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# On the associated primes of the third power of the cover ideal

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An algebraic approach to graph theory involves the study of the edge ideal and the cover ideal of a given graph. While a lot is known for the associated primes of powers of the edge ideal, much less is known for the associated primes of the powers of the cover ideal. The associated primes of the cover ideal and its second power are completely determined. A configuration called a *wheel* is shown to always appear among the associated primes of the third power of the cover ideal.

## 1. Introduction

We start with some definitions and notation, for which we follow [Harris et al. 2008; Villarreal 2001]. A (finite) *graph*  $G$  consists of two finite sets, the vertex set  $V_G$  and the edge set  $E_G$ , whose elements are unordered pairs of vertices. An edge  $\{x_i, x_j\} \in E_G$  is written  $x_i x_j$  (or  $x_j x_i$ ). If  $x_i x_j$  is an edge, we say that the vertices  $x_i$  and  $x_j$  are *adjacent* and that the edge is *incident* to  $x_i$  and  $x_j$ . All our graphs will be *simple*, meaning that the only possible edges are  $x_i x_j$  for  $i \neq j$ .

A subset  $C \subseteq V_G$  is a (*vertex*) *cover* of  $G$  if each edge in  $E_G$  is incident to a vertex in  $C$ . A cover  $C$  is *minimal* if no proper subset of  $C$  is a cover of  $G$ .

The results of this paper are in the area of algebraic graph theory, where algebraic methods are used to investigate properties of graphs. Indeed, a graph  $G$  with vertex set  $V_G = \{x_1, \dots, x_n\}$  can be related to the polynomial ring  $R = k[x_1, \dots, x_n]$ , where  $k$  is a field. In the following we take the liberty of referring to  $x_i$  as a variable in the polynomial ring and as a vertex in the graph  $G$ , without any further specification. Given a ring  $R$ , we denote by  $(f_1, \dots, f_l)$  the ideal of  $R$  generated by the elements  $f_1, \dots, f_l \in R$ .

Two ideals of the polynomial ring  $R = k[x_1, \dots, x_n]$  that have proven most useful in studying the properties of a graph  $G$  with vertex set  $V_G = \{x_1, \dots, x_n\}$

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and edge set  $E_G$  are the *edge ideal*

$$I_G = (x_i x_j \mid x_i x_j \in E_G)$$

and the *cover ideal*

$$J_G = (x_{i_1} \cdots x_{i_k} \mid x_{i_1}, \dots, x_{i_k} \text{ is a minimal cover of } G).$$

Both are square-free monomial ideals, that is, they are generated by monomials in which each variable appears at most one time.

One of the most basic tools in commutative algebra to study an ideal  $I$  of a noetherian ring  $R$  is to compute the finite set of *associated prime ideals* of  $I$ , which is denoted by  $\text{Ass}(R/I)$  (for details, see [Eisenbud 1995]). In the case of a monomial ideal  $L$  in a polynomial ring  $S = k[x_1, \dots, x_n]$ , an element in  $\text{Ass}(S/L)$  is a *monomial prime ideal*, which is an ideal generated by a subset of the variables. Because of this fact we can record the following definition.

**Definition.** Let  $L$  be a monomial ideal in the polynomial ring  $S = k[x_1, \dots, x_n]$  and let  $P = (x_{i_1}, \dots, x_{i_s})$  be a monomial prime ideal. If there exists a monomial  $m$  such that  $x_{i_j} m \in L$  for each  $j = 1, \dots, s$  and  $x_i m \notin L$  for every  $i \neq i_1, \dots, i_s$  then  $P$  is an *associated prime* to  $L$ . We denote by  $\text{Ass}(S/L)$  the set of all associated (monomial) primes of  $L$ .

Chen et al. [2002] gave a constructive method for determining primes associated to the powers of the edge ideal, but much less is known about cover ideals. It is known that, given a graph  $G$  and its cover ideal  $J_G$ , a monomial prime ideal  $P$  is in  $\text{Ass}(S/J_G)$  if and only if  $P = (x_i, x_j)$  and  $x_i x_j$  is an edge of  $G$  (see [Villarreal 2001], for example).

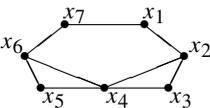
The initial point of our investigation is a result of Francisco, Ha and Van Tuyl (Theorem 1.1 below) describing the associated primes of the ideal  $(J_G)^2$ .

