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FINITE CHAINS INSIDE THIN SUBSETS OF $\mathbb{R}^d$

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In a recent paper, Chan, Łaba, and Pramanik investigated geometric configurations inside thin subsets of Euclidean space possessing measures with Fourier decay properties. In this paper we ask which configurations can be found inside thin sets of a given Hausdorff dimension without any additional assumptions on the structure. We prove that if the Hausdorff dimension of $E \subset \mathbb{R}^d$, $d \geq 2$, is greater than $\frac{1}{2}(d+1)$ then, for each $k \in \mathbb{Z}^+$, there exists a nonempty interval I such that, given any sequence $\{t_1, t_2, \ldots, t_k : t_j \in I\}$, there exists a sequence of distinct points $\{x^j\}_{j=1}^{k+1}$ such that $x^j \in E$ and $|x^{i+1} - x^i| = t_j$ for $1 \leq i \leq k$. In other words, E contains vertices of a chain of arbitrary length with prescribed gaps.

1. Introduction

The problem of determining which geometric configurations one can find inside various subsets of Euclidean space is a classical subject. The basic problem is to understand how large a subset of Euclidean space must be to be sure that it contains the vertices of a congruent and possibly scaled copy of a given polyhedron or another geometric shape. In the case of a finite set, "large" refers to the number of points, while in infinite sets it refers to the Hausdorff dimension or Lebesgue density. The resulting class of problems has been attacked by a variety of authors using combinatorial, number theoretic, ergodic, and Fourier analytic techniques, creating a rich set of ideas and interactions.

We begin with a comprehensive result due to Tamar Ziegler [2006], which generalizes an earlier result due to Furstenberg, Katznelson and Weiss [Furstenberg et al. 1990]. See also [Bourgain 1986].

Theorem 1.1 [Ziegler 2006]. *Let $E \subset \mathbb{R}^d$ be of positive upper Lebesgue density, in the sense that*

$$\limsup_{R \to \infty} \frac{\mathcal{L}^d\{E \cap [-R, R]^d\}}{(2R)^d} > 0,$$

where $\mathcal{L}^d$ denotes the d-dimensional Lebesgue measure. Let E_δ denote the δ-neighborhood of E. Let $V = \{\mathbf{0}, v^1, v^2, \ldots, v^{k-1}\} \subset \mathbb{R}^d$, where $k \geq 2$ is a positive integer. Then there exists $l_0 > 0$ such that, for any $l > l_0$ and any $\delta > 0$, there exists $\{x^1, \ldots, x^k\} \subset E_\delta$ congruent to $lV = \{\mathbf{0}, lv^1, \ldots, lv^{k-1}\}$.

In particular, this result shows that we can recover every simplex similarity type and sufficiently large scaling inside a subset of $\mathbb{R}^d$ of positive upper Lebesgue density. It is reasonable to wonder whether the assumptions of Theorem 1.1 can be weakened, but the following result, due to Maga [2010], shows that

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the conclusion may fail even if we replace the upper Lebesgue density condition with the assumption that the set is of dimension d.

Theorem 1.2 [Maga 2010]. *For any $d \geq 2$ there exists a full-dimensional compact set $A \subset \mathbb{R}^d$ such that A does not contain the vertices of any parallelogram. If $d = 2$ then, given any triple of points $x^1, x^2, x^3, x^j \in A$, there exists a full-dimensional compact set $A \subset \mathbb{R}^2$ such that A does not contain the vertices of any triangle similar to $\triangle x^1 x^2 x^3$.*

In view of Maga's result, it is reasonable to ask whether interesting point configurations can be found inside thin sets under additional structural hypotheses. This question was recently addressed by Chan, Łaba, and Pramanik [Chan et al. 2013]. Before stating their result, we provide two relevant definitions.

Definition 1.3. Fix integers $n \geq 2,\ p \geq 3$ and $m = n\lceil \frac{1}{2}(p+1)\rceil$. Suppose $B_1, \dots, B_p$ are $n \times (m-n)$ matrices.

(a) We say that E contains a p-point $\mathcal{B}$-configuration if there exist vectors $z \in \mathbb{R}^n$ and $w \in \mathbb{R}^{m-n} \setminus \vec{0}$ such that

$$\{z + B_j w\}_{j=1}^{p} \subset E.$$

(b) Moreover, given any finite collection of subspaces $V_1, \dots, V_q \subset \mathbb{R}^{m-n}$ with $\dim(V_i) < m-n$, we say that E contains a nontrivial p-point $\mathcal{B}$-configuration with respect to $(V_1, \dots, V_q)$ if there exist vectors $z \in \mathbb{R}^n$ and $w \in \mathbb{R}^{m-n} \setminus \bigcup_{i=1}^{q} V_i$ such that

$$\{z + B_j w\}_{j=1}^{p} \subset E.$$

Definition 1.4. Fix integers $n \geq 2,\ p \geq 3$ and $m = n\lceil \frac{1}{2}(p+1)\rceil$. We say that a set of $n \times (m-n)$ matrices $\{B_1, \dots, B_p\}$ is nondegenerate if

$$\operatorname{rank}\begin{pmatrix} B_{i_2} - B_{i_1} \\ \vdots \\ B_{i_{m/n}} - B_{i_1} \end{pmatrix} = m - n$$

for any distinct indices $i_1, \dots, i_{m/n} \in \{1, \dots, p\}$.

Theorem 1.5 [Chan et al. 2013]. *Fix integers $n \geq 2,\ p \geq 3$ and $m = n\lceil \frac{1}{2}(p+1)\rceil$. Let $\{B_1, \dots, B_p\}$ be a collection of $n \times (m-n)$ nondegenerate matrices in the sense of Definition 1.4. Then, for any constant C, there exists a positive number $\epsilon_0 = \epsilon_0(C, n, p, B_1, \dots, B_p) \ll 1$ with the following property: Suppose the set $E \subset \mathbb{R}^n$ with $|E| = 0$ supports a positive, finite Radon measure μ with two conditions:*

(a) *Ball condition*: $\sup_{x \in E,\, 0<r<1} \mu(B(x,r))/r^\alpha \leq C$ *if* $n - \epsilon_0 < \alpha < n$.

