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Energy-minimizing unit vector fields

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Given a surface of revolution with boundary, we study the extrinsic energy of smooth tangent unit-length vector fields. Fixing continuous tangent unit-length vector fields on the boundary of the surface of revolution, we ask if there is a unique smooth tangent unit-length vector field continuously achieving the boundary data and minimizing energy amongst all smooth tangent unit-length vector fields also continuously achieving the boundary data.

1. Introduction

Let $\mathcal{S}$ be a surface of revolution given by the parametrization

$$\Phi(\theta, t) = (r(t)\cos\theta, r(t)\sin\theta, t), \quad \theta \in \mathbb{R}, t \in (0, h),$$

where $r(t) \in C^\infty([0, h])$ is positive in $[0, h]$. Let $\mathfrak{X}^1(\mathcal{S})$ be the set of smooth tangent unit-length vector fields on $\mathcal{S}$. For $\boldsymbol{V} \in \mathfrak{X}^1(S)$ we define the *extrinsic* energy of $\boldsymbol{V}$ to be

$$E(\boldsymbol{V}) = \iint_S |D\boldsymbol{V}|^2 \, d\,\text{Area},$$

where $D\boldsymbol{V}$ is the differential of the map $\boldsymbol{V} : \mathcal{S} \to \mathbb{R}^3$. Using the parametrization Φ, we get

$$E(\boldsymbol{V}) = \int_0^h \int_0^{2\pi} \left(\frac{r(t)}{\sqrt{1+r'(t)^2}}\right) \left|\frac{\partial \boldsymbol{V}}{\partial t}\right|^2 + \left(\frac{\sqrt{1+r'(t)^2}}{r(t)}\right) \left|\frac{\partial \boldsymbol{V}}{\partial \theta}\right|^2 \, d\theta \, dt.$$

Suppose V_0 and V_h are continuous unit-length tangent vector fields, defined respectively on $\{\Phi(\theta, 0) : \theta \in \mathbb{R}\}$ and $\{\Phi(\theta, h) : \theta \in \mathbb{R}\}$. For $\boldsymbol{V} \in \mathfrak{X}^1(\mathcal{S})$, we

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write $\boldsymbol{V}|_{\partial\mathscr{S}} = V_0, V_h$ if $\boldsymbol{V}$ continuously achieves the boundary data V_0, V_h on $\mathscr{S}$. Precisely, $\boldsymbol{V}|_{\partial\mathscr{S}} = V_0, V_h$ if for every $\vartheta \in \mathbb{R}$ we have

$$\lim_{(\theta,t)\to(\vartheta,0)} \boldsymbol{V}(\Phi(\theta,t)) = V_0(\Phi(\vartheta,0)), \qquad \lim_{(\theta,t)\to(\vartheta,h)} \boldsymbol{V}(\Phi(\theta,t)) = V_h(\Phi(\vartheta,h)).$$

We pose the following question: *Suppose V_0 and V_h are continuous unit-length tangent vector fields defined respectively on $\{\Phi(\theta,0) : \theta \in \mathbb{R}\}$ and $\{\Phi(\theta,h) : \theta \in \mathbb{R}\}$. Does there exist a unique $\boldsymbol{V} \in \mathfrak{X}^1(\mathscr{S})$ with $\boldsymbol{V}|_{\partial\mathscr{S}} = V_0, V_h$ so that $E(\boldsymbol{V}) < E(\tilde{\boldsymbol{V}})$ for any other $\tilde{\boldsymbol{V}} \in \mathfrak{X}^1(\mathscr{S})$ with $\tilde{\boldsymbol{V}}|_{\partial\mathscr{S}} = V_0, V_h$?*

We give partial answers to the question of existence and uniqueness. Theorem 3.2 shows the existence of minimizers for a certain class of boundary data, and Theorem 4.1 allows us to conclude uniqueness in a parametric sense in general, and outright for the case of the unit cylinder with horizontal boundary data (see Corollary 5.2). Only first-year graduate analysis is needed for most of the results, although some references to regularity of weak solutions to ordinary differential equations and approximations by smooth functions in $W^{1,2}$ is mentioned in the proofs of Theorem 3.2 and Theorem 5.1.

We describe the effect of the shape of $\mathscr{S}$ on the minimizer. Observe that where $r'(t)$ is large $\partial V/\partial t$ can be large in magnitude without paying much in energy. Hence, we can seek to minimize energy by letting $\boldsymbol{V}$ not vary much from the boundary data near $t = 0, h$, and then where $r'(t)$ is large we let $\boldsymbol{V}$ quickly change to a vector field of low energy. In the case of the unit cylinder, Figure 2 (page 448) shows that it is best to steadily homotopy between the boundary data. However, for the surface given by $r(t) = \sin t + 2$ (Figure 3, right), it is better to homotopy to a vector field with low energy in the regions where $r'(t)$ is large, as suggested by Figure 3, left. This illustrates that the $\partial\boldsymbol{V}/\partial t$ term is important, and so we list t-derivatives first in our calculations.

In case of the cylinder $r(t) = 1$ with height h, replacing DV with the covariant derivative of $\boldsymbol{V}$ leaves us to study 2π-periodic harmonic functions defined over $\mathbb{R} \times (0,h)$. In general, intrinsic energy of unit vector fields is also called *total bending*, and has been studied in the more general setting of Riemannian manifolds of any dimension; see [Wiegmink 1995] for an introduction. In [Borrelli et al. 2003], for example, it is shown that the infimum intrinsic energy in the odd-dimensional sphere S^{2k+1} for $k \geq 2$ is given by the energy of the horizontal tangent unit vector field defined on S^{2k+1} except at two antipodal points $\{P, -P\}$. This value, however, is not attained by any smooth tangent unit vector field over S^{2k+1} as shown in [Brito and Walczak 2000].

Minimizing the extrinsic energy over all smooth vector fields can be studied using similar techniques, as will follow. Although the set of vector fields over which we must minimize is larger, we avoid the difficulties arising in the unit-length case by the necessity to work with the angle functions φ introduced in Section 2. Instead, denoting by $\boldsymbol{V}$ a tangent vector field on $\mathcal{S}$ and using the parametrization $\boldsymbol{V}=a(\theta,t)\Phi_t+b(\theta,t)\Phi_\theta$, we work directly with the smooth functions a, b in the general case.

