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Jacobson's refinement of Engel's theorem for Leibniz algebras

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We develop Jacobson's refinement of Engel's Theorem for Leibniz algebras. We then note some consequences of the result.

Since Leibniz algebras were introduced in [Loday 1993] as a noncommutative generalization of Lie algebras, one theme has been to extend Lie algebra results to Leibniz algebras. In particular, Engel's theorem has been extended in [Ayupov and Omirov 1998; Barnes 2011; Patsourakos 2007]. In the second of these works, the classical Engel's theorem is used to give a short proof of the result for Leibniz algebras. The proofs in the other two papers do not use the classical theorem and, therefore, the Lie algebra result is included in the result. In this note, we give two proofs of the generalization to Leibniz algebras of Jacobson's refinement to Engel's theorem, a short proof which uses Jacobson's theorem and a second proof which does not use it. It is interesting to note that the technique of reducing the problem to the special Lie algebra case significantly shortens the proof for the general Leibniz algebras case. This approach has been used in a number of situations [Barnes 2011]. We also note some standard consequences of this theorem. The proofs of the corollaries are exactly as in Lie algebras (see [Kaplansky 1971]). Our result can be used to directly show that the sum of nilpotent ideals is nilpotent, and hence one has a nilpotent radical. In this paper, we consider only finite dimensional algebras and modules over a field $\mathbb{F}$.

An algebra A is called Leibniz if it satisfies $x(yz) = (xy)z + y(xz)$. Denote by R_a and L_a, respectively, right and left multiplication by $a \in A$. Then

$$R_{bc} = R_c R_b + L_b R_c, \tag{1}$$

$$L_b R_c = R_c L_b + R_{bc}, \tag{2}$$

$$L_c L_b = L_{cb} + L_b L_c. \tag{3}$$

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Using (1) and (2) we obtain

$$R_c R_b = -R_c L_b. \tag{4}$$

It is known that $L_b = 0$ if $b = a^i$, $i \geq 2$, where $a^1 = a$ and a^n is defined inductively as $a^{n+1} = aa^n$. Furthermore, for $n > 1$, $R_a^n = (-1)^{n-1} R_a L_a^{n-1}$. Therefore R_a is nilpotent if L_a is nilpotent.

For any set X in an algebra, we let $\langle X \rangle$ denote the algebra generated by X. Using (1), $R_{a^2} = (R_a)^2 + L_a R_a$. Furthermore, the associative algebra generated by all R_b, L_b, $b \in \langle a \rangle$ is equal to $\langle R_a, L_a \rangle$. Suppose that $L_a^{n-1} = 0$. Then $R_a^n = 0$. For any $s \in \langle R_a, L_a \rangle$, s^{2n-1} is a combination of terms with each term having at least $2n-1$ factors. Moreover, each of these factors is either L_a or R_a. Any L_a to the right of the first R_a can be turned into an R_a using (4). Hence, any term with $2n-1$ factors can be converted into a term with either L_a in the first $n-1$ leading positions or R_a in the last n postitions. In either case, the term is 0 and $s^{2n-1} = 0$. Thus $\langle R_a, L_a \rangle$ is nil and hence nilpotent.

Let M be an A-bimodule and let $T_a(m) = am$ and $S_a(m) = ma$, $a \in A$, $m \in M$. The analogues of (1)–(4) hold:

$$S_{bc} = S_c S_b + T_b S_c, \tag{5}$$

$$T_b S_c = S_c T_b + S_{bc}, \tag{6}$$

$$T_c T_b = T_{cb} + T_b T_c, \tag{7}$$

$$S_c S_b = -S_c T_b. \tag{8}$$

These operations have the same properties as L_a and R_a, and the associative algebra $\langle T_a, S_a \rangle$ generated by all T_b, S_b, $b \in \langle a \rangle$ is nilpotent if T_a is nilpotent. We record this as

Lemma. *Let A be a finite dimensional Leibniz algebra and let $a \in A$. Let M be a finite dimensional A-bimodule such that T_a is nilpotent on M. Then S_a is nilpotent, and $\langle S_a, T_a \rangle$, the algebra generated by all S_b, T_b, $b \in \langle a \rangle$, is nilpotent.*

A subset of A which is closed under multiplication is called a Lie set.