(b) *Fourier decay*: $\sup_{\xi \in \mathbb{R}^n} |\hat{\mu}(\xi)|(1+|\xi|)^{\beta/2} \leq C$.

Then:

(i) *E contains a p-point $\mathcal{B}$-configuration in the sense of Definition 1.3(a).*

(ii) *Moreover, for any finite collection of subspaces $V_1, \dots, V_q \subset \mathbb{R}^{m-n}$ with $\dim(V_i) < m-n$, E contains a nontrivial p-point $\mathcal{B}$-configuration with respect to $(V_1, \dots, V_q)$ in the sense of Definition 1.3(b).*

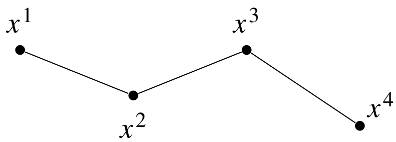

Figure 1. A 3-chain.

One can check that the Chan-Łaba–Pramanik result covers some geometric configurations but not others. For example, their nondegeneracy condition allows them to consider triangles in the plane, but not simplexes in $\mathbb{R}^3$ where three faces meet at one of the vertices at right angles, forming a three-dimensional corner. Most relevant to this paper is the fact that the conditions under which Theorem 1.5 holds are satisfied for chains (see Definition 1.6 below), but the conclusion requires decay properties for the Fourier transform of a measure supported on the underlying set. We shall see that, in the case of chains, such an assumption is not needed and the existence of a wide variety of chains can be established under an explicit dimensional condition alone.

Focus of this article. We establish that a set of sufficiently large Hausdorff dimension, *with no additional assumptions*, contains an arbitrarily long chain with vertices in the set and preassigned admissible gaps.

Definition 1.6 (see Figure 1). A k-chain in $E \subset \mathbb{R}^d$ with gaps $\{t_i\}_{i=1}^k$ is a sequence

$$\{x^1, x^2, \dots, x^{k+1} : x^j \in E,\ |x^{i+1} - x^i| = t_i,\ 1 \le i \le k\}.$$

We say that the chain is *nondegenerate* if all the x^j are distinct.

Our main result is the following:

Theorem 1.7. *Suppose that the Hausdorff dimension of a compact set $E \subset \mathbb{R}^d$, $d \ge 2$, is greater than $\frac{1}{2}(d+1)$. Then, for any $k \ge 1$, there exists an open interval $\tilde{I}$ such that for any $\{t_i\}_{i=1}^k \subset \tilde{I}$ there exists a nondegenerate k-chain in E with gaps $\{t_i\}_{i=1}^k$.*

In the course of establishing Theorem 1.7 we shall prove the following result, which is interesting in its own right and has a number of consequences for Falconer-type problems. See [Falconer 1985; Erdoğan 2005; Wolff 1999] for the background and the latest results pertaining to the Falconer distance problem.

Theorem 1.8. *Suppose that μ is a compactly supported, nonnegative Borel measure such that*

$$\mu(B(x,r)) \le C r^{s_\mu} \tag{1-1}$$

for some $s_\mu \in \left(\frac{1}{2}(d+1), d\right]$, where $B(x,r)$ is the ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$. Then, for any $t_1, \dots, t_k > 0$ and $\epsilon > 0$,

$$\mu \times \mu \times \cdots \times \mu\left\{(x^1, x^2, \dots, x^{k+1}) : t_i - \epsilon \le |x^{i+1} - x^i| \le t_i + \epsilon,\ i = 1, 2, \dots, k\right\} \le C\epsilon^k. \tag{1-2}$$

Corollary 1.9. *Given a compact set $E \subset \mathbb{R}^d$, $d \ge 2$, $k \ge 1$, define*

$$\Delta_k(E) = \{|x^1 - x^2|, |x^2 - x^3|, \dots, |x^k - x^{k+1}| : x^j \in E\}.$$

Suppose that the Hausdorff dimension of E is greater than $\frac{1}{2}(d+1)$. Then

$$\mathscr{L}^k(\Delta_k(E)) > 0.$$

Remark 1.10. Suppose that $E \subset \mathbb{R}^d$ has Hausdorff dimension $s > \frac{1}{2}(d+1)$ and is *Ahlfors–David regular*, i.e., there exists $C > 0$ such that, for every $x \in E$,

$$C^{-1} r^s \leq \mu(B(x,r)) \leq C r^s$$

(where μ is the restriction of the s-dimensional Hausdorff measure to E). Then, using the techniques in [Eswarathasan et al. 2011] along with Theorem 1.8, one can show that, for any sequence of positive real numbers $t_1, t_2, \dots, t_k$, the upper Minkowski dimension of

$$\{(x^1, x^2, \dots, x^{k+1}) \in E^{k+1} : |x^{j+1} - x^j| = t_j,\ 1 \leq j \leq k\}$$

does not exceed $(k+1)\dim_{\mathscr{H}}(E) - k$.

2. Proof of Theorem 1.7 and Theorem 1.8

The strategy for this section is as follows:

We begin by dividing both sides of (1-2) by ϵ^k. The left side becomes

$$\epsilon^{-k} \mu \times \cdots \times \mu\bigl\{(x^1, \dots, x^{k+1}) : t_i - \epsilon \leq |x^{i+1} - x^i| \leq t_i + \epsilon,\ i = 1, 2, \dots, k\bigr\}, \tag{2-1}$$

which can be interpreted as the density of ϵ-approximate chains in $E \times \cdots \times E$.

Theorem 1.8 gives an upper bound on this expression that is independent of ϵ. This is accomplished using an inductive argument on the chain length coupled with repeated application of an earlier result from [Iosevich et al. 2014], in which the authors establish $L^2(\mu)$ mapping properties of certain convolution operators. This upper bound is important in the final section, where we define a measure on the set of chains.

Next, we acquire a lower bound on (2-1). This result was already established in the case $k = 1$ in [Iosevich et al. 2012], where the authors show that the density of ϵ-approximate 1-chains with gap size t is bounded below, independent of ϵ, for all t in a nonempty open interval I. Using a pigeonholing argument, we extend the result in [Iosevich et al. 2012] to obtain a lower bound on (2-1) in the case that every gap is of equal size t for some $t \in I$. To obtain a lower bound on chains with variable gap size, we show that the density of ϵ-approximate k-chains is continuous as a function of gap sizes. Furthermore, we use the lower bound on chains with constant gaps to prove that this continuous function is not identically zero. We conclude that the density of ϵ-approximate k-chains is bounded below, independent of ϵ and independent of the gap sizes, as long as all gap sizes fall within some interval $\tilde{I}$ around t.