2. First variation

We derive a partial differential equation which a minimizing $\boldsymbol{V}$ must solve, using a standard technique from the calculus of variations. First, given $\boldsymbol{V}\in\mathfrak{X}^1(\mathcal{S})$ we can find a function $\varphi(\theta,t)$ so that

$$\boldsymbol{V}(\theta,t)=\begin{pmatrix}-\sin\theta\cos\varphi\\ \cos\theta\cos\varphi\\ 0\end{pmatrix}+\frac{1}{\sqrt{1+r'(t)^2}}\begin{pmatrix}r'(t)\cos\theta\sin\varphi\\ r'(t)\sin\theta\sin\varphi\\ \sin\varphi\end{pmatrix}.$$

Thus, $\varphi(\theta,t)$ measures the angle between $\boldsymbol{V}(\theta,t)$ and the horizontal tangent vector field $(-\sin\theta,\cos\theta,0)$. Our choice of angle function φ is not unique, and may be chosen to be discontinuous. This occurs for example in the proof of Theorem 5.1. Choosing φ continuous may require us to make $|\varphi|$ large. However, $\sin\varphi$, $\cos\varphi$, and $\sin 2\varphi$ will be smooth in $\mathbb{R}\times(0,h)$, continuous even at $t=0,h$, and independent of φ. Using smoothness of $\sin\varphi$, $\cos\varphi$ we can define φ_t, φ_θ smooth in $\mathbb{R}\times(0,h)$ and independent of φ. Whenever $\boldsymbol{V}$ is given by an angle function φ, we shall write $\boldsymbol{V}=\boldsymbol{V}(\varphi)$.

For $\boldsymbol{V}=\boldsymbol{V}(\varphi)$, we can write the energy $E(\boldsymbol{V})=E(\varphi)$ in terms of φ:

$$\begin{aligned}E(\varphi)=\int_0^h\int_0^{2\pi}T(t)(\varphi_t)^2+\Theta(t)(\varphi_\theta)^2\,d\theta\,dt\\ +\int_0^h\int_0^{2\pi}P_c(t)\cos^2\varphi+P_s(t)\sin^2\varphi+Q(t)\,d\theta\,dt\end{aligned}\tag{2-1}$$

where

$$T(t)=\frac{r(t)}{\sqrt{1+r'(t)^2}},\qquad \Theta(t)=\frac{1+r'(t)^2(3+3r'(t)^2+r'(t)^4)}{r(t)(1+r'(t)^2)^{5/2}},$$

$$P_c(t)=\frac{1+4r'(t)^2+2r'(t)^4}{r(t)(1+r'(t)^2)^{5/2}},\qquad P_s(t)=\frac{2r(t)^2r''(t)^2}{r(t)(1+r'(t)^2)^{5/2}},$$

$$\text{and } Q(t)=\frac{r'(t)^2}{r(t)\sqrt{1+r'(t)^2}}.$$

If $\boldsymbol{V}=\boldsymbol{V}(\varphi)$ minimizes energy on $\mathcal{S}$ with respect to the boundary data V_0, V_h, then let $\eta\in C_c^\infty((0,2\pi)\times(0,h))$ (that is, a smooth function with compact support in $(0,2\pi)\times(0,h)$). We then let $\boldsymbol{V}^s\in\mathfrak{X}^1(\mathcal{S})$ be the vector field given by the angle function $\varphi + s\eta$. Then $E(V^s)$ achieves a minimum at $s = 0$, and so

$$\frac{d}{ds} E(V^s)\Big|_{s=0} = 0.$$

Differentiating (2-1) under the integral with respect to s gives:

$$\int_0^h \int_0^{2\pi} 2T(t)\varphi_t \eta_t + 2\Theta(t)\varphi_\theta \eta_\theta - ((P_c(t) - P_s(t)) \sin 2\varphi)\eta \, d\theta \, dt = 0.$$

Since η has compact support in $(0, 2\pi) \times (0, h)$, we may use integration by parts in the first and second terms to get

$$\int_0^h \int_0^{2\pi} [-2(T(t)\varphi_t)_t - 2(\Theta(t)\varphi_\theta)_\theta - (P_c(t) - P_s(t)) \sin 2\varphi]\eta \, d\theta \, dt = 0.$$

We thus have that φ must satisfy the second-order partial differential equation:

$$(T(t)\varphi_t)_t + (\Theta(t)\varphi_\theta)_\theta + (P_c(t) - P_s(t))\left(\frac{\sin 2\varphi}{2}\right) = 0, \tag{2-2}$$

which we call the *Euler–Lagrange equation* associated to the energy $E(\varphi)$.

In case of the cylinder $\mathscr{C}$ with $r(t) = 1$ and height h, the energy (2-1) becomes

$$E(\varphi) = \int_0^h \int_0^{2\pi} (\varphi_t)^2 + (\varphi_\theta)^2 + \cos^2 \varphi \, d\theta \, dt.$$

Equation (2-2) in this case is

$$\varphi_{tt} + \varphi_{\theta\theta} + \frac{\sin 2\varphi}{2} = 0,$$

for which the only constant solutions are $\varphi = k\pi/2$ with $k \in \mathbb{Z}$. Although when k is odd $E(k\pi/2) = 0$, we can show by example that for large h the horizontal vector field $\varphi = \pi$ is not a minimizer. Corollary 5.2 will show that for $h < \sqrt{8}$ the horizontal vector field is a minimizer, and it remains to find the largest h_0 so that this true for all $h < h_0$.

The equation

$$\varphi_{tt} + \varphi_{\theta\theta} + \frac{\sin 2\varphi}{2} = 0$$

is a special case of a form of equations called the *sine-Gordon equations*, which arise in differential geometry and various areas of physics. This particular form arises in the study of ferromagnetics in physics; see [Chen et al. 2004] for example, and in the study of harmonic maps in differential geometry, see [Hu 1982].

3. Existence

In this section we aim to prove the existence of minimizers with boundary data V_0,V_h which make a constant angle with the horizontal vector field $(-\sin\theta, \cos\theta, 0)$.

Lemma 3.1. *Suppose V_0, V_h are continuous tangent unit-length boundary data on $\partial\mathcal{S}$ such that each can be written using a constant angle function. Let $\boldsymbol{V} \in \mathfrak{X}^1(\mathcal{S})$ with $\boldsymbol{V}|_{\partial\mathcal{S}} = V_0$, V_h and $\boldsymbol{V} = \boldsymbol{V}(\varphi)$. If $\varphi_\theta \not\equiv 0$, then there is a vector field $\tilde{\boldsymbol{V}} \in \mathfrak{X}^1(\mathcal{S})$ with $\tilde{\boldsymbol{V}}|_{\partial\mathcal{S}} = V_0$, V_h so that $E(\tilde{\boldsymbol{V}}) < E(\boldsymbol{V})$, and so that we can write $\tilde{\boldsymbol{V}} = \tilde{\boldsymbol{V}}(\tilde{\varphi})$ where $\tilde{\varphi} \in C([0, h]) \cap C^\infty((0, h))$ and $\tilde{\varphi}(0) \in [0, 2\pi)$.*