Let  $G$  be a graph. A *path* in  $G$  is a sequence of distinct vertices  $x_1, x_2, \dots, x_k$  such that  $x_j x_{j+1} \in E_G$  for  $j = 1, 2, \dots, k-1$ . The *length* of such a path is  $k-1$ , one less than the number of vertices. If  $x_k x_1$  is also an edge of  $G$ , we say that the graph  $C$  with vertex set  $\{x_1, x_2, \dots, x_k\}$  and edge set  $\{x_1 x_2, \dots, x_{k-1} x_k, x_k x_1\}$  is a *cycle* (in  $G$ ). A cycle with an odd number of vertices is also called an *odd hole*.

Given a graph  $G$  and a set of vertices  $W \subseteq V_G$ , the *graph generated by  $W$*  has vertex set  $W$  and edge set  $\{xy \mid xy \in E_G, x \in W, y \in W\}$ .

**Theorem 1.1** [Francisco et al. 2010]. *Let  $G$  be a graph with vertex set  $\{x_1, \dots, x_n\}$ , edge set  $E_G$  and cover ideal  $J_G$ . A monomial prime ideal  $P = (x_{i_1}, \dots, x_{i_k})$  of the polynomial ring  $S = k[x_1, \dots, x_n]$  is in the set  $\text{Ass}(S/J_G^2)$  if and only if either*

- $k = 2$  and  $x_{i_1} x_{i_2} \in E_G$ , or
- $k$  is odd and the graph generated by  $x_{i_1}, \dots, x_{i_k}$  is an odd hole.

As an example, if  $G$  is the graph  we will have

$$\text{Ass}(J) =$$

$$\{(x_1, x_2), (x_1, x_7), (x_2, x_3), (x_2, x_4), (x_3, x_4), (x_4, x_5), (x_4, x_6), (x_5, x_6), (x_6, x_7)\}$$

(the associated prime of  $J$  consists of the primes generated by two variables that correspond to the edges of the graph), and

$$\text{Ass}(J^2) = \text{Ass}(J) \cup \{(x_2, x_3, x_4), (x_4, x_5, x_6), (x_1, x_2, x_4, x_6, x_7)\}$$

(the associated prime of  $J_G^2$  contains all the primes that are either generated by two variables corresponding to edges or generated by three variables corresponding to odd cycles of  $G$ ).

In this paper we study the associated primes of the third power of the cover ideal, the ideal  $J_G^3$ . We prove that the primes generated by the variables corresponding to the vertices of a *wheel* (see next definition) always appear among the associated primes of  $J_G^3$ . This result is connected with the coloring number of a graph, as discussed at the end of Section 2.

The algebra system Macaulay2 was used for all the computations in this paper, and in particular in finding the pattern that led to the main theorem.

## 2. Centered odd holes and the main theorem

**Definition.** A graph  $C$  is said to be a *wheel* if  $V_C = V_H \cup \{y\}$ , where  $H$ , called the *rim* of  $C$ , is an odd hole such that the graph generated by  $H$  in  $C$  is  $H$  itself, and  $y$ , called the *center* of  $C$ , is a vertex adjacent in  $C$  to at least three vertices of  $H$  and belonging to at least two odd cycles in  $C$ . (It follows that  $y$  belongs to at least three odd cycles in  $C$ .) The rim  $H$  and center  $y$  are part of the data needed to specify a wheel, as they may not be uniquely determined by  $C$ .

Let  $C$  be a wheel with rim  $H$  and center  $y$ . A vertex  $x \in V_H$  is *radial* if  $xy$  is an edge of  $C$ . Let there be  $k$  radial vertices, labeled consequently  $x_1, \dots, x_k$  in order around the wheel. We leave it to the reader to specify precisely what this means. For  $i = 1, \dots, k - 1$ , we denote by  $l_i$  the length of the path in  $H$  joining  $x_i$  to  $x_{i+1}$  (and not going through any other radial vertex). Similarly  $l_k$  denotes the length of the path in  $H$  from  $x_k$  to  $x_1$ .

For the main theorem we will need the following lemma, where we use the notation  $||$  for the *size* (that is, the number of vertices) of a graph.