In the final section, we address the issue of nondegeneracy. To this end, we reinterpret the density of ϵ-approximate k-chains as a measure supported in E^{k+1} and show that it converges to a new measure, $\Lambda^k_{\vec{t}}$, as $\epsilon \downarrow 0$. This new measure is shown to be supported on "exact" k-chains ($\epsilon = 0$) with admissible gaps. We next show that the measure of the set of degenerate chains is 0, and we conclude that the mass of $\Lambda^k_{\vec{t}}$ is contained in nondegenerate k-chains.

We shall repeatedly use the following result, due to Iosevich, Sawyer, Taylor and Uriarte-Tuero:

Theorem 2.1 [Iosevich et al. 2014]. *Let $T_\lambda f(x) = \lambda * (f\mu)(x)$, where λ and μ are compactly supported, nonnegative Borel measures on $\mathbb{R}^d$. Suppose that μ satisfies (1-1) and, for some $\alpha > 0$,*

$$|\hat{\lambda}(\xi)| \le C|\xi|^{-\alpha}.$$

Suppose that ν is a compactly supported Borel measure supported on $\mathbb{R}^d$ satisfying (1-1) with s_μ replaced by s_ν and suppose that $\alpha > d - s$, where $s = \frac{1}{2}(s_\mu + s_\nu)$. Then

$$\|T_\lambda f\|_{L^2(\nu)} \le c\|f\|_{L^2(\mu)}.$$

In this article, we will use Theorem 2.1 with $\lambda = \sigma$, the surface measure on a $(d-1)$-dimensional sphere in $\mathbb{R}^d$. It is known — see [Stein 1993] — that

$$\hat{\sigma}(\xi) = O(|\xi|^{-(d-1)/2}).$$

Since the proof of Theorem 2.1 is short, we give the argument below for the sake of keeping the presentation as self-contained as possible. It is enough to show that

$$\langle T_{\lambda^\epsilon} f, g\nu\rangle \le C\|f\|_{L^2(\mu)} \cdot \|g\|_{L^2(\nu)}.$$

The left-hand side equals

$$\int \hat{\lambda}^\epsilon(\xi)\widehat{f\mu}(\xi)\widehat{g\nu}(\xi)\, d\xi.$$

By the assumptions of Theorem 2.1, the modulus of this quantity is bounded by

$$C\int |\xi|^{-\alpha}|\widehat{f\mu}(\xi)||\widehat{g\nu}(\xi)|\, d\xi$$

and applying Cauchy–Schwarz bounds this quantity by

$$C\left(\int |\widehat{f\mu}(\xi)|^2|\xi|^{-\alpha_\mu}\, d\xi\right)^{\frac{1}{2}} \cdot \left(\int |\widehat{g\nu}(\xi)|^2|\xi|^{-\alpha_\nu}\, d\xi\right)^{\frac{1}{2}} \tag{2-2}$$

for any $\alpha_\mu, \alpha_\nu > 0$ such that $\alpha = \frac{1}{2}(\alpha_\mu + \alpha_\nu)$.

By Lemma 2.5 below, the quantity (2-2) is bounded by $C\|f\|_{L^2(\mu)} \cdot \|g\|_{L^2(\nu)}$ after choosing, as we may, $\alpha_\mu > d - s_\mu$ and $\alpha_\nu > d - s_\nu$. This completes the proof of Theorem 2.1.

Proof of Theorem 1.8 and Corollary 1.9. Let $\epsilon > 0$. Divide both sides of (1-2) by ϵ^k and note that it suffices to establish the estimate

$$C_k^\epsilon(\mu) = \int \left(\prod_{i=1}^{k} \sigma_{t_i}^\epsilon(x^{i+1} - x^i)\, d\mu(x^i)\right) d\mu(x^{k+1}) \le c^k, \tag{2-3}$$

where c is independent of ϵ and $t_1, \dots, t_k > 0$. Here $\sigma_r^\epsilon(x) = \sigma_r * \rho_\epsilon(x)$, with σ_r the Lebesgue measure on the sphere of radius r, ρ a smooth cut-off function with $\int \rho = 1$ and $\rho_\epsilon(x) = \epsilon^{-d}\rho(x/\epsilon)$. Assume in addition that ρ is nonnegative and that $\rho(x) = \rho(-x)$.

Let σ denote the Lebesgue measure on the $(d-1)$-dimensional sphere in $\mathbb{R}^d$. Set $T_j^\epsilon = T_{\sigma_{t_j}}^\epsilon$, where $T_{\sigma_{t_j}}^\epsilon f(x) = \sigma_{t_j} * (f\mu)(x)$ was introduced in Theorem 2.1. Define

$$f_k^\epsilon(x) = T_k^\epsilon \circ \cdots \circ T_1^\epsilon(1)(x) \tag{2-4}$$

and

$$f_0^\epsilon(x) = 1.$$

It is important to note that $f_k(x)$ depends implicitly on the choices of $t_1, \dots, t_k > 0$, and this choice will be made explicit throughout.

Observe that

$$f_{k+1}^\epsilon = T_{k+1}^\epsilon f_k^\epsilon. \tag{2-5}$$

Rewriting the left-hand side of (2-3), it suffices to show

$$C_k^\epsilon(\mu) = \int f_k^\epsilon(x)\, d\mu(x) \le c^k. \tag{2-6}$$

Using Cauchy–Schwarz (and keeping in mind that $\int d\mu(x) = 1$), we bound the left-hand side of (2-6) by

$$C_k^\epsilon(\mu) = \int f_k^\epsilon(x)\, d\mu(x) \le \|f_k^\epsilon\|_{L^2(\mu)}. \tag{2-7}$$

We now use induction on k to show that

$$\|f_k^\epsilon\|_{L^2(\mu)} \le c^k, \tag{2-8}$$

where c is the constant obtained in Theorem 2.1. For the base case, $k = 0$, we have $\|f_0^\epsilon\|_{L^2(\mu)} = \int d\mu(x) = 1$. Next, we assume inductively that $\|f_k^\epsilon\|_{L^2(\mu)} \le c^k$.