Proof. Suppose $E(\boldsymbol{V}) < \infty$, otherwise we simply take $\tilde{\boldsymbol{V}} = \tilde{\boldsymbol{V}}(\varphi_0 + (t/h)(\varphi_h - \varphi_0))$ where $\varphi_0 \in [0, 2\pi)$, φ_h are constants so that the boundary data $V_0 = V_0(\varphi_0)$ and $V_h = V_h(\varphi_h)$. Let $\boldsymbol{V} = \boldsymbol{V}(\varphi)$, we thus have $\int_0^h \int_0^{2\pi} \Theta(t)(\varphi_\theta)^2 \, d\theta \, dt > 0$. Consider the integrable function

$$f(\theta) = \int_0^h T(t)(\varphi_t)^2 + P_c(t)\cos^2\varphi + P_s(t)\sin^2\varphi + Q(t)\, dt.$$

We then have $\inf_{\theta\in[0,2\pi)} f(\theta) < \infty$. Choose $\theta_0 \in [0, 2\pi)$ so that

$$f(\theta_0) < \inf_{\theta\in[0,2\pi)} f(\theta) + \frac{1}{2\pi} \int_0^h \int_0^{2\pi} \Theta(t)(\varphi_\theta)^2 \, d\theta \, dt.$$

Define $\tilde{\varphi}(\theta, t) = \varphi(\theta_0, t)$, and let $\tilde{\boldsymbol{V}} = \tilde{\boldsymbol{V}}(\tilde{\varphi}) \in \mathfrak{X}^1(S)$. Evidently $\tilde{\boldsymbol{V}}|_{\partial\mathcal{S}} = V_0$, V_h and

$$E(\boldsymbol{V}) = \int_0^{2\pi} f(\theta)\, d\theta + \int_0^h \int_0^{2\pi} \Theta(t)(\varphi_\theta)^2 \, d\theta\, dt > \int_0^{2\pi} f(\theta_0)\, d\theta = E(\tilde{\varphi}) = E(\tilde{\boldsymbol{V}}).$$

Since $\tilde{\varphi}$ only depends on t, we can redefine $\tilde{\varphi}$ so that

$$\tilde{\varphi} \in C([0, h]) \cap C^\infty((0, h)).$$

We can also translate by some $2\pi k$ with $k \in \mathbb{Z}$, without changing the energy $E(\tilde{\varphi})$, so that $\tilde{\varphi}(0) \in [0, 2\pi)$. □

Theorem 3.2. *Suppose $V_0 = V_0(\varphi_0)$, $V_h = V_h(\varphi_h)$ are continuous tangent unit-length boundary data on $\partial\mathcal{S}$, where φ_0, φ_h are constants. Then there exists*

$$\boldsymbol{V} \in \mathfrak{X}^1(\mathcal{S}), \qquad \textit{with } \boldsymbol{V}|_{\partial\mathcal{S}} = V_0,\, V_h,$$

minimizing energy. Moreover,

$$\boldsymbol{V} = \boldsymbol{V}(\varphi), \qquad \textit{with } \varphi \in C^\infty([0, h]).$$

Proof. The argument follows the proof of the existence of minimizers to the Dirichlet energy using weak compactness [Evans 1998, Section 8.2]. Let

$$E = \inf\{E(V) : V \in \mathfrak{X}^1(\mathcal{S}),\, V|_{\partial\mathcal{S}} = V_0,\, V_h\},$$

note that $E < \infty$. Define C_{E+1} to be the set of $\varphi \in C([0,h]) \cap C^\infty((0,h))$ with energy $E(\varphi) \le E+1$ and $\varphi(0) \in [0, 2\pi)$ so that $V(\varphi)|_{\partial\mathcal{S}} = V_0, V_h$. By Lemma 3.1 it suffices to show $E = \inf_{\varphi \in C_{E+1}} E(\varphi)$ is attained. Let $\overline{C}_{E+1}$ be the closure of C_{E+1} in $L^2([0,h])$.

Lemma 3.3. *Every $\overline{\varphi} \in \overline{C}_{E+1}$ is continuous in $[0,h]$ with a weak derivative in $L^2([0,h])$. Moreover, we can find a sequence $\varphi_k \in C_{E+1}$ converging uniformly to $\overline{\varphi}$.*

Proof. Take a sequence $\varphi_k \in C_{E+1}$. Let $T_{\min} = \min_{t\in[0,h]} T(t)$. It follows that the φ_k are equicontinuous in $[0,1]$, since by Cauchy–Schwartz

$$\begin{aligned}|\varphi_k(x) - \varphi_k(y)| = \left|\int_x^y (\varphi_k)_t\,dt\right| &\le \sqrt{|x-y|}\Big(\int_0^h ((\varphi_k)_t)^2\,dt\Big)^{1/2} \\ &= \sqrt{|x-y|}\Big(\int_0^h \frac{T(t)}{T_{\min}}\cdot((\varphi_k)_t)^2\,dt\Big)^{1/2} \le \sqrt{\frac{E+1}{T_{\min}}}\cdot\sqrt{|x-y|}.\end{aligned}$$

Since $0 \le \varphi_k(0) < 2\pi$, there is by Arzelà–Ascoli a subsequence of the φ_k having a uniformly convergent subsequence. Therefore $\overline{C}_{E+1} \subseteq C([0,h])$.

Let $\eta \in C_c^\infty((0,1))$ and $\overline{\varphi} \in \overline{C}_{E+1}$ with $\varphi_k \in C_{E+1}$ converging uniformly to $\overline{\varphi}$. Then

$$\int_0^h \overline{\varphi}\eta_t\,dt = \lim_{k\to\infty}\int_0^h \varphi_k\eta_t\,dt = -\lim_{k\to\infty}\int_0^h (\varphi_k)_t\eta\,dt.$$

However, note that the sequence $(\varphi_k)_t$ is a bounded sequence in $L^2([0,h])$. By Alaoglu's theorem, a subsequence of the $(\varphi_k)_t$ converges weakly to some

$$\overline{\varphi}_t \in L^2([0,h]).$$

We therefore have

$$\int_0^h \overline{\varphi}\eta_t\,dt = -\int_0^h \overline{\varphi}_t\eta\,dt,$$

and so $\overline{\varphi}$ has weak derivative $\overline{\varphi}_t$ in $L^2([0,h])$. □

Returning to the proof of Theorem 3.2, given $\overline{\varphi} \in \overline{C}_{E+1}$ we can define the energy $E(\overline{\varphi})$ by

$$E(\overline{\varphi}) = 2\pi\int_0^h T(t)(\overline{\varphi}_t)^2 + P_c(t)\cos^2\overline{\varphi} + P_s(t)\sin^2\overline{\varphi} + Q(t)\,dt,$$

where $\overline{\varphi}_t$ is the weak derivative in $L^2([0,h])$ of $\overline{\varphi}$. Also define

$$E_{\overline{C}_{E+1}} = \inf_{\overline{\varphi}\in\overline{C}_{E+1}} E(\overline{\varphi}),$$

so that $E_{\overline{C}_{E+1}} \le E$.