**Lemma 2.1.** *Let  $C$  be a wheel with rim  $H$  and center  $y$ , and let  $k$  be its radial number. If  $W$  is a vertex cover for  $C$  that contains  $y$ , then  $|W| \geq |C|/2 + 1$ . If  $W$  is a vertex cover for  $C$  that does not contain  $y$ , then*

$$|W| \geq k + \left\lfloor \frac{l_1 - 1}{2} \right\rfloor + \dots + \left\lfloor \frac{l_k - 1}{2} \right\rfloor.$$

Moreover,

$$k + \left\lfloor \frac{l_1-1}{2} \right\rfloor + \dots + \left\lfloor \frac{l_k-1}{2} \right\rfloor \geq \frac{|C|}{2} + 1. \tag{2-1}$$

*Proof.* Let  $V_H$  be the vertex set of  $H$ . Assume that  $W$  contains the vertex  $y$ . The vertex set  $W \cap V_H$  has to be a vertex cover for  $H$ . Moreover, since  $H$  is an odd hole, the cardinality of  $W \cap V_H$  has to be at least  $(|H| + 1)/2$ , which is equal to  $|C|/2$ . Therefore the cardinality of  $W$  is  $|C|/2 + 1$ .

Assume now that  $W$  does not contain the vertex  $y$ . Let  $x_1, \dots, x_k$  be the radial vertices. Since  $y \notin W$ , all the radial vertices are in  $W$ . As  $W \cap V_H$  is a cover of  $H$ , in the path from  $x_i$  to  $x_{i+1}$  we need at least  $\lfloor (l_i - 1)/2 \rfloor$  vertices, for  $i = 1, \dots, k - 1$ , and we need  $\lfloor (l_k - 1)/2 \rfloor$  vertices for the path from  $x_k$  to  $x_1$ .

To prove (2-1) we write

$$\begin{aligned} k + \left\lfloor \frac{l_1-1}{2} \right\rfloor + \dots + \left\lfloor \frac{l_k-1}{2} \right\rfloor &\geq k + \frac{l_1-1}{2} + \dots + \frac{l_k-1}{2} \geq \frac{l_1}{2} + \dots + \frac{l_k}{2} + \frac{k}{2} \\ &\geq \frac{l_1 + \dots + l_k + 1}{2} + \frac{k-1}{2} \geq \frac{|C|}{2} + 1, \end{aligned}$$

where in the last inequality we used the fact that  $k \geq 3$ . □

In the following we will make an abuse of notation: if  $G$  is a graph with vertices  $x_1, \dots, x_n$  and  $H$  is a subgraph generated by the vertices  $x_{i_1}, \dots, x_{i_k}$ , by  $H$  we also denote the prime monomial ideal  $(x_{i_1}, \dots, x_{i_k})$  in the polynomial ring  $k[x_1, \dots, x_n]$ . Here is our main theorem.

**Theorem 2.2.** *Let  $G$  be a graph with vertex set  $V_G = \{x_1, \dots, x_n\}$  and assume that  $G$  has a subgraph  $C$  which is a wheel. Let  $S = k[x_1, \dots, x_n]$  and let  $J$  be the cover ideal of  $G$ . Then the set  $\text{Ass}(S/J^3)$  is not contained in the set  $\text{Ass}(S/J^2)$ , and in fact  $C \in \text{Ass}(S/J^3) \setminus \text{Ass}(S/J^2)$ .*

*Proof.* By Lemma 2.11 in [Francisco et al. 2011], we may assume that  $G = C$ . Let  $y$  be the center of the wheel  $C$ , and let  $x_1, x_2, \dots, x_k$  be the radial vertices. Denote by  $x_{i_j}$ , for  $j = 1, \dots, l_i - 1$ , the vertices between  $x_i$  and  $x_{i+1}$  if  $i < k$  and the vertices between  $x_k$  and  $x_1$  if  $i = k$ .

That  $C$  is not in  $\text{Ass}(S/J^2)$  follows from Theorem 1.1, since  $C$  is neither an odd hole nor an edge.