We now show that, for any $t_{k+1} > 0$,

$$\|f_{k+1}^\epsilon\|_{L^2(\mu)} \le c^{k+1}.$$

First, use (2-5) to write

$$\|f_{k+1}^\epsilon\|_{L^2(\mu)} = \|T_{k+1}^\epsilon f_k^\epsilon\|_{L^2(\mu)}.$$

Next, use Theorem 2.1 with $\lambda = \sigma$, the Lebesgue measure on the sphere, and $\alpha = \frac{1}{2}(d-1)$ (see the comment immediately following Theorem 2.1 to justify this choice of α) to show that

$$\|T_{k+1}^\epsilon f_k^\epsilon\|_{L^2(\mu)} \le c\|f_k^\epsilon\|_{L^2(\mu)}$$

whenever $s_\mu > d - \alpha = \frac{1}{2}(d+1)$.

We complete the proof by applying the inductive hypothesis. This completes the verification of (2-8).

We now recover Corollary 1.9. Let $s_\mu \in \bigl(\frac{1}{2}(d+1), \dim(E)\bigr)$, and choose a probability measure μ with support contained in E which satisfies (1-1); the existence of such a measure is provided by Frostman's lemma (see [Falconer 1986], [Wolff 2003] or [Mattila 1995]).

Cover $\Delta_k(E)$ with cubes of the form

$$\bigcup_i \prod_{j=1}^{d} (t_{ij}, t_{ij} + \epsilon_i),$$

where $\prod$ denotes the Cartesian product. We have

$$1 = \mu \times \cdots \times \mu(E^{k+1}) \le \sum_i \mu \times \cdots \times \mu\big\{(x^1, \dots, x^{k+1}) : t_{ij} - \epsilon \le |x^{j+1} - x^j| \le t_{ij} + \epsilon_i,\ 1 \le j \le k\big\}.$$

By Theorem 1.8, the expression above is bounded by

$$C \sum_i \epsilon_i^k \tag{2-9}$$

and we conclude that (2-9) is bounded from below by $1/C > 0$. It follows that $\Delta_k(E)$ cannot have measure 0 and the proof of Corollary 1.9 is complete.

We now continue with the proof of Theorem 1.7.

Lower bound on $C_k^\epsilon(\mu)$. Let $s_\mu \in \big(\frac{1}{2}(d+1), \dim(E)\big)$, and choose a probability measure μ with support contained in E which satisfies (1-1).

We now establish the existence of a nonempty open interval $\tilde{I}$ such that

$$\liminf_{\epsilon \to 0} C_k^\epsilon(\mu) > 0, \tag{2-10}$$

where each t_i belongs to $\tilde{I}$ and $C_k^\epsilon(\mu)$ is as in (2-3).

Note that this positive lower bound alone establishes the existence of vertices $x^1, \dots, x^{k+1} \in E$ such that $|x^{i+1} - x^i| = t_i$ for each $i \in \{1, \dots, k\}$ (this follows, for instance, by Cantor's intersection theorem and the compactness of the set E). Extra effort is made in the next section in order to guarantee that we may take $x^1, \dots, x^{k+1}$ distinct.

We first prove the estimate (2-10) in the case that all gaps are equal. This is accomplished using a pigeonholing argument on chains of length one. We then provide a continuity argument to show that the estimate holds for variable gap values t_i belonging to a nonempty open interval $\tilde{I}$. The second argument relies on the first precisely at the point when we show that the said continuous function is not identically equal to zero.

Lower bound for constant gaps. The proof of the estimate (2-10) in the case $k = 1$ was already established in [Iosevich et al. 2012] provided that μ satisfies the ball condition in (1-1) with $\frac{1}{2}(d+1) < s_\mu < \dim_{\mathcal{H}}(E)$. The existence of such measures is established by Frostman's lemma (see, e.g., [Falconer 1986], [Wolff 2003] or [Mattila 1995]).

More specifically, it is demonstrated in [Iosevich et al. 2012] that there exists $c(1) > 0$, $\epsilon_0 > 0$ and a nonempty open interval $I \subset (0, \text{diameter}(E))$ such that, if $t \in I$ and $0 < \epsilon < \epsilon_0$, then

$$C_1^\epsilon = \int \sigma_t^\epsilon * \mu(x)\, d\mu(x) > 2c(1).$$

To establish the estimate (2-10) for longer chains, we rely on the following lemmas:

Lemma 2.2. *Set*

$$G_{t,\epsilon}(1) = \{x \in E : \sigma_t^\epsilon * \mu(x) > c(1)\}.$$

There exists $m(1) \in \mathbb{Z}^+$ such that, if $t \in I$ and $0 < \epsilon < \epsilon_0$, then

$$\mu(G_{t,\epsilon}(1)) \geq 2^{-2m(1)}.$$

Lemma 2.3. *Set*

$$G_{t,\epsilon}(j+1) = \{x \in E : \sigma_t^\epsilon * \mu|_j(x) > c(j+1)\},$$

where $j \in \{1, \ldots, (k-1)\}$, $\mu|_j(x)$ denotes restriction of the measure μ to the set $G_{t,\epsilon}(j)$, and

$$c(j+1) = \tfrac{1}{2}c(j)\mu(G_{t,\epsilon}(j)).$$

Then there exists $m(j+1) \in \mathbb{Z}^+$ such that if $t \in I$ and $0 < \epsilon < \epsilon_0$, then

$$\mu(G_{t,\epsilon}(j+1)) > 2^{-2m(j+1)}.$$

We postpone the proof of Lemmas 2.2 and 2.3 momentarily, and we apply these lemmas to obtain a lower bound on $C_k^\epsilon(\mu)$.

We write

$$C_k^\epsilon(\mu) = \int f_k^\epsilon(x)\,d\mu(x),$$

where f_k^ϵ was introduced in (2-4) and here $t_1 = \cdots = t_k = t$.