We show there is a $\overline{\varphi}\in\overline{C}_{E+1}$ with $E(\overline{\varphi}) = E_{\overline{C}_{E+1}}$. Take a sequence $\overline{\varphi}_k \in \overline{C}_{E+1}$ so that $E(\overline{\varphi}_k) \searrow E_{\overline{C}_{E+1}}$. The sequence $\overline{\varphi}_k$ will also be equicontinuous with $\overline{\varphi}_k(0)\in[0,2\pi)$, and hence a subsequence will converge uniformly to some $\overline{\varphi}\in\overline{C}_{E+1}$. Arguing as in Lemma 3.3, we can show $(\overline{\varphi}_k)_t \to \overline{\varphi}_t$ weakly in $L^2([0,h])$, and since $T(t)$ is bounded in $[0,h]$, we have $T(t)^{\frac{1}{2}}(\overline{\varphi}_k)_t \to T(t)^{\frac{1}{2}}\overline{\varphi}_t$ weakly in $L^2([0,h])$ as well. From this it follows that

$$\int_0^h T(t)(\overline{\varphi}_t)^2\,dt \le \varliminf_{k\to 0}\int_0^h T(t)((\overline{\varphi}_k)_t)^2\,dt,$$

and since $\overline{\varphi}_k \to \overline{\varphi}$ uniformly, we can show

$$\int_0^h P_c(t)\cos^2\overline{\varphi}_k + P_s(t)\sin^2\overline{\varphi}_k\,dt \to \int_0^h P_c(t)\cos^2\overline{\varphi} + P_s(t)\sin^2\overline{\varphi}\,dt.$$

We therefore have $E(\overline{\varphi}) \le \lim_{k\to\infty} E(\overline{\varphi}_k) = E_{\overline{C}_{E+1}}$, and so $E(\overline{\varphi}) = E_{\overline{C}_{E+1}}$.

Now, taking $\overline{\varphi}$, let $\eta\in C_c^\infty((0,h))$ and consider $\overline{\varphi}_s = \overline{\varphi}+s\eta$. Although we may not have $\overline{\varphi}_s\in\overline{C}_{E+1}$, observe that $\overline{\varphi}$ still minimizes the energy over the closure in $L^2([0,h])$ of the set of functions φ as in C_{E+1} except with $E(\varphi)\le E+2$. We can thus conclude $E(\overline{\varphi}) \le E(\overline{\varphi}_s)$ for all sufficiently small s. As in computing the Euler–Lagrange equation (2-2), we have that $\overline{\varphi}$ is a *weak solution* to the second-order ODE in $(0,h)$:

$$(T(t)\overline{\varphi}_t)_t + (P_c(t)-P_s(t))\frac{\sin 2\overline{\varphi}}{2} = 0,$$

meaning that for any $\eta\in C_c^\infty((0,h))$ we have

$$\int_0^h T(t)\overline{\varphi}_t\cdot\eta_t + (P_c(t)-P_s(t))\frac{\sin 2\overline{\varphi}}{2}\cdot\eta\,dt = 0.$$

Using standard regularity theory [Evans 1998, Section 6.3, Theorems 1 and 2], we conclude that $\overline{\varphi}\in C^\infty([0,h])$. $\square$

4. Uniqueness

The following theorem will allow us to conclude uniqueness in certain circumstances. Let

$$T_{\min} = \min_{t\in[0,h]} T(t),\ \Theta_{\min} = \min_{t\in[0,h]}\Theta(t),\qquad P_{c-s} = \sup_{t\in[0,h]}|P_c(t)-P_s(t)|.$$

Theorem 4.1. *Let* $0 < h < \sqrt{\dfrac{8(T_{\min}+\Theta_{\min})}{P_{c-s}}}$, *and suppose that*

$$\varphi \in C^1(\mathbb{R}\times[0,h]) \cap C^2(\mathbb{R}\times(0,h))$$

is 2π*-periodic in* θ *and satisfies the Euler–Lagrange equation* (2-2) *in* $(0,2\pi)\times(0,h)$. *Then* φ *is uniquely determined by its boundary values* $\varphi(\theta,0)$, $\varphi(\theta,h)$.

The requirement that $\varphi(\theta,t)$ is 2π-periodic in θ geometrically means that for each fixed $t\in[0,h]$, as θ increases from 0 to 2π the vector field $\boldsymbol{V}=\boldsymbol{V}(\varphi(\theta,t))$ spins clockwise as many times as it does counterclockwise as measured from the horizontal vector field $(-\sin\theta,\cos\theta,0)$.

To prove the theorem we need first the following Poincaré inequality:

Lemma 4.2. *Suppose* $\varphi\in C^1(\mathbb{R}\times[0,h])$ *satisfies* $\varphi(\theta,0)=\varphi(\theta,h)=0$ *for each* $\theta\in\mathbb{R}$. *Then*

$$\int_0^h\int_0^{2\pi}\varphi^2\,d\theta\,dt \le \frac{h^2}{8}\int_0^h\int_0^{2\pi}(\varphi_t)^2+(\varphi_\theta)^2\,dt\,d\theta.$$

Proof. Writing

$$\varphi(\theta,t)=\int_0^t\frac{\partial}{\partial s}\varphi(\theta,s)\,ds=-\int_t^h\frac{\partial}{\partial s}\varphi(\theta,s)\,ds,$$
$$\int_0^h\varphi^2\,dt=\int_0^{h/2}\varphi^2\,dt+\int_{h/2}^h\varphi^2\,dt,$$

we have

$$\int_0^h\varphi^2\,dt=\int_0^{h/2}\left(\int_0^t\frac{\partial\varphi}{\partial s}\,ds\right)^2dt+\int_{h/2}^h\left(\int_t^h\frac{\partial\varphi}{\partial s}\,ds\right)^2dt.$$

Using Cauchy–Schwartz,

$$\begin{aligned}\int_0^h\varphi^2dt&\le\int_0^{h/2}t\left(\int_0^t\left(\frac{\partial\varphi}{\partial s}\right)^2ds\right)dt+\int_{h/2}^h(h-t)\left(\int_t^h\left(\frac{\partial\varphi}{\partial s}\right)^2ds\right)dt\\&\le\int_0^{h/2}(\varphi_t)^2+(\varphi_\theta)^2\,dt\int_0^{h/2}t\,dt+\int_{h/2}^h(\varphi_t)^2+(\varphi_\theta)^2\,dt\int_{h/2}^h(h-t)\,dt,\end{aligned}$$

which gives $\int_0^h\varphi^2\,dt\le\frac{1}{8}h^2\int_0^h\varphi_t^2+\varphi_\theta^2\,dt$. Integrating with respect to θ gives the result. □

