Innovations in Incidence Geometry

Algebraic, Topological and Combinatorial

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msp

Vol. 17 No. 3 2019

Innovations in Incidence Geometry
Algebraic, Topological and Combinatorial
vol. 17, no. 3, 2019
dx.doi.org/10.2140/iig.2019.17.201
msp

Groups of compact 8-dimensional planes: conditions implying the Lie property

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The automorphism group Σ of a compact topological projective plane with an 8-dimensional point space is a locally compact group. If the dimension of Σ is at least 12, then Σ is known to be a Lie group. For the connected component Δ of Σ it is shown that $\dim \Delta \geq 10$ suffices, if Δ is semisimple or does not fix exactly a nonincident point-line pair or a double-flag. Δ is also a Lie group, if Δ has a compact connected 1-dimensional normal subgroup and $\dim \Delta \geq 11$.

1. Introduction

A systematic study of compact 8-dimensional projective planes began with [Salzmann 1979]. Many of the results obtained in the following 15 years are presented in Chapter 8 of the treatise *Compact projective planes* [Salzmann et al. 1995]. An up-to-date account of more recent contributions to the theme can be found in [Salzmann 2014]. The *classical* model, the projective plane over the quaternion field $\mathbb{H}$, has the automorphism group $\mathrm{PSL}_3\mathbb{H}$ of dimension 35. If $\mathcal{P} = (P, \mathfrak{L})$ is any other compact 8-dimensional plane, then its automorphism group $\Sigma = \mathrm{Aut}\,\mathcal{P}$, taken with the compact-open topology, is a locally compact transformation group of the point space P as well as of the line space $\mathfrak{L}$, and $\dim \Sigma \leq 18$. All planes $\mathcal{P}$ such that $\dim \Sigma \geq 17$ have been described explicitly [Hähl 1986; Salzmann 2014]. The goal is to extend these results and to determine all pairs $(\mathcal{P}, \Delta)$, where Δ is a suitable *connected* subgroup of $\mathrm{Aut}\,\mathcal{P}$. As in the cases of finite projective planes or compact connected planes of smaller dimension, such a classification is possible only if the group — in our case its dimension — is not too small. An important step is to show that Δ is a Lie group. In all known examples, lines are homeomorphic to the 4-sphere $\mathbb{S}_4$, each closed proper subplane is connected and has a point space of dimension 2 or 4, and Σ is even a Lie group. In general, however, it is only known that lines are homotopy equivalent to $\mathbb{S}_4$; it is conceivable that some planes

MSC2010: 22D05, 51H10.
Keywords: topological plane, Lie group.

have compact 0-dimensional subplanes; and it is an open problem whether or not Σ is always a Lie group. According to [Priwitzer 1994], the following theorem holds: *if* $\dim \Sigma \geq 12$, *then* Σ *is a Lie group*. Depending on the structure of a connected subgroup Δ and the configuration $\mathcal{F}_\Delta$ of its fixed elements (points and lines), sharper bounds will be obtained here.

2. Preliminaries and background

This section contains a collection of basic facts. $\mathcal{P} = (P, \mathfrak{L})$ will always be a compact 8-dimensional projective plane if not stated otherwise; Δ denotes a connected closed subgroup of $\operatorname{Aut} \mathcal{P}$.

Notation. The notation is more or less standard and agrees with that in the book [Salzmann et al. 1995]. A *flag* is an incident point-line pair; a *double flag* consists of two points, say u, v, their join uv, and a second line in the *pencil* $\mathfrak{L}_v$. Homeomorphism is indicated by $\approx$. As customary, $\mathrm{Cs}_\Delta\, \Gamma$ or just $\mathrm{Cs}\, \Gamma$ is the centralizer of Γ in Δ. Distinguish between the commutator subgroup Γ' and the connected component Γ^1 of the topological group Γ. The coset space $\Delta/\Gamma = \{\Gamma\delta \mid \delta \in \Delta\}$ has the (covering) dimension $\Delta : \Gamma = \dim \Delta - \dim \Gamma$. The group $\Delta_{[c,A]}$ consists of the axial collineations in Δ with axis A and center c. A collineation group Γ is said to be *straight* if each orbit x^Γ is contained in some line. In this case a theorem of Baer [1946] asserts that either $\Gamma = \Gamma_{[c,A]}$ is a group of axial collineations or the fixed configuration $\mathcal{F}_\Gamma$ is a Baer subplane.

2.1. ***Baer subplanes.*** It is known that *each* 4*-dimensional closed subplane* $\mathcal{B}$ *of a compact* 8*-dimensional plane* $\mathcal{P}$ *is a Baer subplane*; i.e., each point of $\mathcal{P}$ is incident with a line of $\mathcal{B}$ (and dually, each line of $\mathcal{P}$ contains a point of $\mathcal{B}$); see [Salzmann 2003, §3] or [Salzmann et al. 1995, 55.5] for details. Lines of a Baer subplane are homeomorphic to $\mathbb{S}_2$. If $\mathcal{P}$ contains a closed Baer subplane $\mathcal{B}$, it follows easily that the pencil of lines through a point outside $\mathcal{B}$ is a manifold, and hence, the lines of $\mathcal{P}$ are homeomorphic to $\mathbb{S}_4$; see [Salzmann et al. 1995, 53.10] or [Salzmann 2003, 3.7]. By a result of Löwen [1999], any two closed Baer subplanes of $\mathcal{P}$ have a point and a line in common. Generally, $\langle \mathcal{M} \rangle$ will denote the smallest *closed* subplane of $\mathcal{P}$ containing the set $\mathcal{M}$ of points and lines. We write $\mathcal{B} \lessdot \mathcal{P}$ if $\mathcal{B}$ is a Baer subplane.

2.2. ***Stiffness.*** In the classical plane $\mathcal{H}$, the stabilizer $\Lambda = \Sigma_\mathfrak{e}$ of any *frame* $\mathfrak{e}$ (= nondegenerate quadrangle) is isomorphic to $\mathrm{SO}_3\mathbb{R}$; in particular, Λ is compact and $\dim \Lambda = 3$. In any plane, Λ can be identified with the automorphism group of the ternary field H_τ defined with respect to $\mathfrak{e}$. The fixed elements of Λ form a closed subplane $\mathcal{E} = \mathcal{F}_\Lambda$. It is not known if $\mathcal{E}$ is always connected or if Λ is compact in general. Therefore, the following *stiffness* results play an important role:

(1) $\dim \Lambda \le 4$ [Bödi 1994].

(2) *If* $\mathcal{F}_\Lambda$ *is connected or if* Λ *is compact*, *then* $\dim \Lambda \le 3$ [Salzmann et al. 1995, 83.12–13].

(3) *If* $\mathcal{F}_\Lambda$ *is contained in a Baer subplane* $\mathcal{B}$, *then* $\mathcal{F}_\Lambda$ *is connected and the connected component* Λ^1 *of* Λ *is compact* ([Salzmann et al. 1995, 55.4 and 83.9] *or* [Salzmann 1979, (*)]).

(4) *If*, *moreover*, $\mathcal{B}$ *is* Λ*-invariant*, *then* $\dim \Lambda \le 1$ [Salzmann et al. 1995, 83.11],

($\hat{4}$) *if* $\mathcal{F}_\Lambda$ *itself is a Baer subplane*, *then* Λ *is compact* [Salzmann et al. 1995, 83.6].

(5) *If* Λ *is compact*, *then* Λ *is commutative or* $\Lambda^1 \cong \mathrm{SO}_3\mathbb{R}$ [Salzmann 1979, 2(1)].

(6) *The stabilizer* Ω *of a degenerate quadrangle has dimension at most* 7 [Salzmann et al. 1995, 83.17].

(7) *If* $\dim \Omega = 7$, *then* $\Omega^1 \cong e^{\mathbb{R}} \cdot \mathrm{SO}_4\mathbb{R}$ *and lines are* 4*-spheres* [Salzmann 1979, (**)].

(8) *If a subgroup* $\Phi \cong \mathrm{SO}_3\mathbb{R}$ *of* Δ *fixes a line* W, *then each involution in* Φ *is planar. Either* Φ *has no fixed point on* W *or* $\mathcal{F}_\Phi$ *is a* 2*-dimensional subplane* [Salzmann 2010, Observation].

2.3. *Fixed elements.* *The Lefschetz fixed-point theorem implies that each homeomorphism* $\varphi : P \to P$ *has a fixed point.*

(a) *By duality*, *each automorphism of* $\mathcal{P}$ *fixes a point and a line* [Salzmann et al. 1995, 55.19].

(b) *The solvable radical* $\mathrm{P} = \sqrt{\Delta}$ *of* Δ *fixes some element of* $\mathcal{P}$.

(c) *If* $\mathcal{F}_\Delta = \varnothing$, *then* Δ *is semisimple with trivial center*, *or* Δ *induces a simple group on some connected closed* Δ*-invariant subplane.*

Proof. Argument (A) If Θ is a commutative connected normal subgroup of Δ and if $\mathbb{1} \ne \zeta \in \mathrm{Cs}\,\Theta$, then $p^\zeta = p$ for some point p, and either $p^\Theta = p$, or p^Θ is contained in a fixed line of Θ, or p^Θ generates a connected (closed) subplane $\mathcal{S} = \langle p^\Theta \rangle$ and $\zeta|_\mathcal{S} = \mathbb{1}$. In the latter case, $\overline{\Theta} = \Theta|_\mathcal{S} \ne \mathbb{1}$, and $\mathcal{S}$ is a proper subplane of $\mathcal{P}$.

(b) The claim will be proved by induction over the solvable length. Suppose that Δ itself is solvable and that the normal subgroup Θ has no fixed element. Let $\mathcal{S}$ be a proper subplane as given by (A). If $\dim \mathcal{S} = 2$, then $\mathcal{S}$ has no proper closed subplane [Salzmann et al. 1995, 32.7], and Θ has a fixed element in $\mathcal{S}$. If $\mathcal{S}$ is a Baer subplane, then (A) can be applied to $\overline{\Theta}$; again $\mathcal{F}_\Theta \ne \varnothing$, say $p^\Theta = p$. Then $\Theta|_{p^\Delta} = \mathbb{1}$. Either Δ fixed some element or $\mathcal{D} = \langle p^\Delta \rangle$ is a proper subplane. In the latter case, $\Delta|_\mathcal{D} = (\Delta/\Theta)|_\mathcal{D}$ has a fixed element by induction.

