments. □

Proof of the Theorem. Let

$$G(v,u,p)=(\alpha+\alpha_f)pv+u^2-\frac{2(pu-Q)}{\sigma}. \tag{2-20}$$

It follows from (2-1) and (2-20) that

$$\begin{aligned}G(v,u,p)=(\alpha+\alpha_f)(m\sigma+P-\sigma^2v)v+(m-\sigma v)^2\\ -\frac{2}{\sigma}\big((m-\sigma v)(m\sigma+P-\sigma^2v)-Q\big),\end{aligned} \tag{2-21}$$

where m, P, Q are given in (1-14). Therefore, $G(v,u,p)$ is a function of the single variable v, and we simply write it $G(v)$ from now on. It is a quadratic function. Moreover, by (1-14), we have

$$G(v_+)=G(v_-)=0. \tag{2-22}$$

Therefore,

$$G(v)=-\beta(v-v_-)(v-v_+) \tag{2-23}$$

for some constant β. By comparing (2-23) with (2-21), we get $\beta=\sigma^2(1+\alpha+\alpha_f)$. Hence,

$$G(v)=-\sigma^2(1+\alpha+\alpha_f)(v-v_-)(v-v_+). \tag{2-24}$$

So

$$G(v)>0 \tag{2-25}$$

as $v_-<v<v_+$.

In case (1) of the Theorem, we choose a constant v_0 satisfying $v_-<v_0<v_+$ and set $v(0)=v_0$. Then we have from (2-3) that

$$\int_{v_0}^{v}\frac{2\sigma f(v)}{G(v)}\,dv=\xi, \tag{2-26}$$

and consider the expression

$$\int_{v_0}^{v}\frac{2\sigma f(s)}{G(s)}\,dv=F(v),$$

where $F(v)=\xi$.

Also, by the Lemma, and (2-24), we have, if

$$\frac{p_-}{p_+}<1+\frac{2\alpha_f}{\alpha(1+\alpha+\alpha_f)},$$

then $f(v)/G(v) > 0$ for $v_- < v < v_+$, and

$$\int_{v_0}^{v_+} \frac{2\sigma f(v)}{G(v)}\,dv = +\infty, \tag{2-27}$$

$$\int_{v_0}^{v_-} \frac{2\sigma f(v)}{G(v)}\,dv = -\infty. \tag{2-28}$$

Therefore, $\xi = F(v)$ is an increasing mapping of (v_-, v_+) onto (∞_-, ∞_+), which clearly maps v_0 to 0. Thus the inverse mapping $\xi \to v(\xi)$ is a differentiable function (with a positive derivative) and is one-to-one and onto from $(-\infty, +\infty)$ to (v_-, v_+) with $v(0) = v_0$. Moreover, it follows from (2-26) and (2-27) that

$$v(-\infty) = v_-, \quad v(+\infty) = v_+.$$

Therefore the substitution $s = v(t)$ gives

$$\xi = \int_0^{\xi} \frac{2\sigma f(v(t))}{G(v(t))} v'(t)\,dt$$

and differentiation gives

$$1 = \frac{2\sigma f(v(\xi))}{G(v(t))} v'(\xi),$$

and so we have a solution v of (2-3), which proves part (1) of the Theorem.

We prove part (2) as follows. If

$$\frac{p_-}{p_+} \geq 1 + \frac{2\alpha_f}{\alpha(1+\alpha+\alpha_f)}, \tag{2-29}$$

by the Lemma, we know that $v_- < \bar{v} \leq v_+$. In this case, we use the proof by contradiction to prove (2) as follows. Suppose that the problem (1-12) and (1-13) has a solution $v(\xi)$. Since, in this case, $v_- < \bar{v} \leq v_+$, and $f'(v) < 0$, we have $f(v_-) > 0 \geq f(v_+)$. We may write (2-3) as

$$2\sigma f(v)\frac{dv}{d\xi} = G(v). \tag{2-30}$$

Since $v(-\infty) = v_-$ and $f(v) > 0$ for $v_- < v < \bar{v}$ and $G(v) > 0$ for $v_- < v < v_+$, we have $dv/d\xi > 0$ when $v_- < v < \bar{v} \leq v_+$. For a constant v_1 satisfying $v_- < v_1 < \bar{v}$, there exists $\xi_1 \in (-\infty, +\infty)$ such that $v(\xi_1) = v_1$. It follows from (2-30) that

$$\int_{v_1}^{v} \frac{2\sigma f(w)}{G(w)}\,dw = \xi - \xi_1. \tag{2-31}$$

By (1-14), we know that $f(v)$ is a linear function of v in the form

$$f(v) = -k(v - \bar{v}), \tag{2-32}$$

where $k = \sigma^2(1+\alpha)$. It follows from (2-23), and the fact that $v_- < v < \bar{v} \le v_+$ (when (2-29) holds), that