Proof of Theorem 4.1. Suppose $\varphi_1,\varphi_2\in C^1(\mathbb{R}\times[0,h])\cap C^2(\mathbb{R}\times(0,h))$ are solutions to (2-2), both 2π-periodic in θ and satisfying

$$\varphi_1(\theta,0)=\varphi_2(\theta,0),\qquad\varphi_1(\theta,h)=\varphi_2(\theta,h).$$

Multiplying

$$(T(t)(\varphi_1-\varphi_2)_t)_t+(\Theta(t)(\varphi_1-\varphi_2)_\theta)_\theta+(P_c(t)-P_s(t))\Big(\frac{\sin 2\varphi_1}{2}-\frac{\sin 2\varphi_2}{2}\Big)=0$$

by $\varphi_1-\varphi_2$ and integrating gives

$$\int_0^h\int_0^{2\pi}\big[(T(t)(\varphi_1-\varphi_2)_t)_t+(\Theta(t)(\varphi_1-\varphi_2)_\theta)_\theta\big](\varphi_1-\varphi_2) \\ +(P_c(t)-P_s(t))\Big(\frac{\sin 2\varphi_1}{2}-\frac{\sin 2\varphi_2}{2}\Big)(\varphi_1-\varphi_2)\,d\theta\,dt=0.$$

Since $(\varphi_1-\varphi_2)(\theta,0)=(\varphi_1-\varphi_2)(\theta,h)=0$ and φ_1,φ_2 are 2π-periodic in θ, then integration by parts gives:

$$\int_0^h\int_0^{2\pi}T(t)((\varphi_1-\varphi_2)_t)^2+\Theta(t)((\varphi_1-\varphi_2)_\theta)^2\,d\theta\,dt \\ =\int_0^h\int_0^{2\pi}(P_c(t)-P_s(t))\Big(\frac{\sin 2\varphi_1}{2}-\frac{\sin 2\varphi_2}{2}\Big)(\varphi_1-\varphi_2)\,d\theta\,dt.$$

We now use the inequality $|\sin x-\sin y|\le|x-y|$ to get

$$(T_{\min}+\Theta_{\min})\int_0^h\int_0^{2\pi}((\varphi_1-\varphi_2)_t)^2+((\varphi_1-\varphi_2)_\theta)^2\,d\theta\,dt \\ \le P_{c-s}\int_0^h\int_0^{2\pi}(\varphi_1-\varphi_2)^2\,d\theta\,dt.$$

Lemma 4.2 now implies

$$\int_0^h\int_0^{2\pi}((\varphi_1-\varphi_2)_t)^2+((\varphi_1-\varphi_2)_\theta)^2\,d\theta\,dt \\ \le\frac{P_{c-s}}{(T_{\min}+\Theta_{\min})}\frac{h^2}{8}\int_0^h\int_0^{2\pi}((\varphi_1-\varphi_2)_t)^2+((\varphi_1-\varphi_2)_\theta)^2\,d\theta\,dt.$$

When

$$h<\sqrt{\frac{8(T_{\min}+\Theta_{\min})}{P_{c-s}}}$$

we see that $\varphi_1=\varphi_2$ must occur. □

Theorem 4.1 together with Lemma 3.1 imply the following corollary:

Corollary 4.3. *Let*

$$h<\sqrt{\frac{8(T_{\min}+\Theta_{\min})}{P_{c-s}}}$$

and take boundary data V_0, V_h each with constant angle function. Suppose $\boldsymbol{V}=\boldsymbol{V}(\varphi)$ and $\tilde{\boldsymbol{V}}=\tilde{\boldsymbol{V}}(\tilde\varphi)$ are minimizers with $\varphi,\tilde\varphi\in C^\infty([0,h])$. If $\varphi(0)=\tilde\varphi(0)$ and $\varphi(h)=\tilde\varphi(h)$, then $\varphi=\tilde\varphi$ and so $\boldsymbol{V}=\tilde{\boldsymbol{V}}$.

For the boundary data $V_0 = V_0(\pi/2)$ and $V_h = V_h(-\pi/2)$, if $\boldsymbol{V} = \boldsymbol{V}(\varphi)$ is a minimizer then so is $\tilde{\boldsymbol{V}} = \tilde{\boldsymbol{V}}(\pi - \varphi) \neq \boldsymbol{V}$. In the next section we show that uniqueness holds without reference to the angle functions in certain cases.

5. Twisting in the unit cylinder

Recall that in a cylinder or in a frustum of a cone the vector field $\boldsymbol{V} = \boldsymbol{V}(k\pi/2)$ with k odd minimizes energy over all vector fields in $\mathfrak{X}^1(\mathcal{S})$. This allows us to show that minimizers ought not to "twist" too much, if the boundary data does not. We show the case of the cylinder.

Theorem 5.1. *Let $V_0 = V_0(\varphi_0)$ and $V_h = V_h(\varphi_h)$ be continuous boundary data on $\partial\mathscr{C}$. Suppose for some $n \in \mathbb{Z}$ and all $\theta \in \mathbb{R}$ we have*

$$\frac{n\pi}{2} - \frac{\pi}{2} < \varphi_0(\theta), \quad \varphi_h(\theta) < \frac{n\pi}{2} + \frac{\pi}{2}.$$

Then for any $\epsilon > 0$ and $\boldsymbol{V} \in \mathfrak{X}^1(\mathscr{C})$ with $\boldsymbol{V}|_{\partial\mathscr{C}} = V_0, V_h$ and $E(\boldsymbol{V}) < \infty$, there is $\tilde{\boldsymbol{V}} \in \mathfrak{X}^1(\mathscr{C})$ with $\tilde{\boldsymbol{V}}|_{\partial\mathscr{C}} = V_0, V_h$, $E(\tilde{\boldsymbol{V}}) < E(\boldsymbol{V}) + \epsilon$, and so that we can write $\tilde{\boldsymbol{V}} = \tilde{\boldsymbol{V}}(\tilde{\varphi})$ using an angle function $\tilde{\varphi}$ with $n\pi/2 - \pi/2 < \tilde{\varphi} < n\pi/2 + \pi/2$.

We remark that the calculation $\cos^2(\varphi \pm n\pi) = \cos^2(\varphi)$ is used in the proof; the argument as given cannot be used in case φ_0, φ_h have values in a period of length π centered at an angle not of the form $n\pi/2$ with $n \in \mathbb{Z}$.