(c) This will be proved successively for planes $\mathcal{R}$ of dimension 2, 4, and 8. If Δ is not semisimple, then $\mathrm{P} = \sqrt{\Delta} \neq \mathbb{1}$ by definition, and P fixes some element by step (b), say $p^{\mathrm{P}} = p$. Assume also that $\mathcal{F}_{\Delta} = \varnothing$. Then p^{Δ} is not contained in a line and $\langle p^{\Delta} \rangle = \mathcal{S} \leq \mathcal{R}$ is a closed subplane; normality of P implies $\mathrm{P}|_{\mathcal{S}} = \mathbb{1}$. If $\zeta \neq \mathbb{1}$ is a central element of Δ, then (A) yields a common fixed element p of ζ and P, and $\zeta|_{\mathcal{S}} = \mathrm{P}|_{\mathcal{S}} = \mathbb{1}$.

If $\dim \mathcal{R} = 2$, there is no proper closed subplane, $\mathrm{P}|_{\mathcal{R}} = \mathbb{1} = \zeta|_{\mathcal{R}}$, and Δ is semisimple with trivial center, and hence Δ is strictly simple; see [Salzmann et al. 1995, 33.7] or [Salzmann 1967, 5.2]. If $\dim \mathcal{R} = 4$, then $\mathrm{P} \neq \mathbb{1}$ or $\zeta \neq \mathbb{1}$ implies $\mathcal{S} \neq \mathcal{R}$, $\dim \mathcal{S} = 2$, and $\bar{\Delta} = \Delta|_{\mathcal{S}} \neq \mathbb{1}$ is simple. Finally, let $\dim \mathcal{R} = 8$. Then $\mathcal{S} = \langle p^{\Delta} \rangle < \mathcal{R}$, $\dim \mathcal{S} \leq 4$, and $\mathcal{F}_{\bar{\Delta}} = \varnothing$. Either $\dim \mathcal{S} = 2$ and $\Delta|_{\mathcal{S}}$ is simple by what has just been proved, or $\dim \mathcal{S} = 4$ and $\bar{\Delta}$ is semisimple with trivial center. In the latter case $\bar{\Delta}$ is simple by [Salzmann et al. 1995, 71.8]. □

2.4. ***Dimension formula.*** By [Halder 1971] or [Salzmann et al. 1995, 96.10], the following holds for the action of Δ on P or on any closed Δ-invariant subset M of P, and for any point $a \in M$:

$$\dim \Delta = \dim \Delta_a + \dim a^{\Delta} \quad \text{or} \quad \dim a^{\Delta} = \Delta : \Delta_a .$$

2.5. ***Approximation theorem***, see [Salzmann et al. 1995, 93.8].

(a) *Every locally compact group Γ has an open subgroup Δ which is an extension of its connected component Δ^1 by a compact group.*

(b) *If Δ is locally compact and Δ/Δ^1 is compact, then Δ has arbitrarily small compact normal subgroups* N *such that Δ/N is a Lie group.*

(c) *If, moreover,* $\dim \Delta$ *is finite, then* $\dim \mathrm{N} = 0$ *for each sufficiently small subgroup* $\mathrm{N} \leq \Delta$.

2.6. ***Groups with open orbits.*** *Let L be a line of the 8-dimensional plane $\mathcal{P}$, and let Δ be a closed subgroup of* $\mathrm{Aut}\,\mathcal{P}$ *with $L^{\Delta} = L$. If $U \subseteq L$ is a Δ-orbit which is open or, equivalently, satisfies* $\dim U = \dim L$, *then L is a manifold and Δ induces a Lie group on U. It follows that all lines are manifolds homeomorphic to* $\mathbb{S}_4$ (adapted from [Salzmann et al. 1995, 53.2]).

2.7. ***Compact groups on*** $\mathbb{S}_4$ (Richardson). *If a compact connected group Φ acts effectively on the 4-sphere S, and if Φ has an orbit of dimension > 1, then Φ is a Lie group and (Φ, S) is equivalent to the obvious standard action of a subgroup of* $\mathrm{SO}_5\mathbb{R}$ *on* $\mathbb{S}_4$ *or* $\Phi \cong \mathrm{SO}_3\mathbb{R}$ *has no fixed point on S* [Salzmann et al. 1995, 96.34].

2.8. ***Theorem*** (Löwen). *If the connected subgroup Δ of* $\mathrm{Aut}\,\mathcal{P}$ *fixes the line W and if Δ_x is a Lie group for each $x \notin W$, then Δ itself is a Lie group.*

Proof. The following has been shown in [Löwen 1976]. Let (Γ, M) be a locally compact connected transformation group of finite dimension, where $X = M \cup \infty$ is a Peano continuum, all cohomology groups $H^q(X, \mathbb{Q})$ are finite-dimensional, and $H^q(X, \mathbb{Q}) = 0$ for some n and all $q \geq n$; moreover, the Euler characteristic $\chi(X, \mathbb{Q}) \neq 0, 1$. If all stabilizers Γ_x with $x \in M$ are Lie groups, then Γ is a Lie group. This result applies to $(\Delta, P \setminus W)$: by [Salzmann et al. 1995, 51.6, 51.8, 52.12], the one-point compactification X of $P \setminus W$ is homeomorphic to the quotient space P/W, and X is a Peano continuum (i.e., a continuous image of the unit interval); moreover, X is homotopy equivalent to $\mathbb{S}_8$, and X has Euler characteristic $\chi(X) = 2$. □

2.9. *Compact groups.* *Each compact connected group is of the form* $(\mathrm{A} \times \Lambda)/\mathrm{N}$, *where* A *is the connected component of the center and* Λ *is a direct product of compact simply connected almost simple Lie groups*; *the* kernel N *is a compact central subgroup of dimension* $\dim \mathrm{N} = 0$. *A compact connected commutative normal subgroup* Θ *of a connected group* Δ *is contained in the center of* Δ [Salzmann et al. 1995, 93.11, 93.19].

2.10. *Groups of subplanes.* *The automorphism group of every proper connected closed subplane is a Lie group by* [Salzmann et al. 1995, 32.21, 71.2].

2.11. *Lemma.* *Suppose that* Φ *is a compact connected Lie group and that the compact connected* 1*-dimensional group* Θ *is not a Lie group. If* $\Gamma = \Phi\Theta$ *acts effectively on a subspace* M *of the plane, if* $\mathrm{H} = \Phi \cap \Theta$ *is finite, and if* $\Theta_a = \mathbb{1}$ *and* Φ_a *is finite for some* $a \in M$, *then* $\dim a^\Gamma > \dim a^\Phi$.

Proof. First, let $\mathrm{H} = \mathbb{1}$, so that $\Gamma = \Phi \times \Theta$. If $\dim a^\Gamma = \dim a^\Phi$, then the connected component Ξ of $(\Phi\Theta)_a$ satisfies $\dim \Xi = 1$. Consider the restrictions of the projection maps $\pi : \Xi \to \Phi$ and $\varrho : \Xi \to \Theta$. Both maps are continuous homomorphisms. The kernel $\ker \pi$ is contained in $\Theta_a = \mathbb{1}$ and π is injective. Compactness of Φ implies that Ξ is isomorphic to a closed subgroup of Φ; hence, Ξ is a Lie group. From $\ker \varrho \leq \Phi_a$ we infer that $\ker \varrho$ is finite, and [Salzmann et al. 1995, 93.12] shows that ϱ is surjective, but then Θ would be a Lie group contrary to the assumption. In the general case analogous arguments apply to the natural maps $\pi : \Xi \to \Phi/\mathrm{H}$ and $\varrho : \Xi \to \Theta/\mathrm{H}$. □

2.12. *Definition.* For the remainder of this article, we shall call a compact, connected 1-dimensional subgroup of Δ a *serpentine* subgroup. The letter Θ will be reserved for such subgroups. They are 1-tori or, more frequently, solenoids; the latter are not Lie groups.

3. No fixed elements

Suppose in this section that $\mathcal{F}_\Delta = \varnothing$.

3.1. ***Theorem.*** *If* $\dim\Delta \geq 10$, *or if* Δ *is semisimple and* $\dim\Delta \geq 9$, *then* Δ *is a Lie group.*

Proof. By the approximation theorem, there is a compact 0-dimensional central subgroup N such that Δ/N is a Lie group. Suppose that $\mathbb{1} \neq \zeta \in \mathrm{N}$, and let $p^\zeta = p$ be a fixed point of ζ. A slight variation of argument (A) in the proof of 2.3 shows that $\mathcal{E} = \langle p^\Delta \rangle$ is a connected proper subplane.