Now

$$C_k^\epsilon(\mu) = \int f_k^\epsilon(x)\,d\mu(x) = \iint \sigma_t^\epsilon(x-y) f_{k-1}(y)\,d\mu(y)\,d\mu(x).$$

Integrating in x and restricting the variable y to the set $G_{t,\epsilon}(1)$, we write

$$C_k^\epsilon(\mu) \geq \int_{G_{t,\epsilon}(1)} \sigma_t^\epsilon * \mu(y) f_{k-1}(y)\,d\mu(y) \geq c(1)\int_{G_{t,\epsilon}(1)} f_{k-1}(y)\,d\mu(y) = c(1)\int f_{k-1}(y)\,d\mu_1(y).$$

To achieve a lower bound, we iterate this process. For each $j \in \{2, \ldots, k-1\}$ we have

$$\begin{aligned}\int f_{k-j}^\epsilon(x)\,d\mu_j(x) = \iint \sigma_t^\epsilon(x-y) f_{k-j-1}(y)\,d\mu(y)\,d\mu_j(x) &\geq \int_{G_{t,\epsilon}(j+1)} \sigma_t^\epsilon * \mu_j(y) f_{k-j-1}(y)\,d\mu(y)\\ &\geq c(j+1)\int_{G_{t,\epsilon}(j+1)} f_{k-j-1}(y)\,d\mu(y)\\ &= c(j+1)\int f_{k-j-1}(y)\,d\mu_{j+1}(y).\end{aligned}$$

It follows that

$$C_k^\epsilon(\mu) \geq \biggl(\prod_{j=1}^{k-1} c(i)\biggr)\iint \sigma_t^\epsilon(x-y)\,d\mu_{k-1}(y)\,d\mu(x) \geq \biggl(\prod_{j=1}^{k} c(i)\biggr)\mu(G_{t,\epsilon}(k)),$$

and we are done in light of Lemma 2.3.

Given Lemmas 2.2 and 2.3, we have shown that, for all $t \in I$ and all $0 < \epsilon < \epsilon_0$, we have

$$\liminf_{\epsilon \to 0} C_k^{\epsilon}(\mu) > 0, \tag{2-11}$$

where all gap lengths $t_1, \ldots, t_k$ are constantly equal to t. This concludes the proof of the estimate (2-10) in the case of constant gaps.

We now proceed to the proofs of Lemmas 2.2 and 2.3.

Proof of Lemma 2.2. We write

$$2c(1) < \int \sigma_t^{\epsilon} * \mu(x)\,d\mu(x) \le \left(\int_{(G_{t,\epsilon}(1))^c} \sigma_t^{\epsilon} * \mu(x)\,d\mu(x)\right) + \left(\int_{G_{t,\epsilon}(1)} \sigma_t^{\epsilon} * \mu(x)\,d\mu(x)\right) = \mathcal{I} + \mathcal{II},$$

where A^c denotes the complement of a set $A \subset E$.

We first observe that

$$\mathcal{I} \le c(1).$$

Next we estimate $\mathcal{II}$. Let $m \in \mathbb{Z}^+$ and write

$$G_{t,\epsilon}(1) = \{x \in E : c(1) < \sigma_t^{\epsilon} * \mu(x) \le 2^m\} \cup \{x \in E : 2^m \le \sigma_t^{\epsilon} * \mu(x)\}.$$

Then

$$\begin{aligned}\mathcal{II} &= \int_{\{x \in E : c(1) < \sigma_t^{\epsilon} * \mu(x) \le 2^m\}} \sigma_t^{\epsilon} * \mu(x)\,d\mu(x) + \int_{\{x \in E : 2^m \le \sigma_t^{\epsilon} * \mu(x)\}} \sigma_t^{\epsilon} * \mu(x)\,d\mu(x)\\ &\le 2^m \mu(G_{t,\epsilon}(1)) + \sum_{l=m} 2^{l+1} \cdot \mu\big(\{x \in E : 2^l \le \sigma_t^{\epsilon} * \mu(x) \le 2^{l+1}\}\big).\end{aligned}$$

We use Theorem 2.1 to estimate

$$\mu\big(\{x \in E : 2^l \le \sigma_t^{\epsilon} * \mu(x) \le 2^{l+1}\}\big) \le c_d \cdot 2^{-2l},$$

where the constant c_d depends only on the ambient dimension d. Now,

$$\mathcal{II} \le 2^m \mu(G_{t,\epsilon}(1)) + 2c_d \cdot \sum_{l=m} 2^l \cdot 2^{-2l} \lesssim 2^m \mu(G_{t,\epsilon}(1)) + 2^{-m}.$$

It follows that

$$2c(1) \le \mathcal{I} + \mathcal{II} \lesssim c(1) + 2^m \mu(G_{t,\epsilon}(1)) + 2^{-m}.$$

Taking $m \in \mathbb{Z}^+$ large enough, we conclude that

$$\mu(G_{t,\epsilon}(1)) \ge 2^{-2m}. \qquad \square$$

Proof of Lemma 2.3. We prove the lemma by induction on j. The base case, $j = 1$, was established in Lemma 2.2. Next, assume that there exists $m(j) \in \mathbb{Z}^+$ such that

$$2^{-m(j)} < \mu(G_{t,\epsilon}(j))$$

for all $0 < \epsilon < \epsilon_0$ and $t \in I$.

By the definition of $G_{t,\epsilon}(j)$,

$$c(j)\mu(G_{t,\epsilon}(j)) < \int_{G_{t,\epsilon}(j)} \sigma_t^{\epsilon} * \mu|_{G_{t,\epsilon}(j-1)}(x)\, d\mu(x).$$

Set $c(j+1) = \frac{1}{2}c(j)\mu(G_{t,\epsilon}(j))$. By assumption, $2c(j+1) = c(j)\mu(G_{t,\epsilon}(j)) \geq c(j)2^{-m(j)}$, and in particular this quantity is positive. Next, we obtain a bound from above:

$$\begin{aligned}
\int_{G_{t,\epsilon}(j)} \sigma_t^{\epsilon} * \mu|_{G_{t,\epsilon}(j-1)}(x)\, d\mu(x) &\leq \int_{G_{t,\epsilon}(j)} \sigma_t^{\epsilon} * \mu(x)\, d\mu(x) \\
&= \int \sigma_t^{\epsilon} * \mu|_j(x)\, d\mu(x) \\
&= \left(\int_{(G_{t,\epsilon}(j+1))^c} \sigma_t^{\epsilon} * \mu|_j(x)\, d\mu(x)\right) + \left(\int_{G_{t,\epsilon}(j+1)} \sigma_t^{\epsilon} * \mu|_j(x)\, d\mu(x)\right) \\
&= \mathcal{I} + \mathcal{II}.
\end{aligned}$$