Proof. Take a vector field $\boldsymbol{V} \in \mathfrak{X}^1(\mathscr{C})$ with boundary data V_0, V_h, and write $\boldsymbol{V} = \boldsymbol{V}(\varphi)$ using an angle function satisfying $n\pi/2 - \pi \le \varphi < n\pi/2 + \pi$. We choose φ to be smooth at all points where $\varphi \neq n\pi/2 - \pi$, so that in particular φ is smooth near $t = 0, h$.

Suppose $\{(\theta, t) : |\varphi(\theta, t) - n\pi/2| \ge \pi/2\} = \{(\theta, t) : \cos(\varphi(\theta, t) - n\pi/2) \le 0\}$ is a nonempty set. Applying Sard's theorem to the smooth function $\cos(\varphi - n\pi/2)$ in $[0, 2\pi] \times [0, h]$, we can choose $\theta_1 < \pi/2$ so that $|\varphi_0(\theta) - n\pi/2|$, $|\varphi_h(\theta) - n\pi/2| < \theta_1$ for all $\theta \in [0, 2\pi]$ and so that $\{(\theta, t) \in [0, 2\pi] \times [0, h] : |\varphi(\theta, t) - n\pi/2| = \theta_1\}$ is a finite collection of closed Jordan curves together with Jordan arcs with endpoints at $\{0, 2\pi\} \times (0, h)$. See Figure 1 for example.

Let

$$A_{<\theta_1} = \{(\theta, t) \in (0, 2\pi) \times (0, h) : |\varphi - n\pi/2| < \theta_1\}.$$

Necessarily, φ is smooth in $A_{<\theta_1}$. Also let $A_{>\theta_1} = \{(\theta, t) \in (0, 2\pi) \times (0, h) : |\varphi - n\pi/2| > \theta_1\}$. We then have $\overline{A_{>\theta_1}} \subset [0, 2\pi] \times (0, h)$ (see the shaded region in Figure 1 for example).

Define the function

$$R_{\theta_1} : \left[\frac{n\pi}{2} - \pi, \frac{n\pi}{2} + \pi\right) \to \left[\frac{n\pi}{2} - \theta_1, \frac{n\pi}{2} + \theta_1\right]$$

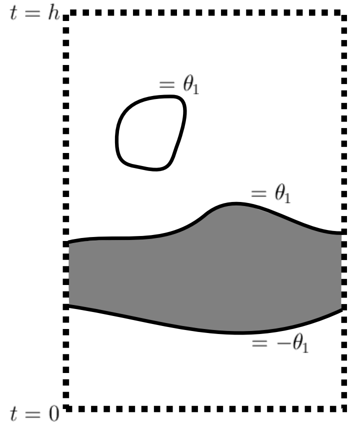

Figure 1. Sard's theorem for $\varphi(\theta, t) - n\pi/2 = \pm\theta_1$. In this case φ is discontinuous in the shaded region, which is $A_{>\theta_1}$.

by

$$R_{\theta_1}(x) = \begin{cases} -\dfrac{\theta_1}{\pi - \theta_1}\Big(x - \Big(\dfrac{n\pi}{2} - \pi\Big)\Big) + \dfrac{n\pi}{2} & \text{for } x \in \Big[\dfrac{n\pi}{2} - \pi, \dfrac{n\pi}{2} - \theta_1\Big], \\ x & \text{for } x \in \Big(\dfrac{n\pi}{2} - \theta_1, \dfrac{n\pi}{2} + \theta_1\Big], \\ -\dfrac{\theta_1}{\pi - \theta_1}\Big(x - \Big(\dfrac{n\pi}{2} + \pi\Big)\Big) + \dfrac{n\pi}{2} & \text{for } x \in \Big(\dfrac{n\pi}{2} + \theta_1, \dfrac{n\pi}{2} + \pi\Big). \end{cases}$$

Considering $R_{\theta_1}(\varphi(\theta, t))$, we see that $R_{\theta_1} \circ \varphi = \varphi$ for $(\theta, t) \in A_{<\theta_1}$. Furthermore, we can immediately see that $R_{\theta_1} \circ \varphi$ is Lipschitz near every point with $\varphi(\theta, t) \neq n\pi/2 - \pi$. However, note that the function defined by

$$\begin{cases} \varphi(\theta, t) + \pi & \text{if } \varphi(\theta, t) < n\pi/2, \\ \varphi(\theta, t) - \pi & \text{if } \varphi(\theta, t) > n\pi/2, \end{cases}$$

is smooth at points where $\varphi(\theta, t) = n\pi/2 - \pi$. Hence, $R_{\theta_1} \circ \varphi$ is Lipschitz in $(0, 2\pi) \times (0, h)$.

Next, since $R_{\theta_1} \circ \varphi$ is Lipschitz, it is differentiable almost everywhere [Evans 1998, Theorem 6, Section 5.8], and we may still define the energy of $R_{\theta_1} \circ \varphi$ by

$$E(R_{\theta_1} \circ \varphi) = \int_0^h \int_0^{2\pi} ((R_{\theta_1} \circ \varphi)_t)^2 + ((R_{\theta_1} \circ \varphi)_\theta)^2 + \cos^2 R_{\theta_1} \circ \varphi \, d\theta \, dt.$$

Observe then that

$$E(R_{\theta_1} \circ \varphi) = \int_{A_{<\theta_1}} (\varphi_t)^2 + (\varphi_\theta)^2 + \cos^2 \varphi \, d\theta \, dt \\ + \Big(\frac{\theta_1}{\pi - \theta_1}\Big)^2 \int_{A_{>\theta_1}} (\varphi_t)^2 + (\varphi_\theta)^2 \, d\theta \, dt + \int_{A_{>\theta_1}} \cos^2 R_{\theta_1} \circ \varphi \, d\theta \, dt.$$

Choose a sequence $\theta_k \nearrow \pi/2$ so that we have regions $A_{<\theta_k}, A_{>\theta_k}$ as above (Sard's theorem says this can be done for almost every θ near $\pi/2$). Note that the sets $A_{<\theta_k} \to \{|\varphi - (n\pi/2)| < \pi/2\}$, $A_{>\theta_k} \to \{|\varphi - (n\pi/2)| \geq \pi/2\}$, and the functions $R_{\theta_k} \circ \varphi \to R_{\pi/2} \circ \varphi$ pointwise. Since $E(V) < \infty$ we have, by the dominated convergence theorem

$$\lim_{k\to\infty} E(R_{\theta_k} \circ \varphi) = \int_{\{|\varphi-(n\pi/2)|<\pi/2\}} (\varphi_t)^2 + (\varphi_\theta)^2 + \cos^2\varphi \, d\theta\, dt \\ + \int_{\{|\varphi-(n\pi/2)|\geq\pi/2\}} (\varphi_t)^2 + (\varphi_\theta)^2 \, d\theta\, dt + \int_{\{|\varphi-(n\pi/2)|\geq\pi/2\}} \cos^2 R_{\pi/2} \circ \varphi \, d\theta\, dt.$$

Since

$$\cos^2\left(-\left(\varphi - \left(\frac{n\pi}{2} - \pi\right)\right) + \frac{n\pi}{2}\right) = \cos^2\left(-\left(\varphi - \left(\frac{n\pi}{2} + \pi\right)\right) + \frac{n\pi}{2}\right) = \cos^2\varphi,$$

we have $\cos^2 R_{\pi/2} \circ \varphi = \cos^2\varphi$. Thus $\lim_{k\to\infty} E(R_{\theta_k} \circ \varphi) = E(V)$.