(a) If $\dim\mathcal{E} = 2$, then Δ induces on $\mathcal{E}$ a group $\Delta^* = \Delta/\mathrm{K}$ of dimension at most 8, and stiffness yields $\dim\mathrm{K} \leq 3$. Hence, $\dim\mathrm{K} \geq 1$ and $\Delta : \mathrm{K} \geq 6$. In particular, $\mathcal{E}$ is isomorphic to the classical real projective plane [Salzmann et al. 1995, 33.6], and Δ^* is a subgroup of $\mathrm{SL}_3\mathbb{R}$. As Δ^* has no fixed element, Δ^* is simple by [Salzmann 1967, 5.2] or [Salzmann et al. 1995, 33.1] (see also 2.3 above), and then $\dim\Delta^* = 8$, $\Delta^* \cong \mathrm{SL}_3\mathbb{R}$. If Δ is semisimple, the kernel K is also simple, and $\dim\mathrm{K} = 3$. In any case, $\dim\Delta \geq 10$ and $\dim\mathrm{K} \geq 2$. Because N induces a Lie group on $\mathcal{E}$ (see 2.10 or [Salzmann et al. 1995, 32.21]), we may assume that $\mathrm{N} < \mathrm{K}$. Either $\mathcal{F}_\zeta \lessdot \mathcal{P}$ for some $\zeta \in \mathrm{N} \setminus \{\mathbb{1}\}$, or N acts freely on the set of *exterior* points (points not belonging to $\mathcal{E}$). In the first case, the stiffness result (4) would imply $\dim\mathrm{K} \leq 1$. Hence, $\mathrm{N}_z = \mathbb{1}$ for each exterior point z on an *interior* line L (a line of $\mathcal{E}$). If $\dim\Delta_L - \dim\Delta_z = 4$, then Δ_L induces a Lie group on the orbit z^{Δ_L} by 2.6. Therefore, N is finite, and Δ would be a Lie group. Consequently $\Delta : \Delta_z \leq 2+3$. Choose two interior points $a, b \notin L$ and consider the stabilizer $\Omega = \Delta_{z,a,b}$; it fixes also the point $L \cap ab$ and hence 3 collinear points of $\mathcal{E}$. Linear algebra shows that Ω fixes all interior points of ab; moreover, $\dim\Omega \geq 1$ and $\Omega|_{z^\mathrm{N}} = \mathbb{1}$. Thus, $\mathcal{F}_\Omega$ is a connected proper subplane of dimension 2 or 4, and N acts effectively on $\mathcal{F}_\Omega$. From 2.10 it follows that N is a Lie group, and so is Δ.

(b) Finally, let $\mathcal{E} \lessdot \mathcal{P}$ and note that $\Delta^* = \Delta|_\mathcal{E}$ has no fixed element. According to [Salzmann et al. 1995, 71.4, 71.8], the group $\Delta^* = \Delta/\mathrm{K}$ is strictly simple. Stiffness shows $\dim\mathrm{K} \leq 1$ and $\Delta : \mathrm{K} > 8$ (since $\dim\Delta \geq 10$ or $\dim\mathrm{K} = 0$). All possibilities for Δ^* are listed in [Salzmann et al. 1995, 71.8]; only $\mathrm{PSL}_3\mathbb{C}$ has dimension > 8. Hence, $\dim\Delta \geq 16$, and Δ is a Lie group by [Priwitzer 1994]. □

Remark. Previously 3.1 was only known for $\dim\Delta \geq 11$; see [Salzmann 2010, Theorem 1.1] or [Salzmann 2014, 2.1].

3.2. ***Compact normal subgroup.*** *If* Δ *has a serpentine normal subgroup* Θ *and if* $\dim\Delta \geq 9$, *then* Δ *is a Lie group or, conceivably,* $\Delta \cong \mathrm{SL}_3\mathbb{R}\times\Theta$ *induces the full collineation group on some invariant* 2*-dimensional desarguesian subplane.*

Proof. The proof follows the scheme of the previous one, and the same notation will be used. If Δ is not a Lie group, then $\dim \Delta = 9$ by 3.1.

(a) Let $\mathcal{E} = \langle p^\Delta \rangle$ be a 2-dimensional subplane. Again $\mathcal{E}$ is the classical real plane and $\Delta^* = \Delta/\mathrm{K} \cong \mathrm{SL}_3\mathbb{R}$ is simple. Hence, $\Theta\mathrm{N} \le \mathrm{K}$. Either $\Delta' \cong \Delta^*$ or Δ' is a twofold covering of $\mathrm{SL}_3\mathbb{R}$. In the *first case*, each involution in Δ' is a reflection of $\mathcal{P}$ (if $\mathcal{F}_\beta \lessdot \mathcal{P}$ for some involution β, then N induces a Lie group on $\mathcal{F}_\beta$ by 2.10, the induced map $\beta|_\mathcal{E}$ is a reflection, $\langle \mathcal{E}, \mathcal{F}_\beta \rangle = \mathcal{P}$, and Δ would be a Lie group). Consequently, there is a translation group $\Delta_{[L,L]} \cong \mathbb{R}^2$ for each *interior* axis L. It remains an open problem whether or not Θ must be a Lie group in this situation.

In the *second case*, the center of Δ' contains an involution ι such that $\mathcal{F}_\iota \lessdot \mathcal{P}$, and the lines of $\mathcal{P}$ are homeomorphic to $\mathbb{S}_4$ (see 2.1). Moreover, $\Theta|_{\mathcal{F}_\iota} = \mathbb{1}$ by the stiffness property [Salzmann et al. 1995, 71.7(a)] or by [Grundhöfer and Salzmann 1990, XI.9.3] (recall that $\Theta|_\mathcal{E} = \mathbb{1}$). Hence, Θ acts freely on the set of points not belonging to $\mathcal{F}_\iota$. Let L be a line of $\mathcal{E}$ and put $L' = L \setminus \mathcal{F}_\iota$. The group Δ' has a subgroup $\Upsilon \cong \mathrm{SU}_2\mathbb{C}$, and the connected component Φ of Υ_L is a torus. As L' is dense in L, it follows that Φ acts effectively on L' (note that $\mathcal{E}$ is classical). Let $p \in L'$ such that $p^\Phi \ne p$, $\dim \Phi_p = 0$, and Φ_p is finite. We have $\Phi \cap \Theta = \langle \iota \rangle$. Therefore, Lemma 2.11 applies and shows that $1 = \dim p^\Phi < \dim p^{\Phi\Theta} = 2$. By [Salzmann et al. 1995, 96.24] or 2.7 above Δ is a Lie group.

(b) If $\langle p^\Delta \rangle = \mathcal{C} \lessdot \mathcal{P}$, the lines of $\mathcal{P}$ are 4-spheres. From 2.3 and 2.9 it follows that $\Theta|_\mathcal{C} = \mathbb{1}$ and that $\Delta|_\mathcal{C}$ is semisimple of dimension 8. By 2.3(c) and [Salzmann et al. 1995, 71.8], the group $\Delta^* = \Delta|_\mathcal{C}$ is isomorphic to $\mathrm{SL}_3\mathbb{R}$ or to $\mathrm{PSU}_3(\mathbb{C}, r)$, $r \le 1$. For each of the unitary groups, there is an interior line L such that $\mathrm{SU}_2\mathbb{C}$ acts nontrivially on the set L' of exterior points of L. In particular, a maximal compact subgroup of Δ_L has an orbit of dimension > 1 on L'. Recall that N acts freely on the set of exterior points. By Richardson's theorem, Δ_L induces a Lie group on L'. Hence, N and Δ are Lie groups. If $\Delta^* \cong \mathrm{SL}_3\mathbb{R}$, then there exists a Δ-invariant 2-dimensional subplane of $\mathcal{C}$ [Salzmann et al. 1995, 72.3], and $\dim L^\Delta = 2$ for a suitable line L. Hence, $\Delta'|_{L'}$ contains a circle group Φ. Again Φ acts effectively on L'. The proof can now be completed exactly as at the end of step (a). □

3.3. ***Normal vector group.*** *If* $\mathcal{F}_\Delta = \varnothing$, *if* Δ *has a minimal normal vector subgroup* Ξ, *and if* $\dim \Delta \ge 7$, *then* Δ *is a Lie group.*

Proof. From 2.3 it follows that $\mathcal{F} = \mathcal{F}_\Xi$ is a proper connected Δ-invariant subplane. There is a compact group $\mathrm{N} \lhd \Delta$ such that Δ/N is a Lie group. We may assume that $\dim \mathrm{N} = 0$, that N is not a Lie group, and that $\mathrm{N}|_\mathcal{F} = \mathbb{1}$. Note that $\dim \Delta \le 9$ by 3.1. If $\mathcal{F} \lessdot \mathcal{P}$, then $\Xi|_\mathcal{F} = \mathbb{1}$ by definition, and Ξ would be compact by stiffness. Hence, $\mathcal{F}$ is a 2-dimensional subplane.

(a) First, let $\dim \Delta = 9$. Then the induced group $\Delta|_{\mathcal{F}} = \Delta/\mathrm{K}$ is simple by 2.3; in fact, $\Delta/\mathrm{K} \cong \mathrm{SL}_3\mathbb{R}$. A maximal compact subgroup Φ of Δ is connected by the Malcev–Iwasawa theorem, and $\mathrm{N} < \Phi$. Consequently, $\dim \Phi = 4$ (or $\Phi \cong \mathrm{SO}_3\mathbb{R}$ by [Salzmann et al. 1995, 93.12]), Φ is a product of Φ' and a compact group $\Theta = \mathrm{K}^1$, $\Xi \cap \Theta = \mathbb{1}$, and Ξ would be contained in Δ/K^1, which is locally isomorphic to $\mathrm{SL}_3\mathbb{R}$.

(b) In the cases $\dim \Delta \in \{7, 8\}$ the induced group Δ/K is simple by 2.3, and then $\Delta : \mathrm{K} = 3$ [Salzmann et al. 1995, 33.6,7], but $\dim \mathrm{K} \leq 3$ by stiffness. □

Remark. If $\mathcal{F}_\Delta$ is not empty, Ξ can be a group of axial collineations, and in the case $\Xi \lhd \Delta$ there are no sharper results than in general.

4. Exactly one fixed element

Up to duality, we may assume that $\mathcal{F}_\Delta$ consists of a line W.

4.1. ***Semisimple groups.*** *If the semisimple group Δ fixes exactly one line and possibly some points on this line, and if* $\dim \Delta > 3$, *then Δ is a Lie group* [Salzmann 2010, Theorem 1.3].

4.2. ***Theorem.*** *If $\mathcal{F}_\Delta = \{W\}$ and if* $\dim \Delta \geq 9$, *then Δ is a Lie group.*

Proof. (a) Again there exist arbitrarily small compact central subgroups $\mathrm{N} \leq \Delta$ of dimension 0 such that Δ/N is a Lie group; see 2.5. If N acts freely on $P \setminus W$, then each stabilizer Δ_x with $x \notin W$ is a Lie group because $\Delta_x \cap \mathrm{N} = \mathbb{1}$, and Δ is a Lie group by 2.8.