To show that  $C$  is in  $\text{Ass}(S/J^3)$  we need to find a monomial  $c$  such that  $c \notin J^3$  and  $xc \in J^3$  for each vertex  $x$  of  $C$ . Let  $c$  be the monomial

$$c = y^2 \prod_{i=1, \dots, k} x_i^2 \prod_{\substack{i=1, \dots, k \\ j=1, \dots, l_i-1}} x_{ij}^a, \quad \text{where } a = \begin{cases} 1 & \text{if } j \text{ is odd,} \\ 2 & \text{if } j \text{ is even.} \end{cases}$$

To show that  $c$  is the desired monomial, we first prove that

$$\deg c = k + 2 + n + \left\lfloor \frac{l_1-1}{2} \right\rfloor + \dots + \left\lfloor \frac{l_k-1}{2} \right\rfloor. \tag{2-2}$$

Let  $n$  be the size of  $H$ . For a monomial  $m$  we denote by  $\deg m$  the degree of  $m$ . In computing  $\deg c$ , the contribution from the variables  $y$  and  $x_i$ , for  $i = 1, \dots, k$ , is given by  $2k + 2$ . For  $i = 1, \dots, k - 1$ , between  $x_i$  and  $x_{i+1}$ , there are  $l_i - 1$  vertices, and there are  $l_k - 1$  vertices between  $x_k$  and  $x_1$ . Given an integer  $s$ , there are  $\lfloor s/2 \rfloor$  even integers and  $\lceil s/2 \rceil$  odd integers between 1 and  $s$ . Therefore, in computing  $\deg c$ , the contribution from the variables  $x_{ij}$  is given by

$$2 \left\lfloor \frac{l_1 - 1}{2} \right\rfloor + \dots + 2 \left\lfloor \frac{l_k - 1}{2} \right\rfloor + \left\lceil \frac{l_1 - 1}{2} \right\rceil + \dots + \left\lceil \frac{l_k - 1}{2} \right\rceil.$$

The degree of the monomial  $c$  is therefore equal to

$$\begin{aligned} & 2k + 2 + 2 \left\lfloor \frac{l_1 - 1}{2} \right\rfloor + \dots + 2 \left\lfloor \frac{l_k - 1}{2} \right\rfloor + \left\lceil \frac{l_1 - 1}{2} \right\rceil + \dots + \left\lceil \frac{l_k - 1}{2} \right\rceil \\ &= 2k + 2 + \left( \left\lfloor \frac{l_1 - 1}{2} \right\rfloor + \left\lceil \frac{l_1 - 1}{2} \right\rceil \right) + \dots + \left( \left\lfloor \frac{l_k - 1}{2} \right\rfloor + \left\lceil \frac{l_k - 1}{2} \right\rceil \right) \\ & \quad + \left\lfloor \frac{l_1 - 1}{2} \right\rfloor + \dots + \left\lfloor \frac{l_k - 1}{2} \right\rfloor \\ &= k + 2 + k + (l_1 - 1) + \dots + (l_k - 1) + \left\lfloor \frac{l_1 - 1}{2} \right\rfloor + \dots + \left\lfloor \frac{l_k - 1}{2} \right\rfloor \\ &= k + 2 + l_1 + \dots + l_k + \left\lfloor \frac{l_1 - 1}{2} \right\rfloor + \dots + \left\lfloor \frac{l_k - 1}{2} \right\rfloor \\ &= k + 2 + n + \left\lfloor \frac{l_1 - 1}{2} \right\rfloor + \dots + \left\lfloor \frac{l_k - 1}{2} \right\rfloor. \end{aligned}$$

The last line establishes (2-2).

To prove that  $c$  does not belong to  $J^3$ , we first show the strict inequality

$$\deg c < 2 \left( \frac{|C|}{2} + 1 \right) + k + \left\lfloor \frac{l_1 - 1}{2} \right\rfloor + \dots + \left\lfloor \frac{l_k - 1}{2} \right\rfloor. \quad (2-3)$$

For suppose this inequality is not satisfied. Then (2-2) gives

$$k + 2 + n + \left\lfloor \frac{l_1 - 1}{2} \right\rfloor + \dots + \left\lfloor \frac{l_k - 1}{2} \right\rfloor \geq 2 \left( \frac{|C|}{2} + 1 \right) + k + \left\lfloor \frac{l_1 - 1}{2} \right\rfloor + \dots + \left\lfloor \frac{l_k - 1}{2} \right\rfloor,$$

which means that

$$2 + n \geq 2 \left( \frac{|C|}{2} + 1 \right).$$

But  $|C| = |H| + 1 = n + 1$ . Thus

$$2 + n \geq 2 \left( \frac{n+1}{2} + 1 \right) = n + 2 + 1,$$

which is impossible. Therefore (2-3) holds.