Note that $R_{\theta_k} \circ \varphi$ is Lipschitz with derivatives in $L^2_{\mathrm{loc}}(\mathbb{R} \times [0, h])$. In other words,

$$R_{\theta_k} \circ \varphi \in W^{1,2}_{\mathrm{loc}}(\mathbb{R} \times [0, h])$$

(see [Evans 1998, Section 5.2.2]) and we can find a θ-periodic function

$$\varphi^\infty \in C(\mathbb{R} \times [0, h]) \cap C^\infty(\mathbb{R} \times (0, h)),$$

so that $\|(R_{\theta_k} \circ \varphi) - \varphi^\infty\|_{W^{1,2}([0,2\pi]\times[0,h])}$ is as small as we please. For this, see [Evans 1998, Section 5.3]; in particular we can apply Theorem 3 of Section 5.3.3 to a bounded smooth region $U \subset \mathbb{R} \times (0, h)$ containing $(0, 2\pi) \times (0, h)$, and we can ensure our approximating functions are θ-periodic.

However, φ^∞ may not have the correct boundary data, and so we fix φ^∞ as follows. Choose $\sigma > 0$ so that $R_{\theta_k} \circ \varphi = \varphi$ for $t \in [0, 2\sigma) \cup (h - 2\sigma, h]$. Pick a smooth function g with $0 \leq g \leq 1$ so that $g(t) = 1$ for $t \in [0, \sigma) \cup (h - \sigma, h]$, $g(t) = 0$ for $t \in (2\sigma, h - 2\sigma)$, and $|g'(t)| \leq 2/\sigma$. Define

$$\tilde{\varphi}(\theta, t) = g(t) R_{\theta_k}(\varphi(\theta, t)) + (1 - g(t))\varphi^\infty(\theta, t).$$

We now compute $E(\tilde{\varphi})$. First,

$$\tilde{\varphi}_t = (\varphi^\infty)_t + g_t((R_{\theta_k} \circ \varphi) - \varphi^\infty) + g((R_{\theta_k} \circ \varphi) - \varphi^\infty)_t.$$

By the inequality $(x + y)^2 \leq (1 + \epsilon)x^2 + ((\epsilon + 1)/\epsilon)y^2$ we have

$$(\tilde{\varphi}_t)^2 \leq (1+\epsilon)((\varphi^\infty)_t)^2 + \left(\frac{\epsilon+1}{\epsilon}\right)\bigl(g_t((R_{\theta_k} \circ \varphi) - \varphi^\infty) + g((R_{\theta_k} \circ \varphi) - \varphi^\infty)_t\bigr)^2.$$

Then $(x + y)^2 \leq 2x^2 + 2y^2$ along with $|g_t| \leq 2/\sigma$ imply

$$(\tilde{\varphi}_t)^2 \leq (1+\epsilon)((\varphi^\infty)_t)^2 + \left(\frac{\epsilon+1}{\epsilon}\right)\Bigl(\frac{8}{\sigma^2}((R_{\theta_k} \circ \varphi) - \varphi^\infty)^2 + 2(((R_{\theta_k} \circ \varphi) - \varphi^\infty)_t)^2\Bigr).$$

Second, we similarly have

$$(\tilde{\varphi}_\theta)^2 \le (1+\epsilon)((\varphi^\infty)_\theta)^2 + \left(\frac{\epsilon+1}{\epsilon}\right)(((R_{\theta_k}\circ\varphi) - \varphi^\infty)_\theta)^2.$$

Third, since $|\cos^2 x - \cos^2 y| \le 2|x-y|$, we have

$$\cos^2\tilde{\varphi} \le \cos^2\varphi^\infty + 2|(R_{\theta_k}\circ\varphi) - \varphi^\infty|.$$

We therefore have by the definition of $\|(R_{\theta_k}\circ\varphi) - \varphi^\infty\|_{W^{1,2}([0,2\pi]\times[0,h])}$

$$E(\tilde{\varphi}) \le (1+\epsilon)E(\varphi^\infty) + \left[\left(\frac{\epsilon+1}{\epsilon}\right)\left(\frac{8}{\sigma^2}+3\right)+2\right]\|(R_{\theta_k}\circ\varphi) - \varphi^\infty\|^2_{W^{1,2}([0,2\pi]\times[0,h])}.$$

Given $\epsilon > 0$, we can choose $R_{\theta_k}\circ\varphi$ so that $E(R_{\theta_k}\circ\varphi) < E(\varphi)+\epsilon$. We can then choose φ^∞ with $\|(R_{\theta_k}\circ\varphi) - \varphi^\infty\|^2_{W^{1,2}([0,2\pi]\times[0,h])}$ sufficiently small so that $E(\varphi^\infty) < E(\varphi)+\epsilon$ as well. Since σ depends only on φ, we can make $E(\tilde{\varphi})$ as close to $E(\varphi)$ as we like. □

We remark that even given Theorem 5.1, we cannot immediately argue as in Theorem 3.2 to show the existence of a minimizer in case the boundary data does not twist too much. To see this, take a sequence φ_k as in Theorem 5.1 with $E(\varphi_k)$ converging to the infimum energy. Unlike in the proof of Theorem 3.2, it is unclear whether the sequence φ_k is equicontinuous. Although we can conclude the φ_k converge to some function φ *weakly* in L^2, it is not clear whether the sequence $\cos\varphi_k$ converges weakly to $\cos\varphi$. Thus, we cannot conclude that $E(\varphi)$ is the infimum energy.