(b) If $x^\zeta = x \notin W$ for some $\zeta \in \mathrm{N} \setminus \{\mathbb{1}\}$, then x^Δ is not contained in a line, $\zeta|_{x^\Delta} = \mathbb{1}$, and $\mathcal{E} = \langle x^\Delta, W\rangle$ is a proper connected subplane. Assume in this step that $\mathcal{E}$ is 2-dimensional. In this case the claim follows by similar arguments as in 3.1(a): let $\Delta^* = \Delta|_{\mathcal{E}} = \Delta/\mathrm{K}$. Then $\Delta : \mathrm{K} \leq 6$ by [Salzmann 1967, 3.19] or [Salzmann et al. 1995, 33.6] together with the dimension formula 2.4, and $\dim \mathrm{K} \leq 3$ by stiffness. It follows that $\dim \mathrm{K} = 3$, $\Delta : \mathrm{K} = 6$, and $\mathcal{E} \setminus W$ is the classical real affine plane [Salzmann 1967, 4.3]. As Δ^* is a Lie group, we may assume that $\mathrm{N} < \mathrm{K}$. Again N acts freely on the set of exterior points. The remainder of the proof is as in 3.1(a) with W instead of L.

(c) If Δ is not a Lie group, the case $\mathcal{E} \lessdot \mathcal{P}$ will lead to a contradiction. Write again $\Delta^* = \Delta|_{\mathcal{E}} = \Delta/\mathrm{K}$. Note that K is compact and acts freely on the set of points not in $\mathcal{E}$. If Δ is transitive on $W \cap \mathcal{E} \approx \mathbb{S}_2$, then a maximal compact subgroup of Δ induces a Lie group on W by 2.7. Hence, K and Δ are Lie groups. Therefore, Δ has a 1-dimensional orbit $V \subset W \cap \mathcal{E}$. Brouwer's theorem [Salzmann et al. 1995, 96.30] (see also [Hofmann 1965]) shows that $\Delta|_V = \Delta/\Gamma$ has dimension at most 3. Consequently $\dim \Gamma \geq 6$. Choose a point $v \in V$, a line L in $\mathcal{E}$ with $v \in L$, and an *exterior* point $z \in L$. By 2.6 we have $\dim z^{\Gamma_L} < 4$. Note that $\Lambda = (\Gamma_{L,z})^1$ fixes V pointwise and that $\dim \Lambda > 0$. Because N acts freely on $L \setminus \mathcal{E}$ and $\mathrm{N} \leq \mathrm{Cs}\,\Delta$, it follows that $\mathcal{F}_\Lambda$ is a proper connected subplane. Now N is a Lie group by 2.10. □

5. Collinear fixed points

Suppose in this section that Δ fixes a unique line W and one or more points on W.

5.1. ***Theorem.*** *Let $\mathcal{F}_\Delta = \{v, W\}$ be a flag. If* $\dim \Delta \geq 10$, *then Δ is a Lie group.*

Proof. By the approximation theorem, there is a compact 0-dimensional normal subgroup N such that Δ/N is a Lie group. Because of 2.8 we may assume that $x^\zeta = x$ for some $\zeta \in \mathrm{N} \setminus \{\mathbb{1}\}$ and some $x \notin W$. As x^Δ is not contained in a line and $\zeta|_{x^\Delta} = \mathbb{1}$, it follows that $\mathcal{C} = \langle x^\Delta, v, W\rangle$ is a proper connected subplane. If $\mathcal{C}$ is 2-dimensional, then $\dim \Delta|_\mathcal{C} \leq 5$ and $\dim \Delta \leq 8$ by stiffness. Therefore, $\mathcal{C}$ is a Δ-invariant Baer subplane. The induced group $\Delta|_\mathcal{C} = \Delta/\mathrm{K}$ is a Lie group by 2.10. Hence, it may be supposed that $\mathrm{N} \leq \mathrm{K}$. Obviously, K acts freely on the set of *exterior* points (points not in $\mathcal{C}$), and $\dim \mathrm{K} \leq 1$ by stiffness. Thus, $\Delta : \mathrm{K} \geq 9$, and $\mathcal{C}$ is isomorphic to the classical complex plane [Salzmann et al. 1995, 72.8]. Choose *interior* points $u, w \in W$, an interior line L in the pencil $\mathfrak{L}_v$, and an *exterior* point $z \in L$. If N is not a Lie group, then the connected component Λ of $\Delta_{u,w,z}$ has positive dimension by 2.6, because $\Delta : \Delta_{u,w,L} \leq 6$. Note that $z^\mathrm{N} \subset \mathcal{F}_\Lambda$ and that Λ fixes all interior points of W, so that $\mathcal{F}_\Lambda$ is a connected proper subplane. Now N is a Lie group by 2.10. □

5.2. ***Theorem.*** *If $\mathcal{F}_\Delta = \langle u, v\rangle$ and if* $\dim \Delta \geq 8$, *then Δ is a Lie group.*

For a proof see [Salzmann 2017, Lemma 6.0′].

5.3. ***Proposition.*** *If Δ fixes at least* 3 *distinct points and exactly* 1 *line, and if* $\dim \Delta \geq 8$, *then Δ is a Lie group.*

Remark. This follows from 5.2. An easy proof is given in [Salzmann 2017, 7.0′].

5.4. ***Compact normal subgroup.*** *Suppose that $\mathcal{F}_\Delta$ is a flag and that Δ has a serpentine normal subgroup Θ. If* $\dim \Delta \geq 9$, *then Δ is a Lie group.*

Proof. This can be proved in a similar way as 5.1 and the first arguments are the same. Again there is a Δ-invariant Baer subplane $\mathcal{C}$ and $\Delta|_\mathcal{C} = \Delta/\mathrm{K}$ is a Lie group. Note that $\Theta \leq \mathrm{Cs}\,\Delta$ by 2.9 and that $\Theta|_\mathcal{C}$ is a Lie group.

(a) $\Theta^* = \Theta|_\mathcal{C}$ *does not contain any involution*: as $\mathcal{F}_\Delta$ is a flag, there is no reflection in Θ^*. If ι is a planar involution in Θ^*, then $\mathcal{C} \cap \mathcal{F}_\iota$ is a Δ-invariant 2-dimensional subplane and stiffness implies $\dim \Delta \leq 5 + 1$. Hence, $\Theta^* = \mathbb{1}$ and $\Theta \leq \mathrm{K}$.

(b) Choose an *interior* line $L \in \mathfrak{L}_v$, and *exterior* points $x \in L$ and $z \in W$. The kernel K acts freely on the set of all exterior points. Result 2.6 implies that $1 \le \dim x^{\mathrm{K}}, \dim z^{\mathrm{K}} \le 3$, so that $\Lambda = (\Delta_{x,z})^1$ has positive dimension. Recall that $\mathrm{N} \le \mathrm{K}$. Put $\Gamma = \Theta\mathrm{N}$ and $\mathcal{E} = \langle x^\Gamma, z^\Gamma \rangle$. Then $\mathcal{E} \le \mathcal{F}_\Lambda$ is a proper connected subplane, Γ acts faithfully on $\mathcal{E}$, and Γ, N, and Δ are Lie groups (2.10). □

5.5. ***Compact normal subgroup.*** *Assume that* $\mathcal{F}_\Delta = \langle u, v, w \rangle$. *If* $\dim \Delta \ge 7$, *and if* Δ *has a serpentine normal subgroup* Θ, *then* Δ *is a Lie group.*

Proof. If Δ is not a Lie group, there exists a point $p \notin W = uv$ such that $\mathcal{E} = \langle p^\Delta, u, v, w \rangle$ is a 2- or 4-dimensional subplane; see steps (a) and (b) in the proof of 4.2. Put $\Delta|_\mathcal{E} = \Delta/\mathrm{K}$. In the first case, $\Delta : \mathrm{K} \le 3$ and $\dim \mathrm{K} \le 3$ by the dimension formula and stiffness. Therefore, $\mathcal{E} \lessdot \mathcal{P}$ and lines are homeomorphic to $\mathbb{S}_4$. Recall that $\Theta \le \mathrm{Cs}\,\Delta$ and that $\Theta|_\mathcal{E}$ is a Lie group, either a torus or trivial. A torus would contain a reflection [Salzmann et al. 1995, 55.21(c)], and Δ would fix some point $c \notin W$. Hence, $\mathcal{E} = \mathcal{F}_\Theta$ and $\Theta \le \mathrm{K}$. There is a compact central subgroup $\mathrm{N} < \Delta$ such that Δ/N is a Lie group and $\mathrm{N} \le \mathrm{K}$. As $\mathcal{E}$ is maximal in $\mathcal{P}$, the kernel K acts freely on the set of points outside $\mathcal{E}$ (the exterior points). Let x be an exterior point on an *interior* line L in the pencil $\mathfrak{L}_v$. Because of 2.6, we have $\Delta_L : \Delta_x < 4$. Hence, $\Lambda = (\Delta_x)^1$ satisfies $\dim \Lambda \ge 2$. Stiffness implies that $\mathcal{F}_\Lambda$ is 2-dimensional. KN acts freely on $\mathcal{F}_\Lambda$, and N is a Lie group by 2.10, but then Δ is also a Lie group. □

Arguments a little more intricate show that even the following is true:

5.6. ***Compact normal subgroup.*** *Assume that* $\mathcal{F}_\Delta = \langle u, v \rangle$. *If* $\dim \Delta \ge 7$, *and if* Δ *has a serpentine normal subgroup* Θ, *then* Δ *is a Lie group.*

Proof. Suppose that Δ is not a Lie group. Again there is a point $p \notin W = uv$ such that $\mathcal{E} = \langle p^\Delta, u, v \rangle$ is a proper connected subplane; see steps (a) and (b) in the proof of 4.2. Put $\Delta|_\mathcal{E} = \Delta/\mathrm{K}$. There is a compact central subgroup $\mathrm{N} < \Delta$ of dimension $\dim \mathrm{N} = 0$ such that Δ/N is a Lie group and $\mathrm{N} \le \mathrm{K}$.