Theorem (Jacobson's refinement of Engel's theorem for Leibniz algebras). *Let A be a finite dimensional Leibniz algebra and M be a finite dimensional A-bimodule. Let C be a Lie set in A such that $A = \langle C \rangle$. Suppose that T_c is nilpotent for each $c \in C$. Then, for all $a \in A$, the associative algebra $B = \langle S_a, T_a \rangle$ is nilpotent. Consequently B acts nilpotently on M, and there exists $m \in M$, $m \neq 0$, such that $am = ma = 0$ for all $a \in A$.*

Proof 1 (*using the Lie result*). If M is irreducible, then either $MA = 0$ or $ma = -am$ for all a in A and all m in M from [Barnes 2011, Lemma 1.9]. Since left multiplication of A on M gives a Lie module, the Jacobson refinement to Engel's theorem yields that A acts nilpotently on M on the left and hence on M as a bimodule. If M is not irreducible, then A acts nilpotently on the irreducible factors in a composition series of M and hence on M. □

Proof 2 (*independent of the Lie result*). Let $x \in C$. Then T_x is nilpotent and the associative algebra generated by T_b and S_b for all $b \in \langle x \rangle$ is nilpotent by the lemma. Since $\{a \mid aM = 0 = Ma\}$ is an ideal in A, we may assume that A acts faithfully on M.

Let D be a Lie subset of C such that $\langle D \rangle$ acts nilpotently on M, and $\langle D \rangle$ is maximal with these properties. If $C \subseteq \langle D \rangle$, then $A = \langle C \rangle = \langle D \rangle$, and we are done. Thus suppose that $C \nsubseteq \langle D \rangle$, and we will obtain a contradiction.

Let $E = \langle D \rangle \cap C$. E is a Lie set since both $\langle D \rangle$ and C are Lie sets. Since $D \subseteq \langle D \rangle$ and $D \subseteq C$, it follows that $D \subseteq E$ and $\langle D \rangle \subseteq \langle E \rangle$. Since $E \subseteq \langle D \rangle$, $\langle E \rangle \subseteq \langle D \rangle$ and $\langle D \rangle = \langle E \rangle$.

Let $\dim(M) = n$. Since $\langle D \rangle = \langle E \rangle$ acts nilpotently on M, $\sigma_1 \cdots \sigma_n = 0$ where $\sigma_i = S_{d_i}$ or T_{d_i} for $d_i \in E$. Then:
$\sigma_1 \cdots \sigma_i \tau \sigma_{i+1} \cdots \sigma_{2n-1} = 0$ where $\tau = S_a$ or T_a, $a \in A$, for all i.

If x is any product in A with $2n$ terms, of which $2n - 1$ come from E, then S_x and T_x are linear combinations of elements as in the last paragraph. Hence $S_x = T_x = 0$, which implies that $x = 0$, since the representation is faithful.

There exists a smallest positive integer j such that $\tau_1 \cdots \tau_j C \subseteq \langle E \rangle$ for all $\tau_1, \ldots, \tau_j$ with $\tau_i = R_{d_i}$ or L_{d_i} where $d_i \in E$. Then there exists an expression $z = \tau_{d_1} \cdots \tau_{d_{j-1}} x \notin \langle E \rangle$ for some $x \in C$ and $d_i \in E$. Note that $z \in C$ since C is a Lie set. Consider zE. Now, zC, $Cz \subseteq C$ and $z\langle E \rangle$, $\langle E \rangle z \subseteq \langle E \rangle$. Therefore zE, $Ez \subseteq E$. Then $z^n E$, $Ez^n \subseteq E$ for all positive integers n, using induction and the defining identity for Leibniz algebras. Then $F = \{z^n, n \geq 1\} \cup E$ is a Lie set contained in C, and since $z \notin \langle E \rangle$, it follows that $\langle E \rangle \subsetneq \langle F \rangle$.

It remains to show that $\langle F \rangle$ acts nilpotently on M. Define $M_0 = 0$ and

$$M_i = \{m \in M \mid Em, mE \subseteq M_{i-1}\}.$$

Since E acts nilpotently on M, $M_k = M$ for some k. We show zM_i, $M_i z \subseteq M_i$. Clearly $zM_0 = M_0 z = 0$. Suppose that z acts invariantly on M_i for all $i < t$. For $m \in M_t$, $d \in E$, $(zm)d = z(md) - m(zd) \in zM_{t-1} + mE \subseteq M_{t-1}$ with similar expressions for $(mz)d$, $d(mz)$ and $d(zm)$. Thus z acts invariantly on each M_i, and hence z^2 does also. Thus $\langle z \rangle$ acts invariantly on each M_i. But $\langle z \rangle$ acts nilpotently on M by the lemma. Hence F acts nilpotently on M, which is a contradiction. □

We obtain the abstract version of the theorem.

Corollary 1. *Let C be a Lie set in a Leibniz algebra A such that $\langle C \rangle = A$ and L_c is nilpotent for all $c \in C$. Then A is nilpotent.*

The following are extensions of results from [Jacobson 1955], whose proofs are the same as in the Lie algebra case.

Corollary 2. *If T is an automorphism of A of order p and has no nonzero fixed points, then A is nilpotent.*

Corollary 3. *If D is a nonsingular derivation of A over a field of characteristic* 0, *then A is nilpotent.*

Corollary 4. *If B and C are nilpotent ideals of A, then $B + C$ is a nilpotent ideal of A.*

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