First we observe that

$$\mathcal{I} \leq c(j+1).$$

Next, we estimate $\mathcal{II}$. Let $m \in \mathbb{Z}^+$ and write

$$G_{t,\epsilon}(j+1) = \{x \in E : c(j+1) < \sigma_t^{\epsilon} * \mu|_j(x) \leq 2^m\} \cup \{x \in E : 2^m \leq \sigma_t^{\epsilon} * \mu|_j(x)\}.$$

Then

$$\begin{aligned}
\mathcal{II} &= \int_{\{x\in E: c(j+1)<\sigma_t^{\epsilon}*\mu|_j(x)\leq 2^m\}} \sigma_t^{\epsilon} * \mu|_j(x)\, d\mu(x) + \int_{\{x\in E: 2^m \leq \sigma_t^{\epsilon}*\mu(x)\}} \sigma_t^{\epsilon} * \mu|_j(x)\, d\mu(x) \\
&\leq 2^m \cdot \mu(G_{t,\epsilon}(j+1)) + \sum_{l=m} 2^{l+1} \cdot \mu\big(\{x \in E : 2^l \leq \sigma_t^{\epsilon} * \mu|_j(x) \leq 2^{l+1}\}\big).
\end{aligned}$$

We use Theorem 2.1 to estimate

$$\mu\big(\{x \in E : 2^l \leq \sigma_t^{\epsilon} * \mu|_j(x) \leq 2^{l+1}\}\big) \leq c_d \cdot 2^{-2l},$$

where the constant c_d depends only on the ambient dimension d and the choice of the measure μ. Now,

$$\mathcal{II} \leq 2^m \mu(G_{t,\epsilon}(j+1)) + 2c_d \cdot \sum_{l=m} 2^l \cdot 2^{-2l} \lesssim 2^m \mu(G_{t,\epsilon}(j+1)) + 2^{-m}.$$

It follows that

$$2c(j+1) \leq \mathcal{I} + \mathcal{II} \lesssim c(j+1) + 2^m \mu(G_{t,\epsilon}(j+1)) + 2^{-m}.$$

Taking $m \in \mathbb{Z}^+$ large enough, we conclude that

$$\mu(G_{t,\epsilon}(j+1)) \geq 2^{-2m}. \qquad \square$$

Lower bound for variable gaps. We now verify (2-10) in the case of variable gap lengths. In more detail, we show that, for all $k \in \mathbb{Z}^+$ and for values of t_i in a nonempty open interval $\tilde{I}$, we have

$$\liminf_{\epsilon \to 0} \int f_k^\epsilon(x)\, d\mu(x) > 0, \tag{2-12}$$

where f_k^ϵ is as defined in (2-4) with $0 < t_1, \dots, t_k \in \tilde{I}$.

The following lemma captures the strategy of the proof and establishes (2-12).

Lemma 2.4. *We have*

$$C_k^\epsilon(\mu) = \int f_k^\epsilon(x)\, d\mu(x) = M_k(t_1, \dots, t_k) - \sum_{j=1}^{k} R_{k,j}^\epsilon(t_1, \dots, t_k), \tag{2-13}$$

where

$$M_k(t_1, t_2, \dots, t_k) = \int \hat{\sigma}_{t_k}(\xi)\, \widehat{f_{k-1}\mu}(-\xi)\hat{\mu}(\xi)\, d\xi \tag{2-14}$$

is continuous and bounded below by a positive constant (*independent of* ϵ) *on* $\tilde{I} \times \cdots \times \tilde{I}$ *for a nonempty open interval* $\tilde{I}$, *and*

$$R_{k,j}^\epsilon(t_1, t_2, \dots, t_k) = \int \hat{\sigma}(t_j\xi)(1 - \hat{\rho}(\epsilon\xi))\, \widehat{f_{j-1}\mu}(\xi)\, \widehat{g_{j+1}^\epsilon\mu}(-\xi)\, d\xi = \mathbb{O}(\epsilon^{\alpha(s-(d+1)/2)}) \tag{2-15}$$

for some $\alpha > 0$.

In proving the lemma, we utilize the notation

$$g_j^\epsilon(x) = T_j^\epsilon \circ \cdots \circ T_k^\epsilon(1)(x) \tag{2-16}$$

and

$$g_{k+1}(x) = 1. \tag{2-17}$$

It is important to note that $g_j(x)$ depends implicitly on the choices of $t_1, \dots, t_k > 0$, and this choice will be made explicit throughout.

First, we demonstrate (2-13) with repeated use of Fourier inversion. We again employ a variant of the argument in [Iosevich et al. 2012]. Write

$$\int f_k^\epsilon(x)\, d\mu(x) = \iint \sigma_{t_1}^\epsilon(x-y) g_2^\epsilon(y)\, d\mu(x)\, d\mu(y) = \iint (\sigma_{t_1} * \rho_\epsilon)(x-y) g_2^\epsilon(y)\, d\mu(x)\, d\mu(y).$$

Using Fourier inversion and properties of the Fourier transform, this is equal to

$$\iiint e^{2\pi i (x-y)\cdot\xi} \hat{\sigma}_{t_1}(\xi)\hat{\rho}_\epsilon(\xi) g_2^\epsilon(y)\, d\mu(x)\, d\mu(y)\, d\xi.$$

Simplifying further, we write

$$\begin{aligned}\int f_k^\epsilon(x)\,d\mu(x) &= \int \hat{\sigma}_{t_1}(\xi)\hat{\rho}(\epsilon\xi)\hat{\mu}(\xi)\widehat{g_2^\epsilon\mu}(-\xi)\,d\xi \\ &= \int \hat{\sigma}_{t_1}(\xi)\hat{\mu}(\xi)\widehat{g_2^\epsilon\mu}(-\xi)\,d\xi + \int \hat{\sigma}_{t_1}(\xi)(1-\hat{\rho}(\epsilon\xi))\hat{\mu}(\xi)\widehat{g_2^\epsilon\mu}(-\xi)\,d\xi \\ &= \int \hat{\sigma}_{t_1}(\xi)\hat{\mu}(\xi)\widehat{g_2^\epsilon\mu}(-\xi)\,d\xi + R_{k,1}^\epsilon(t_1,t_2,\dots,t_k).\end{aligned}$$