Let us show that (2-3) implies that  $c \notin J^3$ . Assume otherwise; then  $c = hm_1m_2m_3$  with  $m_i \in J$  for  $i = 1, 2, 3$ . Since  $m_i \in J$ , the variables that appear in each  $m_i$  correspond to a minimal cover of  $C$ . Lemma 2.1 says that such a cover has at least  $|C|/2 + 1$  vertices if it contains  $y$  and at least  $k + \lfloor (l_1 - 1)/2 \rfloor + \dots + \lfloor (l_k - 1)/2 \rfloor$

— a number at least as large as  $|C|/2 + 1$  — if not. Using the fact that at least one of the three covers must *not* contain  $y$ , we thus obtain

$$\begin{aligned} \deg c &= \deg h + \deg m_1 + \deg m_2 + \deg m_3 \\ &\geq \deg h + 2\left(\left\lfloor \frac{|C|}{2} + 1 \right\rfloor\right) + k + \left\lfloor \frac{l_1 - 1}{2} \right\rfloor + \dots + \left\lfloor \frac{l_k - 1}{2} \right\rfloor. \end{aligned}$$

This contradicts (2-3) (since  $\deg h \geq 0$ ) and so shows that  $c \notin J^3$ .

To finish the proof of Theorem 2.2 we need to show that for every vertex  $x \in V_C$  we have  $xc \in J^3$ .

Let  $x$  be any vertex of  $H$  and relabel the vertices of  $H$  starting from  $x = t_1$  clockwise  $t_2, \dots, t_n$ , where  $n$  is the size of  $H$ . We can write  $xc = m_1 m_2 m_3$ , where

$$m_1 = y \prod_{i \text{ odd}} t_i, \quad m_2 = y t_1 \prod_{i \text{ even}} t_i, \quad m_3 = \prod_{i=1, \dots, k} x_i \prod_{\substack{i=1, \dots, k \\ j \text{ even}}} x_{ij}.$$

Note that  $m_1$  and  $m_2$  correspond to covers, as they contain  $y$  and every other vertex of  $H$ . Also  $m_3$  corresponds to a cover as all the  $x_i$  are included, and therefore all the edges connecting  $y$  to  $H$  are covered, and every other vertex in the path from  $x_i$  to  $x_{i+1}$  is included.

Finally we need to write  $yc = m_1 m_2 m_3$  with  $m_i \in J$  for  $i = 1, 2, 3$ . For this assume that  $x_1$  is such that the path from  $x_k$  to  $x_1$  is odd. Relabel the vertices  $x_1 = t_1$  and then clockwise to  $t_n$ . Let

$$m_1 = y \prod_{i \text{ odd}} t_i.$$

Note that  $m_1$  will give a cover as we are considering every other vertex in the odd cycle and the vertex  $y$ . Now let  $l$  the least even number so that  $t_l$  corresponds to a radial vertex  $x_g$ , for some  $g$ . Set

$$m_2 = y \prod_{\substack{l \leq i \leq n \\ i \text{ even}}} t_i \prod_{\substack{1 \leq i \leq l \\ i \text{ odd}}} t_i.$$

Because we are considering every other vertex from  $t_1$  to  $t_{l-1}$ , every other vertex from  $t_l$ , and the center  $y$ , the monomial  $m_2$  corresponds to a cover of the wheel.

Finally

$$m_3 = y x_g x_{g+1} \dots x_k \prod_{\substack{i=g, \dots, k \\ j \text{ even}}} x_{ij} \prod_{\substack{i=1, \dots, l-1 \\ i \text{ even}}} t_i.$$

Also  $m_3$  gives a cover as it contains every other vertex from  $t_2$  to  $t_l = x_g$ , every other vertex from  $x_i$  to  $x_{i+1}$ , for  $i = g, \dots, k - 1$ , every other vertex from  $x_k$  to  $x_1$ ,

and the center  $y$ . Notice that  $x_1$  is missing from the monomial  $m_3$  but the vertex  $y$  is listed in the monomial as for the vertex preceding  $x_1$ , because of the assumption that the path  $x_k, \dots, x_1$  in  $H$  is odd. □

For every ideal  $I$  in a polynomial ring  $S$  (or a more general ring), one can compute the sequence of sets  $\text{Ass}(S/I^n)$  for  $n \in \mathbb{N}$ . Brodmann [1979] proved, in much greater generality, that there exists a positive integer  $a$  such that

$$\bigcup_{i=1}^{a_I} \text{Ass}(S/I^i) = \bigcup_{i=1}^{\infty} \text{Ass}(S/I^i). \tag{2-4}$$

Very little is known about the value of  $a_I$ . In [Francisco et al. 2011], the authors give an upper bound for  $a_I$  in the case that  $I$  is an edge ideal for a graph.