$$\int_{v_1}^{\bar{v}} \frac{2\sigma f(w)}{G(w)}\,dw < +\infty. \tag{2-33}$$

We let

$$\bar{\xi} = \xi_1 + \int_{v_1}^{\bar{v}} \frac{2\sigma f(w)}{G(w)}\,dw.$$

By (2-33),

$$-\infty < \bar{\xi} < \infty. \tag{2-34}$$

It follows from (2-31) that, as

$$v(\xi) \to \bar{v}, \quad \xi \to \bar{\xi}. \tag{2-35}$$

This will lead to a contradiction by the following argument. By (2-30), (2-32) and (2-22), we have

$$\frac{dv}{d\xi} = \frac{G(v)}{2\sigma f(v)} = \frac{\beta}{2\sigma}\frac{(v-v_-)(v-v_+)}{v-\bar{v}}. \tag{2-36}$$

Therefore, if

$$\frac{p_-}{p_+} > 1 + \frac{2\alpha_f}{\alpha(1+\alpha+\alpha_f)},$$

then $\bar{v} < v_+$, and then by (2-35) and (2-36), we have $dv(\xi)/d\xi \to +\infty$ as $\xi \to \bar{\xi}$. This is a contradiction due to (2-34) because the solution $v(\xi)$ is smooth, so its derivative cannot tend to $+\infty$ for finite ξ.

If

$$\frac{p_-}{p_+} = 1 + \frac{2\alpha_f}{\alpha(1+\alpha+\alpha_f)},$$

then $\bar{v} = v_+$, and then (2-36) reduces to

$$\frac{dv}{d\xi} = \frac{\beta}{2\sigma}(v-v_-). \tag{2-37}$$

So $dv/(v-v_-) = (\beta/2\sigma)\,d\xi$. In this case, (2-31) becomes

$$\int_{v_1}^{v} \frac{dw}{w-v_-}\,dw = \frac{\beta}{2\sigma}(\xi-\xi_1). \tag{2-38}$$

This implies that

$$\ln(v-v_-) - \ln(v_1-v_-) = (\beta/2\sigma)(\xi-\xi_1).$$

Solving this for v, we obtain

$$v(\xi) = v_- + (v_1 - v_-)e^{(\beta/2\sigma)(\xi-\xi_1)}.$$

Hence, $v(\xi) \to \infty$ as $\xi \to \infty$. Therefore, it is impossible that $v(+\infty) = v_+$. This is a contradiction. Thus, part (2) of the Theorem is proved by the above argument.

We can prove part (3) as follows. We have already proved that, if

$$\frac{p_-}{p_+} < 1 + \frac{2\alpha_f}{\alpha(1+\alpha+\alpha_f)},$$

the problem (1-13) and (1-12) has a solution. In this case, $v'(\xi) > 0$ is an easy consequence of the above argument in (i). So $v_- < v(\xi) < v_+$ for $-\infty < \xi < +\infty$. Next, we prove (1-17). We may write (2-26) as, in view of (2-26) and (2-24),

$$\int_{v_0}^{v} \frac{2f(w)}{(w-v_-)(w-v_+)}\,dw = -\sigma(1+\alpha+\alpha_f)\xi. \tag{2-39}$$

It is easy to verify that, by noting that $f(w) = (1+\alpha/2)(\sigma m + P) - \sigma^2(1+\alpha)w$ (see (1-14)),

$$\frac{2f(w)}{(w-v_-)(w-v_+)} = \frac{-2f(v_-)}{(w-v_-)}\frac{1}{(v_+-v_-)} + \frac{2f(v_+)}{(w-v_+)}\frac{1}{(v_+-v_-)}. \tag{2-40}$$

Equation (1-17) then follows from (2-39) and (2-40). From (1-17), we can easily get the bounds for $v_+ - v(\xi)$ in (1-20). Similarly, we can get the bounds for $v(\xi) - v_-$ in (1-21). By (1-17), we have

$$\left(\frac{2f(v_+)}{v_+-v} + \frac{2f(v_-)}{v-v_-}\right)v'(\xi) = -\sigma(1+\alpha+\alpha_f)(v_+-v_-).$$

Therefore, the bounds for $v'(\xi)$ in (1-20) and (1-21) can be derived from the bounds of $v_+ - v(\xi)$ and $v(\xi) - v_-$ which we have just proved. This finishes the proof of the Theorem. □

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xwang1994@gmail.com *Sidwell Friends Upper School, 3825 Wisconsin Avenue, Washington, DC 20016, United States*

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