Corollary 5.2. *The minimizer V with horizontal boundary data on $\mathscr{C}$ is unique for $h < \sqrt{8}$.*

Proof. Let $V = V(\varphi)$ be a minimizer, so φ must be θ-independent, and we can write $\varphi(0)=0$. Now, if $\varphi(h)=0$, then by Theorem 4.1 we get $\varphi=0$. If instead, suppose $\varphi(h)=2\pi$, then let

$$t_0 = \inf\left\{t\in[0,h] : \varphi(t) = \frac{\pi}{2}\right\} \quad \text{and} \quad t_1 = \sup\left\{t\in[0,h] : \varphi(t) = \frac{5\pi}{2}\right\}.$$

Define $\tilde{\varphi}(t) = -\varphi(t)$ for $0 \le t < t_0$, $\tilde{\varphi}(t) = -(\pi/2)$ for $t_0 \le t < t_1$, and $\tilde{\varphi}(t) = \varphi(t) - 2\pi$ for $t_1 \le t \le h$. In this case note $E(\tilde{\varphi}) < E(\varphi)$, and we can smooth $\tilde{\varphi}$ and still conclude the same. This is a contradiction, and so we must have $\varphi = 0$. □

6. Computer approximations

In this section we present two numerical approximations of solutions to (2-2) for two surfaces of revolution. To sidestep the possibility of suffering Runge's phenomenon [Runge 1901], our numerical approximations sample Chebyshev points; these are points which cluster near the boundary of $[0,2\pi]\times[0,h]$. To handle

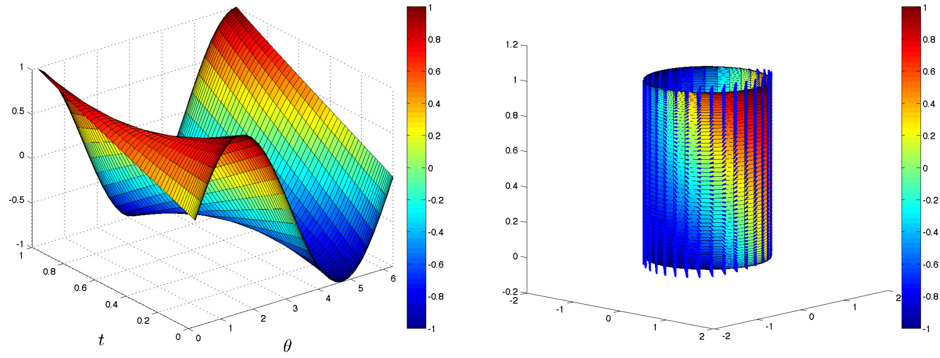

Figure 2. Left: plot of $\varphi(\theta, t)$ for $\varphi(\theta, 0) = \sin\theta$, $\varphi(\theta, 1) = \cos\theta$. Right: plot of $\boldsymbol{V}(\varphi)$ for the same φ.

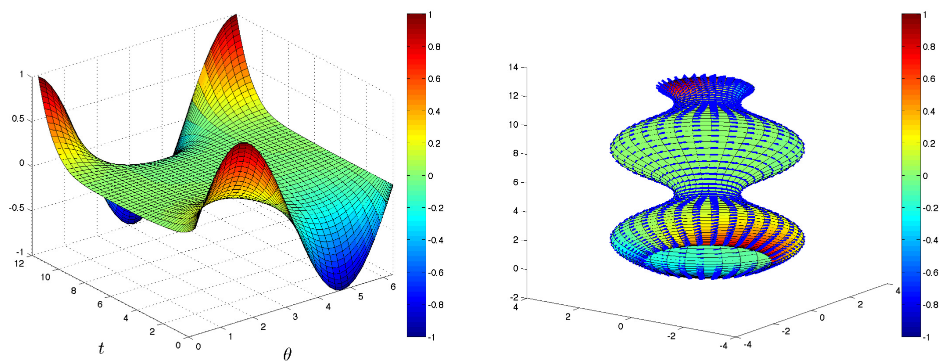

Figure 3. Left: plot of $\varphi(\theta, t)$ for $r(t) = \sin(t) + 2$. Right: plot of $\boldsymbol{V}(\varphi)$ for the same φ.

periodicity in the θ variable, we borrow some theory about Fourier discretization matrices from [Trefethen 2000]. These matrices allow us to solve our differential equation on the interior of the cylinder $(\mathbb{R} \bmod 2\pi) \times [0, h]$ while leaving the boundary conditions fixed.

Our program allows us to input a height h, a radius function $r(t)$, and two functions $\varphi_0(\theta)$ and $\varphi_h(\theta)$ that describe the boundary conditions, and finds a very close approximation of a function $\varphi(\theta, t)$ which satisfies (2-2) with boundary data φ_0, φ_h over $[0, 2\pi] \times [0, h]$.

First, we take the unit cylinder with unit height, and we take boundary data $\varphi_0(\theta) = \sin(\theta)$ and $\varphi_1(\theta) = \cos(\theta)$. We plot the solution $\varphi(\theta, t)$ and also $\boldsymbol{V} = \boldsymbol{V}(\varphi)$ in Figure 2. Second, we take the surface with $r(t) = \sin(t) + 2$, and set $\varphi_0(\theta) = \sin(\theta)$, $\varphi_{12}(\theta) = \cos(\theta)$. We again plot $\varphi(\theta, t)$ and $\boldsymbol{V} = \boldsymbol{V}(\varphi)$ in Figure 3.

7. Future projects

There are a number of projects well-suited for future VIGRE at Rice internships for undergraduates. We mention a few.

The first problem is to extend Theorem 3.2 to the case when the boundary data V_0, V_h cannot be written using constant angle functions. A preliminary challenge is to show the existence of $\boldsymbol{V} \in \mathfrak{X}^1(\mathcal{S})$ with $\boldsymbol{V}|_{\partial\mathcal{S}} = V_0$, V_h in the case of general continuous boundary data. (When V_0, V_h are smooth, this can be done using an argument similar to the end of the proof of Theorem 5.1.) Theorem 5.1 provides a first step in showing the existence of minimizers, at least if we assume V_0, V_h do not twist too much.

Related to Theorem 5.1 is finding the largest h_0 so that Corollary 5.2 continues to hold with $h < h_0$. This is related to the analogous question for Theorem 4.1 and Lemma 4.2, and similar to the well-studied question of finding the optimal constant in the usual Poincaré inequality [Bebendorf 2003].

Another direction is to consider the following inverse problem: given an angle function $\varphi(\theta, t)$, find the surface of revolution $\mathcal{S}$ such that $\boldsymbol{V} = \boldsymbol{V}(\varphi)$ minimizes energy with respect to the boundary data $\boldsymbol{V}(\varphi(\theta, 0))$, $\boldsymbol{V}(\varphi(\theta, h))$. A different problem with a similar flavor is to find, given an angle function $\varphi(\theta, t)$, which surface of revolution $\mathcal{S}$ is such that $E(\varphi)$ is the least.

The torus of revolution also provides a fountain of projects, by asking which smooth tangent unit-length vector field minimizes energy. Some work has been done by the authors in this direction [Rosales et al. 2010], most notably in computing the relationship between the radii of the tube and the distance to the center of the tube of the torus with the energies of the normalizations of the coordinate vector fields, when the torus is given the usual parametrization.

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