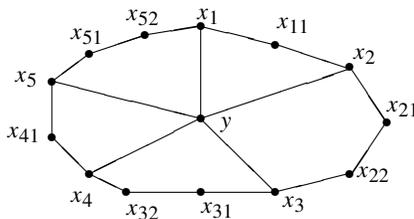
The value of  $a_J$ , where  $J$  is the cover ideal of a graph  $G$ , is related to the coloring number of  $G$ , that is, the least number of colors that one needs to color the vertices of  $G$  so that two adjacent vertices always have different colors. We denote the coloring number of  $G$  by  $\chi(G)$ . It is shown in [Francisco et al. 2011] that, in (2-4),  $a_J \geq \chi(G) - 1$  when  $J$  is the cover ideal of  $G$ . The same paper gives examples for which  $a_J > \chi(G) - 1$ . Centered odd holes are an infinite family of such examples.

**Corollary 2.3.** *Let  $C$  be a wheel with cover ideal  $J$ . If  $C$  has a vertex that is not radial, then  $a_J \geq \chi(C)$ .*

*Proof.* Because  $C$  contains an odd hole, one needs at least three colors for the vertexes of  $C$ . We first show that  $\chi(C) = 3$ . Let  $\{a, b, c\}$  be a list of three colors. Assume that  $x$  is a vertex of  $C$  which is not radial. Color the vertex  $x$  and the center  $y$  with  $c$ , and finally color the remaining vertices alternating  $a$  and  $b$ .

The main theorem implies that  $a_J \geq 3$ . □

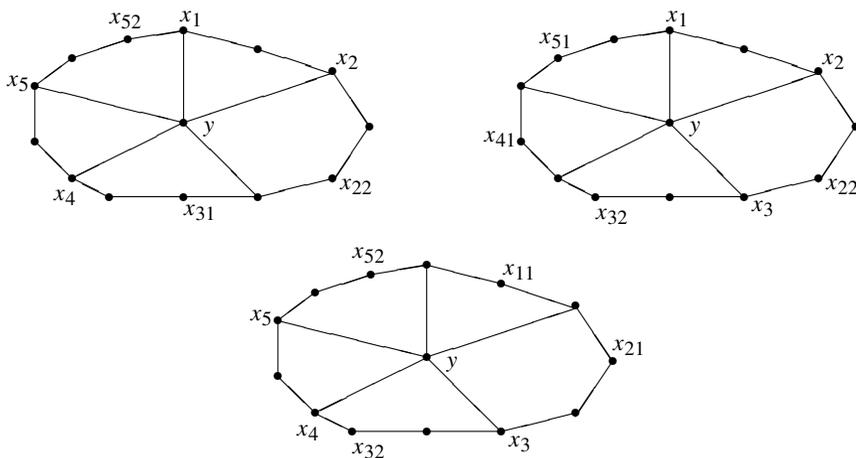
We finish the paper with an example that illustrates the idea behind the proof of the main theorem. Consider this wheel:



The monomial  $c$  used in the proof of the main theorem is given by

$$c = x_1^2 x_2^2 x_3^2 x_4^2 x_5^2 x_{22}^2 x_{32}^2 x_{52}^2 x_{11} x_{21} x_{31} x_{41} x_{51}.$$

We can write  $yc = m_1m_2m_3$ , where the monomials  $m_1$ ,  $m_2$ , and  $m_3$  correspond to the following covers:



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[kimberly.kesting@student.fairfield.edu](mailto:kimberly.kesting@student.fairfield.edu) *Department of Mathematics and Computer Science, Fairfield University, Fairfield, CT 06824, United States*

[james.pozzi@student.fairfield.edu](mailto:james.pozzi@student.fairfield.edu) *Department of Mathematics and Computer Science, Fairfield University, Fairfield, CT 06824, United States*

[jstriuli@fairfield.edu](mailto:jstriuli@fairfield.edu) *Department of Mathematics and Computer Science, Fairfield University, Fairfield, CT 06824, United States*

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