(a) If $\mathcal{E}$ is 2-dimensional, then $\dim \mathrm{K} = 3$ and $\Delta : \mathrm{K} = 4$. From [Salzmann et al. 1995, 33.9] it follows that $\mathcal{E}$ is the classical real plane; moreover, each compact subgroup of $\Delta|_\mathcal{E}$ is trivial, and $\Theta|_\mathcal{E} = \mathbb{1}$. Let L be a line of $\mathcal{E}$ in the pencil $\mathfrak{L}_v$ and consider a point $x \in L \setminus \mathcal{E}$ and a third point $w \in uv \cap \mathcal{E}$. Then $\Lambda = \Delta_{x,w}$ has positive dimension and fixes each point of $uv \cap \mathcal{E}$. Hence, $\mathcal{F}_\Lambda$ is a proper connected subplane, and $\mathrm{N}|_{\mathcal{F}_\Lambda}$ is a Lie group by 2.10. This is true for each choice of x. As $\mathcal{P}$ is generated by $\mathcal{E}$ and at most two of such subplanes, N itself is a Lie group, and so is Δ.

(b) Thus, $\mathcal{E} \lessdot \mathcal{P}$ and lines are homeomorphic to $\mathbb{S}_4$ by 2.1. Recall that $\Theta \le \mathrm{Cs}\,\Delta$. Again $\Theta|_\mathcal{E}$ is a compact Lie group by 2.10, and $\Theta|_\mathcal{E}$ is either a torus or trivial. In the first case, the involution in $\Theta|_\mathcal{E}$ is a reflection by [Salzmann et al. 1995, 55.21(c)], and Δ would fix its center and axis. Hence, $\Theta|_{\mathcal{E}} = \mathbb{1}$ and $\mathcal{F}_\Theta = \mathcal{E}$. Choose L, x, and w as in step (a). Because of 2.6, we have $\dim \Delta_x \geq 2$. Put $\Lambda = \Delta_{x,w}$ and note that $\Theta\mathrm{N}$ acts freely on $L \setminus \mathcal{E}$. It follows that $\mathcal{F}_\Lambda$ is connected and that N acts effectively on $\mathcal{F}_\Lambda$. Hence, $\mathcal{F}_\Lambda = \mathcal{P}$ and $\Delta_{x,w} = \mathbb{1}$ for each admissible w. Therefore, Δ_x is sharply transitive on a cylinder and Δ_x has a torus subgroup Ψ. If the involution $\iota \in \Psi$ is planar, then $\Theta\mathrm{N}$ acts effectively on $\mathcal{F}_\iota$, and N would be a Lie group. Thus, ι is a reflection, its axis is L and its center is u. Interchanging the roles of u and v, we find also a torus subgroup $\Phi < \Delta$ such that the involution $\sigma \in \Phi$ has the center v. We have $\Delta_{w,L} : \Delta_{w,x} \leq \dim x^\Delta \leq 3$ and $\dim L^{\Delta_w} = \Delta_w : \Delta_{w,L} \geq 5 - 3$. Consequently Δ is transitive on the set of admissible lines L, which is homeomorphic to $\mathbb{R}^2$. Therefore, Φ fixes one of the lines L. This follows, e.g., from the much more general result [Poncet 1959, Théorème a]. The axis of σ is an interior line in $\mathfrak{L}_u$ and $\sigma \notin \Phi_x$ so that Φ_x is finite. As $L^\Delta \approx \mathbb{R}^2$ is simply connected, a maximal compact subgroup X of Δ_L is connected [Salzmann et al. 1995, 93.10], and X induces a connected group $\overline{\mathrm{X}}$ on $L \setminus \mathcal{E}$. The group Φ yields a torus $\overline{\Phi} \leq \overline{\mathrm{X}}$. If $\dim \overline{\mathrm{X}} = 2$, then $\overline{\mathrm{X}} = \overline{\Phi}\Theta$ by [Salzmann et al. 1995, 93.12], and $\mathrm{N} < \Theta$. Moreover, $\overline{\Phi} \cap \Theta = \mathbb{1}$ because Φ acts effectively on $\mathcal{E}$, and $\dim x^{\Phi\Theta} > 1$ by 2.11. If $\dim \overline{\mathrm{X}} > 2$, then $\dim x^{\mathrm{X}} \geq 2$ because $\mathrm{X}_x \leq \Delta_x$ and X_x is a torus. In both cases, X is a Lie group by [Salzmann et al. 1995, 96.24], and then Δ is also a Lie group. $\square$

6. Nonincident fixed elements

If Δ fixes a nonincident point-line pair (and possibly further elements), then Löwen's criterion 2.8 does not apply.

6.1. ***Proposition.*** *If Δ fixes a line W and if Δ is transitive on W, then Δ is a Lie group* [Priwitzer 1994, 2.1].

Alternative proof. By [Hofmann and Kramer 2015, Corollary 5.5], the induced group $\Delta|_W$ is a Lie group and W is a manifold; in fact, $W \approx \mathbb{S}_4$ [Salzmann et al. 1995, 52.3]. From [Salzmann et al. 1995, 96.19–22] it follows that $\Delta|_W$ has a transitive subgroup $\mathrm{SO}_5\mathbb{R}$. The Malcev–Iwasawa theorem [Salzmann et al. 1995, 93.10] implies that a maximal compact subgroup Φ of Δ is connected. The result [Salzmann et al. 1995, 55.40] shows that Φ has a subgroup $\Upsilon \cong \mathrm{Spin}_5\mathbb{R}$. The central involution in Υ is a reflection with some center $a \notin W$. It suffices to show that Φ is a Lie group. By the approximation theorem, there is an arbitrarily small central subgroup $\mathrm{N} < \Phi$ such that Φ/N is a Lie group. As N centralizes each stabilizer Υ_z with $z \in W$, we conclude that $\mathrm{N}|_W = \mathbb{1}$, i.e., N consists of homologies with axis W and center a. Select a point $v \in W$ and consider the action of Φ_v on the line av. Note that $\Upsilon_v \cong \mathrm{Spin}_4\mathbb{R}$ fixes a second point $u \in W$, and that Υ_v has no subgroup of dimension 5. Put $\Upsilon_v|_{av} = \Upsilon_v/\mathrm{K}$. The homology group K has dimension at most 3. Hence, Υ_v has an orbit on av of dimension > 1, and

Richardson's theorem applies to $\Phi_v|_{av}$. In particular, Φ_v induces a Lie group on av, and then N is a Lie group. □

6.2. ***Semisimple groups.*** *Suppose that $\mathcal{F}_\Delta$ is a nonincident point-line pair $\{a, W\}$, Δ is semisimple, and* $\dim \Delta \geq 10$*. Then Δ is a Lie group.*

Proof. By [Priwitzer 1994] we may assume that $\dim \Delta < 12$.

<u>Case 1</u> ($\dim \Delta = 11$). Then $\Delta = \Gamma\Psi$ is a product of two almost simple factors, where $\dim \Gamma = 3$.

(a) Suppose that Δ is not a Lie group, and denote the center of Δ by Z. If $\Gamma\mathrm{Z}|_W \neq \mathbb{1}$, then there is a point p such that $\mathcal{G} = \langle p^{\Gamma\mathrm{Z}}, a, W\rangle$ is a connected subplane (note that $\Gamma|_W = \mathbb{1}$ implies $p^\Gamma \neq p$). If $\dim p^\Psi = 8$, then Δ would be a Lie group by [Salzmann et al. 1995, 53.2]. Therefore, $\Psi_p \neq \mathbb{1}$ and $\Psi_p|_{\mathcal{G}} = \mathbb{1}$, so that $\mathcal{G}$ is a proper subplane (in fact a Baer subplane) and $\Gamma\mathrm{Z}|_{\mathcal{G}}$ is a Lie group (see 2.10). Thus, $\mathcal{G} = \mathcal{F}_\zeta$ for some $\zeta \in \mathrm{Z}$. Consequently $\mathcal{G}^\Delta = \mathcal{G}$, but Δ cannot act on the 4-dimensional plane $\mathcal{G}$ [Salzmann et al. 1995, 71.8].

(b) Hence, $\Gamma\mathrm{Z} \leq \Delta_{[a,W]}$. From [Salzmann et al. 1995, 61.2] it follows that the almost simple group Γ is compact. By [Salzmann et al. 1995, 55.32(ii)], the homology group Γ does not contain a pair of commuting involutions. Hence, $\Gamma \cong \mathrm{SU}_2\mathbb{C}$. Moreover, Γ has 3-dimensional orbits on any line av, $v \in W$. The group Ψ acts almost effectively on W and Ψ is not a Lie group. Therefore, $\Psi|_W \cong \mathrm{PSU}_3(\mathbb{C}, 1)$. In fact, $\Psi|_W$ is strictly simple because $\mathrm{Z}|_W = \mathbb{1}$, and $\Psi|_W$ is different from $\mathrm{PSL}_3\mathbb{R}$ and from the compact group $\mathrm{PSU}_3(\mathbb{C}, 0)$ because these groups admit only finite coverings and Ψ is not a Lie group. The kernel K of the canonical map $\kappa : \Psi \to \Psi|_W$ is contained in Z. Let Φ be a maximal compact subgroup of Ψ. Then Φ is connected, $\Phi^\kappa \cong \mathrm{U}_2\mathbb{C}$, and $\dim \Phi = 4$. As Ψ is not a Lie group, it follows that K is compact. If lines are manifolds, then Richardson's theorem as stated in [Salzmann et al. 1995, 96.34] applies and shows that Φ has two fixed points on W. Let $v^\Phi = v \in W$. Then a maximal compact subgroup Ω of Δ fixes v, and Ω is connected by the Malcev–Iwasawa theorem [Salzmann et al. 1995, 93.10]. Now $\Omega|_{av}$ is a Lie group by 2.7, and so are $\mathrm{Z} \leq \Omega$ and Δ. Thus, lines are not manifolds, and 2.6 implies that all orbits of Δ on W have dimension < 4.