With repeated use of Fourier inversion, we get

$$\begin{aligned}\int f_k^\epsilon(x)\,d\mu(x) &= \int \hat{\sigma}_{t_j}(\xi)\cdot\widehat{f_{j-1}\mu}(-\xi)\cdot\widehat{g_{j+1}^\epsilon\mu}(\xi)\,d\xi + \sum_{l=1}^{j} R_{k,l}^\epsilon(t_1,t_2,\dots,t_k) \\ &\;\;\vdots \\ &= \int \hat{\sigma}_{t_k}(\xi)\cdot\widehat{f_{k-1}\mu}(-\xi)\cdot\hat{\mu}(\xi)\,d\xi + \sum_{l=1}^{k} R_{k,l}^\epsilon(t_1,t_2,\dots,t_k) \\ &= M_k(t_1,t_2,\dots,t_k) + \sum_{l=1}^{k} R_{k,l}^\epsilon(t_1,t_2,\dots,t_k).\end{aligned}$$

We now prove that $M_k(t_1,t_2,\dots,t_k)$ is continuous on any compact set away from $(t_1,\dots,t_k)=\vec{0}$ and that

$$R_{k,j}^\epsilon(t_1,\dots,t_k) = \mathbb{O}(\epsilon^{\alpha(s-(d+1)/2)}). \tag{2-18}$$

Once these are established, we observe that the lower bound on constant chains established in (2-11) combined with (2-18) implies that $M_k(t_1,\dots,t_k)$ is positive when $t_1=\cdots=t_k=t$ for any given $t\in I$. Fixing any such $t\in I$, it will then follow by continuity that $M_k(t_1,\dots,t_k)$ is bounded from below on $\tilde{I}\times\cdots\times\tilde{I}$, where $\tilde{I}$ is a nonempty interval.

We now use the dominated convergence theorem to verify the continuity of $M_k(t_1,\dots,t_k)$ on any compact set away from $(t_1,\dots,t_k)=\vec{0}$. Let $t_1,\dots,t_k>0$. Using properties of the Fourier transform and recalling the definition of f_j from (2-4) and g_j from (2-16), we write

$$M_k(t_1,t_2,\dots,t_k) = \int \hat{\sigma}_{t_j}(\xi)\cdot\widehat{f_{j-1}\mu}(-\xi)\cdot\widehat{g_{j+1}\mu}(\xi)\,d\xi$$

for any $j\in\{1,\dots,k\}$.

Let $h_1,\dots,h_k\in\mathbb{R}$ be such that $(h_1,\dots,h_k)\downarrow 0$. Let

$$\tilde{f}_j = T_{t_j+h_j}\circ\cdots\circ T_{t_1+h_1}(1) \quad\text{and}\quad \tilde{g}_j = T_{t_j+h_j}\circ\cdots\circ T_{t_k+h_k}(1).$$

We have

$$M_k(t_1+h_1,t_2+h_2,\dots,t_k+h_k) = \int \hat{\sigma}_{t_j+h_j}(\xi)\cdot\widehat{\tilde{f}_{j-1}\mu}(-\xi)\cdot\widehat{\tilde{g}_{j+1}\mu}(\xi)\,d\xi.$$

The integrand goes to 0 as h_j goes to 0. Now, for t_j in a compact set, the expression above is bounded by

$$C(t_j)\int |\xi|^{-(d-1)/2}|\widehat{\tilde{f}_{j-1}\mu}(-\xi)||\widehat{\tilde{g}_{j+1}\mu}(\xi)|\,d\xi.$$

To proceed, we will utilize the following calculation:

Lemma 2.5. *Let μ be a compactly supported Borel measure such that $\mu(B(x,r)) \le Cr^s$ for some $s \in (0, d)$. Suppose that $\alpha > d - s$. Then, for $f \in L^2(\mu)$,*

$$\int |\widehat{f\mu}(\xi)|^2|\xi|^{-\alpha}\,d\xi \le C'\|f\|^2_{L^2(\mu)}. \tag{2-19}$$

To prove Lemma 2.5, observe that

$$\int |\widehat{f\mu}(\xi)|^2|\xi|^{-\alpha}\,d\xi = C\iint f(x)f(y)|x-y|^{-d+\alpha}\,d\mu(x)\,d\mu(y) = \langle Tf, f\rangle, \tag{2-20}$$

where

$$Tf(x) = \int |x-y|^{-d+\alpha} f(y)\,d\mu(y)$$

and the inner product above is with respect to $L^2(\mu)$. The positive constant C appearing in (2-20) depends only on the ambient dimension d. Observe that

$$\int |x-y|^{-d+\alpha}\,d\mu(y) \approx \sum_{j>0} 2^{j(d-\alpha)}\int_{|x-y|\approx 2^{-j}} d\mu(y) \le C\sum_{j>0} 2^{j(d-\alpha-s)} \le C'$$

since $\alpha > d - s$.

By symmetry, $\int |x-y|^{-d+\alpha}\,d\mu(x) \le C'$. It follows by using Schur's test [1911] — see also Lemma 7.5 in [Wolff 2003] — that

$$\|Tf\|_{L^2(\mu)} \le C'\|f\|_{L^2(\mu)}.$$

This implies the conclusion of Lemma 2.5 by applying the Cauchy–Schwarz inequality to (2-20). We note that Lemma 2.5 can also be recovered from the fractal Plancherel estimate due to R. Strichartz [1990]. See also Theorem 7.4 in [Wolff 2003], where a similar statement is proved by the same method as above.