(c) The structure theorem 2.9 shows that Φ' is a Lie group. In fact, $\Phi' \cong \mathrm{SU}_2\mathbb{C}$ because $\Phi'^\kappa \ncong \mathrm{SO}_3\mathbb{R}$. The restriction of κ to Φ' is an isomorphism, the involution $\omega \in \Phi'$ is in the center of Φ, and ω is not planar (or lines would be manifolds); moreover, ω is not a reflection with axis W. Hence, $\omega \in \Delta_{[u,av]}$ for suitable points $u, v \in W$. Choose a maximal compact subgroup Ω of Δ such that $\Phi \leq \Omega$, so that Ω fixes u and v. Both Φ' and Γ act effectively on au; the product of their involutions is a reflection in $\Delta_{[v,au]}$. Hence, $\Phi'\Gamma|_{au} \cong \mathrm{SO}_4\mathbb{R}$. From $\dim \Phi = 4$ it follows that $\dim \Omega = 7$. The structure theorem of compact groups [Salzmann et al. 1995, 93.11] shows that Ω is a product of the connected component Θ of its center and the groups Φ' and Γ. Let U be some nontrivial orbit of Ω on au and note that $\dim U < 4$; in fact, $\dim U = 3$ because Γ acts freely on U. By [Salzmann et al. 1995, 96.13] we have $\dim \Omega|_U \le 6$. Consequently Ω has a 1-dimensional normal subgroup acting trivially on U. The only possible kernel contains Θ, but $\Theta|_U \ne \mathbb{1}$ since Z acts freely on U. This contradiction proves that $\dim \Delta \ne 11$.

Case 2 ($\dim \Delta = 10$). Then $\Delta/\mathrm{Z} \cong \mathrm{PSp}_4\mathbb{R} \cong \mathrm{O}'_5(\mathbb{R}, 2)$; note that the other two 10-dimensional simple groups have simply connected double coverings [Salzmann et al. 1995, 94.33] and hence cannot be images of non-Lie groups.

(a) The center Z acts freely on $C = \{x \in P \setminus W \mid x \ne a\}$: suppose that $p^\zeta = p$ for some $p \in C$ and $\zeta \in \mathrm{Z} \setminus \{\mathbb{1}\}$. Then $\zeta|_{p^\Delta} = \mathbb{1}$, by assumption p^Δ is not contained in a line, and $\mathcal{D} = \langle a, p^\Delta, W\rangle$ is a proper connected subplane. The induced group $\Delta|_\mathcal{D}$ is locally isomorphic to $\mathrm{Sp}_4\mathbb{R}$, and $\mathcal{D}$ is a Baer subplane, but then $\dim \Delta|_\mathcal{D} \le 8$ because Δ fixes $a, W \in \mathcal{D}$. (According to [Salzmann et al. 1995, 72.8] a 4-dimensional plane with a group of dimension > 8 is classical, and $\Delta|_\mathcal{D}$ would be contained in $\mathrm{GL}_2\mathbb{C}$; see also [Salzmann 1971, 8.1].)

(b) If Δ contains a planar involution β, then Z induces a Lie group on $\mathcal{F}_\beta$, $\mathcal{F}_\beta = \mathcal{F}_\zeta$ for some $\zeta \in \mathrm{Z}$, and $\mathcal{F}_\zeta$ would be a Δ-invariant Baer subplane. This is impossible for the same reasons as in step (a).

(c) As Δ/Z has a subgroup $\mathrm{SO}_3\mathbb{R}$, the structure theorem 2.9 shows that Δ has a subgroup $\Phi \cong \mathrm{SU}_2\mathbb{C}$: in the case $\Phi \cong \mathrm{SO}_3\mathbb{R}$ one of 3 pairwise commuting reflections of Φ would have the axis W [Salzmann et al. 1995, 55.35], but $\mathrm{SO}_3\mathbb{R}$ is simple.

(d) Suppose that lines are manifolds. Then $W \approx \mathbb{S}_4$ by [Salzmann et al. 1995, 52.3]. Some orbit of Φ on W has dimension at least 2. Consequently Δ induces a Lie group Δ/K on W (use Richardson's theorem 2.7). The structure of Δ shows that a maximal compact subgroup Ω of Δ is 4-dimensional. As $\mathrm{K} \le \mathrm{Z}$ and $\dim \mathrm{Z} = 0$, it follows that $\dim \Omega/\mathrm{K} = 4$. Note that $\Omega' = \Phi \cong \mathrm{SU}_2\mathbb{C}$. Richardson's theorem as stated in [Salzmann et al. 1995, 96.34] shows that either $\Phi|_W \cong \Phi$ has exactly two fixed points $u, v \in W$, where v is the center of the involution $\iota \in \Phi$, or $\Phi|_W \cong \mathrm{SO}_3\mathbb{R}$ has a circle of fixed points and the central involution $\iota \in \Phi$ is a reflection with axis W. In any case, there is a point $v \in W$ such that $v^\Phi = v$ and $\Phi|_{av} \cong \Phi$. By 2.7 each orbit c^Φ with $a, v \ne c \in av$ is a 3-sphere. It follows that the orbit space av/Φ is a closed interval J. The compact group $\mathrm{K} \le \Delta_{[a,W]}$ induces a group of order-preserving homeomorphisms on J. Each endpoint $b = c^\Phi$ of an orbit $x^\mathrm{K} \subset J$ is a fixed element of K. Hence, K maps c^Φ onto itself. As K is central, $c^\kappa = c^{\varphi(\kappa)}$ defines an injective continuous homomorphism $\mathrm{K} \to \Phi$. Consequently K is finite and Ω would be a Lie group.

(e) Thus, lines are not manifolds, and by 2.6 each orbit of (a subgroup of) Δ on a line has dimension at most 3. The group Ω is a product $\Theta\Phi$, where Θ is the connected component of the center of Ω, $\Theta \cap \Phi \le \langle\sigma\rangle$ is trivial or generated by the involution $\sigma \in \Phi$, and Θ is not a Lie group.

(f) Suppose that σ is not a reflection with axis W. Step (b) shows that σ has some center $u \in W$ and an axis av with $v \in W$. Consider an arbitrary point $z \in Y := W \setminus \{u, v\}$. We have $\dim \Phi_z = 0$, and Φ_z is finite. With [Salzmann et al. 1995, 93.6] it follows that $\dim \Delta_z = 7$ and $\dim \Delta_z\Phi = 10$. Therefore, $\Delta = \Delta_z\Phi$ and $z^\Delta = z^\Phi$. Thus, $Y^\Delta = Y$ and $\{u, v\}$ would be Δ-invariant, but $\mathcal{F}_\Delta = \{a, W\}$.

(g) Hence, $\sigma \in \Delta_{[a,W]}$. Recall that a maximal compact subgroup $\Omega = \Theta\Phi$ of Δ has dimension $\dim\Omega = 4$. If $z^\Delta \subseteq W$ is a nontrivial orbit, and if $\Delta|_{z^\Delta} = \Delta/\mathrm{K}$, then the kernel K is contained in Z (because Δ is almost simple). Therefore, Ω acts almost effectively on z^Δ. By [Salzmann et al. 1995, 96.13(a)] either $z^\Omega = z$ or $\dim z^\Omega = 3$. Consequently, $\dim z^\Delta = 3$ for each $z \in W$. (Note that $z^\Delta \ne z$. If $\dim z^\Delta < 3$, then $\Omega^\delta|_{z^\Delta} = \mathbb{1}$ for all $\delta \in \Delta$. As Δ is generated by all conjugates of Ω, this is impossible.)

(h) Θ *has* (at least) 2 *fixed points* $u, v \in W$. This follows from [Löwen 1976, Lemma 1 or 2]; see also 2.8 above.

(i) By 2.5, there is a sufficiently small compact central subgroup Ξ of Δ such that Δ/Ξ is a Lie group. Put $\mathrm{N} = \Theta \cap \Xi$. Then Θ/N is a Lie group, and so is Ω/N. Hence, Δ/N is also a Lie group. Denote the canonical map $\Delta \to \Delta/\mathrm{N}$ by λ. The quotient space $M = \Delta^\lambda/(\Delta_v)^\lambda$ is a manifold, and M can be written in the form

$$\{\{\mathrm{N}\gamma \mid \gamma \in \Delta_v\}\mathrm{N}\delta \mid \delta \in \Delta\} = \{\Delta_v\delta \mid \delta \in \Delta\} = \Delta/\Delta_v \approx v^\Delta,$$

since $\mathrm{N} < \Theta < \Delta_v$. Therefore, v^Δ is a 3-manifold. If $v^\Omega \ne v$, then [Salzmann et al. 1995, 96.11(a)] implies $v^\Omega = v^\Delta$. As $\Theta \le \mathrm{Cs}\,\Omega$, we have $\Theta|_{v^\Omega} = \mathbb{1}$ and hence $\Theta|_{v^\Delta} = \mathbb{1}$, i.e., Θ is in the kernel of the action of Δ on M. This kernel is contained in Z because Δ is almost simple. Consequently $\dim\Theta = 0$, a contradiction showing that $v^\Omega = v$.

(j) Consider the action of Ω and of Φ on $K := av \setminus \{a, v\}$. The only involution in Φ is the reflection σ with axis W. Therefore, $\dim\Phi_c = 0$ for each $c \in K$, and the compact group Φ_c is finite. Let $\Gamma = (\Delta_v)^1$ and note that $\dim\Gamma = 7$, $\dim c^\Gamma = \dim c^\Phi = 3$, $\dim\Gamma_c = 4$, $\dim\Gamma_c\Phi = 7$, and hence $\Gamma = \Gamma_c\Phi$, $c^\Theta \subseteq c^\Gamma = c^\Phi$. As Δ/Z is a Lie group and $\mathrm{Z}_c = \mathbb{1}$ by step (a), it follows that the stabilizer $\Pi = \Theta_c$ is a Lie group. The condition $c^\vartheta = c^{\varphi(\vartheta)}$ defines a continuous injective isomorphism of the compact group Θ/Π onto a closed subgroup of Φ. Hence, Θ/Π is a Lie group, and so are Θ and Δ. □

6.3. ***Compact normal subgroup.*** *Suppose that $\mathcal{F}_\Delta = \{a, W\}$ is a nonincident point-line pair. If Δ has a serpentine normal subgroup Θ and if* $\dim \Delta \geq 11$, *then Δ is a Lie group.*

Proof. (a) Θ is contained in the center $\mathrm{Z} = \mathrm{Cs}\,\Delta$ (see 2.9), and Δ/Z is a Lie group. Assume that Z is not a Lie group. If $\mathrm{Z}|_W \neq \mathbb{1}$, there is some point $p \notin W$ such that $p^{\mathrm{Z}} \nsubseteq ap$, and $\Delta_p|_{\langle p^{\mathrm{Z}}\rangle} = \mathbb{1}$. From [Salzmann et al. 1995, 53.2] it follows that $\dim \Delta_p \geq 4$. Thus, $\mathcal{E} = \langle p^{\mathrm{Z}}\rangle$ is a proper connected subplane, and $\mathrm{Z}|_{\mathcal{E}}$ is a Lie group by 2.10. Therefore, $\zeta|_{\mathcal{E}} = \mathbb{1}$ for some $\zeta \in \mathrm{Z} \setminus \{\mathbb{1}\}$. In particular, $p^\zeta = p$, $\zeta|_{p^\Delta} = \mathbb{1}$, $\dim p^\Delta \leq 4$, and $\dim \Delta_p \geq 7$. This contradicts stiffness and proves that $\mathrm{Z} \leq \Delta_{[a,W]}$.

(b) By assumption, Δ has no fixed point on W, and 6.1 shows that Δ is not transitive on W. Hence, there is some orbit $V = v^\Delta \subset W$ such that $0 < \dim V < 4$. Choose points $u, w \in V$ and $c \in av \setminus \{a, v\}$ and note that $\dim c^{\Delta_v} < 4$ by 2.6. If $\Lambda = \Delta_{c,u,w} \neq \mathbb{1}$, then $\mathcal{F}_\Lambda$ is a proper connected subplane, Z acts freely on $\mathcal{F}_\Lambda$, and Z would be a Lie group by 2.10. We have $\dim \Delta_c \geq 5$ and $\dim u^{\Delta_c} = 3$ for each $u \in V \setminus \{v\}$. Consequently Δ is doubly transitive on V.

(c) By [Salzmann et al. 1995, 96.16–17], either V is compact and the induced group $\Delta^* = \Delta|_V$ is isomorphic to one of the simple groups $\mathrm{PSL}_4\mathbb{R}$, $\mathrm{O}'_5(\mathbb{R}, 1)$, or $\mathrm{PSU}_3(\mathbb{C}, 1)$, or Δ^* is an extension of $\mathbb{R}^3 \approx V$ by a transitive linear group. In the first case $\dim \Delta > 15$ and Δ is a Lie group. In the last case, $\dim w^{\Delta_{u,v}} \leq 1$, $\Lambda \neq \mathbb{1}$, and Δ is also a Lie group. Only two possibilities remain: Δ^* is a simple group of dimension 10 or 8.

(d) If $\dim \Delta^* = 10$, then a maximal semisimple subgroup Ψ of Δ is isomorphic to the simple group $\mathrm{O}'_5(\mathbb{R}, 1)$ or to its double cover $\mathrm{U}_2(\mathbb{H}, 1)$; a maximal compact subgroup Φ of Ψ is isomorphic to $\mathrm{SO}_4\mathbb{R}$ or to $\mathrm{Spin}_4\mathbb{R}$. Accordingly $\Phi_v \cong \mathrm{SO}_3\mathbb{R}$ or $\Phi_v \cong \mathrm{Spin}_3\mathbb{R}$. In the first case, Φ_v would contain a reflection with axis W, but $\mathrm{SO}_3\mathbb{R}$ is simple. Hence, $\Upsilon = \Phi_v$ is simply connected. The involution $\omega \in \Upsilon$ is contained in $\Delta_{[a,W]}$, and each orbit c^Υ, $c \in av \setminus \{a, v\}$, is 3-dimensional. Hence, $\omega \notin \Upsilon_c$ and Υ_c is finite. Moreover, $\Theta_c = \mathbb{1}$ and $\Upsilon \cap \Theta \leq \langle\omega\rangle$. Lemma 2.11, applied to $\Upsilon\Theta$, shows that $\dim c^{\Upsilon\Theta} = 4$. By 2.7 the group Θ is a Lie group and so is Δ.

(e) Finally, let $\Delta^* = \Delta/\mathrm{K} \cong \mathrm{PSU}_3(\mathbb{C}, 1)$. Note that the central group Θ is contained in K. There exists an 8-dimensional semisimple subgroup Ψ of Δ (see [Salzmann et al. 1995, 94.27] or apply Levi's theorem [Salzmann et al. 1995, 94.28] to a Lie approximation of Δ). Consequently $\mathrm{K} = \sqrt{\Delta}$ is the radical, $\Delta = \Psi\mathrm{K}$, and $\mathrm{K} \leq \mathrm{Cs}_\Delta \Psi$. Suppose that $z^{\mathrm{K}} \neq z \in W$, let $c \in az \setminus \{a, z\}$, and put $\Lambda = \Psi_c$. If $\dim \Lambda = 0$, then $\dim c^\Delta = 8$, and Δ would be a Lie group by [Salzmann et al. 1995, 53.2]. As Λ fixes a connected set of points on W, it follows that $\mathcal{E} = \mathcal{F}_\Lambda$ is a connected proper subplane, and $\mathcal{E}^\Theta = \mathcal{E}$ because $\Theta \leq \mathrm{Cs}\,\Lambda$. The fact that $\Theta|_V = \mathbb{1}$ implies that Θ acts effectively on $\mathcal{E}$, so that Θ would be a Lie group by 2.10 above. Therefore, $\mathrm{K} \leq \Delta_{[a,W]}$, and K contains a compact connected subgroup of dimension at least 2 by [Salzmann et al. 1995, 61.2]. If lines are manifolds, the claim follows from Richardson's theorem 2.7. In the other case, 2.6 shows $\dim z^{\Delta} < 4$ for each $z \in W$. In fact, Δ is doubly transitive on each orbit $z^{\Delta} \subseteq W$; see step (b) of the present proof. Moreover, all transformation groups $(\Delta/\mathrm{K}, U)$, where U is an orbit of Δ on W, are equivalent to $(\mathrm{PSU}_3(\mathbb{C}, 1), \mathbb{S}_3)$ by [Salzmann et al. 1995, 96.17(b)]. Consequently, Δ_v has a fixed point in each of these orbits. Let again $c \in av \setminus \{a, v\}$. Then $\dim c^{\Delta_v} < 4$, $\Lambda = \Delta_c$ fixes a quadrangle, and $\dim \Lambda \geq 5$. This contradicts stiffness and completes the proof. □

7. Fixed double flag

Throughout this section, let $\mathcal{F}_\Delta = \langle u, v, av\rangle$ be a *double flag*.

7.0. ***Fact.*** *If a semisimple group Δ fixes a double flag, then* $\dim \Delta \leq 10$ [Salzmann 2014, 6.1].

7.1. ***Semisimple groups.*** *Suppose that $\mathcal{F}_\Delta$ is a double flag. If Δ is semisimple and if* $\dim \Delta \geq 10$, *then Δ is a Lie group.*

Proof. (a) We have $\dim \Delta = 10$ by 7.0, and Δ is almost simple. Let Φ be a maximal compact subgroup of Δ. If Δ is not a Lie group, then Δ maps onto $\mathrm{PSp}_4\mathbb{R}$ (or else Φ is locally isomorphic to $\mathrm{SO}_k\mathbb{R}$, $k \in \{4, 5\}$, and Δ would be a Lie group). Hence, Φ' is locally isomorphic to $\mathrm{SU}_2\mathbb{C}$. The center Z of Δ is an infinite compact 0-dimensional subgroup, and Z acts freely on $P \setminus (uv \cup av)$: if $x^\zeta = x$ for some x not on a fixed line and $\zeta \in \mathrm{Z} \setminus \{\mathbb{1}\}$, then $\zeta|_{\langle x^\Delta\rangle} = \mathbb{1}$ and $\langle x^\Delta\rangle$ is a proper connected subplane, but the almost simple group Δ cannot act on this subplane [Salzmann et al. 1995, 71.8]. By the Malcev–Iwasawa theorem $\mathrm{Z} \leq \Phi$.

(b) *Any involution $\sigma \in \Phi$ is a reflection with axis av*; *in particular*, $\Phi' \cong \mathrm{SU}_2\mathbb{C}$ *and* $\Phi'|_{av} \cong \mathrm{SO}_3\mathbb{R}$. In fact, σ is not planar (or else Z would induce a Lie group on $\mathcal{F}_\sigma$ and the kernel of the induced action would not act freely on $P \setminus (uv \cup av)$). If $\sigma \in \Delta_{[a,uv]}$, then $\sigma^\Delta\sigma$ would be a normal subgroup of translations of dimension $\Delta : \Delta_a$. Hence, $\sigma \in \Delta_{[u,av]}$.

(c) Z *consists of homologies with axis av*. Suppose that $a^{\mathrm{Z}} \neq a$. Then $\dim \Delta_a \leq 7$ by [Salzmann 1979, (*)] or [Salzmann et al. 1995, 83.17], and $d = \dim a^\Delta \geq 3$. It follows that $av \approx \mathbb{S}_4$: in the case $d = 3$, $\dim \Delta_a = 7$ [Salzmann 1979, (**)]; otherwise apply 2.6. Moreover, 2.6 implies that $\Phi|_{av}$ is a Lie group, since Φ' has an orbit of dimension > 1 on av. More precisely, $\Phi|_{av} \cong \mathrm{SO}_3\mathbb{R}$ and $\Theta = \sqrt{\Phi}$ acts trivially on av; see the explicit form of Richardson's theorem in [Salzmann et al. 1995, 96.34].

(d) Δ *acts faithfully on uv, in particular*, $\Phi_{[uv]} = \mathbb{1}$: this holds since Δ is almost simple and $\Delta_{[uv]} \leq \mathrm{Z} \leq \Delta_{[av]}$.

(e) Recall that $\Phi' \cong \mathrm{SU}_2\mathbb{C}$ and that $\Phi = \Phi'\Theta$ is not a Lie group. If $\dim z^{\Phi'} = 2$ for some $z \in uv \setminus \{u, v\}$, then Φ'_z would contain an involution σ, but σ is a reflection in $\Phi_{[u,av]}$. Hence, $\dim z^{\Phi'} = 3 \leq \dim z^{\Phi}$. Note that all the assumptions of Lemma 2.11 are satisfied by Φ instead of Γ; in fact, $\Phi' \cap \Theta \leq \langle\sigma\rangle$, $\Theta \leq \Delta_{[u,av]}$, and $\Theta_z = \mathbb{1}$; moreover, $\dim \Phi'_z = 0$ and Φ'_z is finite. Consequently $\dim z^{\Phi} = 4$ and 2.6 implies that Δ is a Lie group. □

7.2. *Compact normal subgroup.* *If* Δ *has a serpentine normal subgroup* Θ, *and if* $\dim \Delta \geq 11$, *then* Δ *is a Lie group.*

Proof. Assume that Δ is not a Lie group. By the approximation theorem, there is a compact subgroup $\mathrm{N} \triangleleft \Delta$ such that Δ/N is a Lie group and $\dim \mathrm{N} = 0$. From 2.9 it follows that $\Gamma := \Theta\mathrm{N} \leq \mathrm{Cs}\,\Delta$.

(a) If Γ is straight, then $\mathcal{F}_\Gamma \lessdot \mathcal{P}$ or Γ is a group of axial collineations with fixed center and axis in $\mathcal{F}_\Delta$ [Baer 1946]. In the first case, Δ induces on $\mathcal{F}_\Gamma$ a group of dimension at most 6, and $\dim \Delta \leq 7$ by stiffness. Letting $a \in \mathcal{F}_\Gamma$, we get $\dim \Delta_a \leq 5$.

(b) If Γ has the center v, then the axis passes through u and is fixed by Δ, i.e., $\Gamma \leq \Delta_{[v,uv]}$ and $\Gamma_a = \mathbb{1}$. From 2.6 it follows that there is a suitable point a such that $\dim a^{\Delta} < 4$. Let $z \in uv \setminus \{u, v\}$. The group Γ acts effectively on the *connected* subplane $\mathcal{D} = \langle a^{\Gamma}, z, u\rangle$ and $\Delta_{a,z}|_{\mathcal{D}} = \mathbb{1}$. In the cases $\mathcal{D} < \mathcal{P}$ both Γ and Δ would be Lie groups by 2.10. Therefore, $\Delta_{a,z} = \mathbb{1}$, $\dim \Delta \leq 7$, and $\dim \Delta_a \leq 4$.

(c) If Γ has the center u, then the axis of Γ is av. For a given point a there are points $z \in uv$ and $b \in au$ such that $\dim z^{\Delta}$, $\dim b^{\Delta} < 4$. As Γ is not a Lie group, the connected subplane $\mathcal{D} = \langle a, b, v, z^{\Gamma}\rangle$ coincides with $\mathcal{P}$. Consequently $\Delta_{a,b,z} = \mathbb{1}$, so that $\dim \Delta_a \leq 6$ and $\dim \Delta \leq 10$.

(d) If Γ is not straight, there is a point x such that $\mathcal{E} = \langle x^{\Gamma}, u, v, av\rangle$ is a connected subplane and $\Delta_x|_{\mathcal{E}} = \mathbb{1}$. In particular, $\Gamma_x|_{\mathcal{E}} = \mathbb{1}$ and Γ acts effectively on $\mathcal{E}$. Again $\mathcal{E} = \mathcal{P}$, and then $\dim \Delta \leq 7$ by 2.6. Similarly, $\dim \Delta_a \leq 6$. □

Remark. In any case, $\dim \Delta_a \leq 6$. This proves 8.2.

8. Fixed triangle

Let $\mathcal{F}_\Delta = \{a, u, v\}$ be a triangle.

8.0. *Theorem.* *If* $\dim \Delta \geq 10$, *then* Δ *is a Lie group.*

Proof. If Δ is not a Lie group, then 2.6 implies that Δ has only orbits of dimension at most 3 on two sides of the fixed triangle, say on uv and av. Hence, $\dim \Delta_z = 7$ for $z \in uv \setminus \{u, v\}$, and [Salzmann 1979, (**)] applies to Δ_z. Choose $c \in av \setminus \{a, v\}$ and put $x = az \cap cu$. Then $\dim \Delta_{c,z} \geq 4$, but 2.2(7) or [Salzmann 1979, (**)] asserts that $\Delta_x \cong \mathrm{SO}_3\mathbb{R}$, a contradiction. □

8.1. *Semisimple groups.* *If $\mathcal{F}_\Delta$ is a triangle, if Δ is semisimple, and if* $\dim\Delta \geq 9$, *then Δ is a Lie group.*

Proof. Suppose that Δ is not a Lie group. Only the case $\dim\Delta = 9$ has to be considered. Then Δ has a 3-dimensional factor Γ which is not a Lie group. Either the complement Ψ of Γ is locally isomorphic to $\mathrm{SL}_2\mathbb{C}$, or Ψ is a product of two 3-dimensional factors. Let $D = P \setminus (au \cup av \cup uv)$.

(a) *The center* Z *of Δ acts freely on D*: if $x^\zeta = x \in D$ for some $\zeta \in \mathrm{Z} \setminus \{\mathbb{1}\}$, then $\langle x^\Delta \rangle$ is a proper subplane, and $\dim\Delta_x \geq 5$ contrary to stiffness 2.2.

(b) $\Gamma|_{uv} \neq \mathbb{1}$ *and* $\Gamma/\mathrm{Z} \cong \mathrm{PSL}_2\mathbb{R}$: in the case $\Gamma \leq \Delta_{[a,uv]}$ it would follow from [Salzmann et al. 1995, 61.2] that Γ is compact and hence a Lie group. For the same reason, Γ acts nontrivially on the other sides of the fixed triangle.

(c) *There is at most one fixed line, say uv, such that* $\mathrm{Z}|_{uv}$ *is a Lie group*: otherwise Γ itself would be a Lie group.

(d) $\dim x^\Delta \leq 6$ *for each $x \in D$, and* $\dim\Delta_x \geq 3$: as $\mathrm{Z}|_{au}$ and $\mathrm{Z}|_{av}$ are not Lie groups, 2.6 implies that all orbits on these two sides of the fixed triangle have dimension < 4.

(e) *There is some $p \in D$ such that* $(\mathrm{Z}\Psi)_p = \mathbb{1}$, *and* $\Lambda = (\Delta_p)^1$ *satisfies* $\dim\Lambda = 3$; *moreover*, $(\Gamma\mathrm{Z})_p = \mathbb{1}$: if $p^\Gamma \not\subseteq ap$ (such a point p exists by step (b)), then $\langle p^\Gamma \rangle$ is a connected subplane, and $\langle p^{\Gamma\mathrm{Z}} \rangle = \mathcal{P}$, or else Z would be a Lie group by 2.10. On the other hand, $(\mathrm{Z}\Psi)_p|_{p^{\Gamma\mathrm{Z}}} = \mathbb{1}$, $\dim p^\Psi = 6$, $\dim\Delta_p = 3$ by step (d), $\langle p^\Psi \rangle = \mathcal{P}$, and $(\Gamma\mathrm{Z})_p|_{p^\Psi} = \mathbb{1}$.

(f) $\Lambda \cong \Gamma/\mathrm{Z}$ *and any involution $\iota \in \Lambda$ is planar*: consider the canonical epimorphism $\kappa : \Delta \to \Delta/\mathrm{Z}$ and note that $\Delta^\kappa = \Gamma^\kappa \times \Psi^\kappa$. Let π be the projection onto the first factor. Then $\kappa : \Lambda \cong \Lambda^\kappa$ since $\Lambda \cap \mathrm{Z} = \mathbb{1}$. The restriction $\pi : \Lambda^\kappa \to \Gamma^\kappa$ is injective because $\Lambda \cap \Psi\mathrm{Z} = \mathbb{1}$, and it is surjective since $\dim\Lambda = \dim\Gamma = 3$ [Salzmann et al. 1995, 93.12]. A reflection in Λ would have one of the fixed lines as axis, but Λ is simple; moreover, ι fixes a nondegenerate quadrangle. Therefore, ι is indeed planar. Now Z acts effectively on $\mathcal{F}_\iota$ by step (a), and Z is a Lie group contrary to the assumption. □

8.2. *Compact normal subgroup.* *If Δ has a serpentine normal subgroup Θ, and if* $\dim\Delta \geq 7$, *then Δ is a Lie group* (see the remark after 7.2).

Summary

The following table lists our conditions implying that Δ is a Lie group. There are always three conditions to be combined: the first column specifies the fixed configuration $\mathcal{F}_\Delta$, the first row lists possible assumptions on the structure of Δ, and in the body of the table, a lower bound for $\dim\Delta$ is given. The abbreviations in the first line mean, in this order, that Δ is semisimple, that Δ contains a serpentine normal subgroup in the sense of 2.12, that Δ contains a normal vector group, or that no condition is imposed on the structure of Δ.

$\mathcal{F}_\Delta$	Δ s-s	$\Theta \lhd \Delta$	$\mathbb{R}^t \lhd \Delta$	Δ arbitr.	references
$\varnothing$	9	9*	7	10	3.1, 3.2, 3.3
$\{W\}$	4			9	4.1, 4.2
flag	4	9		10	4.1, 5.4, 5.1
$\langle u, v\rangle$	4	7		8	4.1, 5.6, 5.2
$\langle u, v, w\rangle$	4	7		8	4.1, 5.5, 5.3
$\{o, W\}$	10	11		12	6.2, 6.3, [Priwitzer 1994]
$\langle u, v, ov\rangle$	10	11		12	7.1, 7.2, [Priwitzer 1994]
$\langle o, u, v\rangle$	9	7		10	8.1, 8.2, 8.0
arbitrary	10	11		12	[Priwitzer 1994]

Here 9* means that also $\Delta \cong \mathrm{SL}_3\mathbb{R}\times\Theta$ is conceivable.

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Received 18 Nov 2018. Revised 19 Mar 2019.

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Innovation in Incidence Geometry

Vol. 17 No. 3 2019