We already established, using [Iosevich et al. 2014], that finite compositions of the operators T_l applied to $L^2(\mu)$ functions are in $L_2(\mu)$. Using the Cauchy–Schwarz inequality and in light of Lemma 2.5, $M_k(t_1+h_1, t_2+h_2, \dots, t_k+h_k)$ is bounded. We proceed by applying the dominated convergence theorem. We have

$$\begin{aligned}\lim_{h_j\downarrow 0} M_k&(t_1+h_1, t_2+h_2, \dots, t_k+h_k)\\ &= \int \hat{\sigma}_{t_j}(\xi)\cdot\widehat{\tilde{g}_{j-1}\mu}(-\xi)\cdot\widehat{\tilde{f}_{j+1}\mu}(\xi)\,d\xi\\ &= \int \hat{\sigma}_{t_j}(\xi)\cdot(T_{t_{j-1}+h_{j-1}}\circ\cdots\circ T_{t_1+h_1}(1)\cdot\mu)\widehat{\ }(-\xi)\cdot(T_{t_{j+1}+h_{j+1}}\circ\cdots\circ T_{t_k+h_k}(1)\cdot\mu)\widehat{\ }(\xi)\,d\xi.\end{aligned}$$

We then rewrite the procedure, isolating $\hat{\sigma}_{t_j}$ for each $j \in \{1, \dots, k\}$, and repeat the process above a total of k times.

Bounding the remainder. Next, we wish to show that $\lim_{\epsilon\downarrow 0} R_k^\epsilon(t_1, \dots, t_k) = 0$. Fix $\epsilon > 0$. Recall that $R_k^\epsilon(t_1, \dots, t_k)$ is equal to

$$\int (1 - \hat{\rho}(\epsilon\xi))\hat{\sigma}(t\xi)\hat{\mu}(\xi)\,\widehat{f_k\mu}(-\xi)\,d\xi.$$

We consider the integral over $|\xi| < (1/\epsilon)^\alpha$ and the integral over $|\xi| > (1/\epsilon)^\alpha$ separately, where $\alpha \in (0, 1)$ will be determined. Assume that $s > \frac{1}{2}(d+1)$.

Lemma 2.6. *Let $\rho : \mathbb{R}^d \to \mathbb{R}$ satisfy the following properties: $\rho \geq 0$, $\rho(x) = \rho(-x)$, the support of ρ is contained in $\{x : |x| < c\}$, and $\int \rho = 1$. Then*

$$0 \leq 1 - \hat{\rho}(\xi) \leq 2\pi c|\xi|.$$

To prove Lemma 2.6, write

$$\hat{\rho}(\xi) = \int \cos(2\pi x \cdot \xi)\rho(x)\,dx.$$

We observe that $\cos x + |x| > 1$, and conclude that the lemma follows when $|x| < c$. It follows that

$$\int_{|\xi|<(1/\epsilon)^\alpha} |\hat{\rho}(\epsilon\xi) - 1|\,|\hat{\sigma}(t\xi)|\,|\hat{\mu}(\xi)|\,|\widehat{f_k\mu}(-\xi)|\,d\xi \lesssim \epsilon^{1-\alpha} \int |\hat{\sigma}(t\xi)|\,|\hat{\mu}(\xi)|\,|\widehat{f_k\mu}(-\xi)|\,d\xi \lesssim \epsilon^{1-\alpha},$$

where the last line is justified in the estimation of $M_k(t)$ above.

It remains to estimate the quantity

$$\int_{|\xi|>(1/\epsilon)^\alpha} |\hat{\sigma}(t\xi)|\,|\hat{\mu}(\xi)|\,|\widehat{f_k\mu}(-\xi)|\,d\xi.$$

Proceeding as in the estimation of $M_k(t)$ above, we bound the integral above by

$$Ct^{-(d-1)/2} \int_{|\xi|>(1/\epsilon)^\alpha} |\xi|^{-(d-1)/2}|\hat{\mu}(\xi)|\,|\widehat{f_k\mu}(-\xi)|\,d\xi$$

and then use Cauchy–Schwarz to bound it further by

$$Ct^{-(d-1)/2}\left(\int_{|\xi|>(1/\epsilon)^\alpha} |\xi|^{-(d-1)/2}|\hat{\mu}(\xi)|^2\,d\xi\right)^{\frac{1}{2}}\left(\int_{|\xi|>(1/\epsilon)^\alpha} |\xi|^{-(d-1)/2}|\widehat{f_k\mu}(\xi)|^2\,d\xi\right)^{\frac{1}{2}}.$$

We have already shown that the second integral is finite. The first integral is bounded by

$$\sum_{j>\alpha\log_2(1/\epsilon)} 2^{-j(d-1)/2} \int_{2^j\leq|\xi|<2^{j+1}} |\hat{\mu}(\xi)|^2\,d\xi.$$

We may choose a smooth cut-off function ψ such that the inner integral is bounded by

$$\int |\hat{\mu}(\xi)|^2\hat{\psi}(2^{-j}\xi)\,d\xi.$$

By Fourier inversion, this integral is equal to

$$2^{dj} \iint \psi(2^j(x-y))\,d\mu(x)\,d\mu(y) \leq C2^{j(d-s)}.$$

Returning to the sum, we now have the estimate

$$C \sum_{j>\alpha \log_2(1/\epsilon)} 2^{-j(d-1)/2} \cdot 2^{j(d-s)} \le C \sum_{j>\alpha \log_2(1/\epsilon)} 2^{j(d+1)/2-s}.$$

As long as $s > \frac{1}{2}(d+1)$, this is $\ll \epsilon^{\alpha(s-(d+1)/2)}$. Thus $R_k^\epsilon(t_1, \dots, t_k)$ tends to 0 with ϵ as long as $\dim_{\mathcal{H}}(E) > \frac{1}{2}(d+1)$.

In conclusion, we have

$$\lim_{\epsilon \downarrow 0} \int \left(\prod_{j=1}^{k} \sigma_{t_j}^{\epsilon}(x^{i+1} - x^i)\, d\mu(x^i) \right) d\mu(x^{k+1}) > c_k > 0 \tag{2-21}$$

for all $t_j \in \tilde{I}$.

To complete the proof of Theorem 1.7, it remains to verify that E contains a nondegenerate k-chain with prescribed gaps. This is the topic of the next section.

3. Nondegeneracy

An important issue we have not yet addressed is that the chains we have found may be degenerate. As an extreme example, consider the case where $t_i = 1$ for all i. Then included in our chain count are chains which simply bounce back and forth between two different points. We now take steps to ensure that we can indeed find chains with distinct vertices.

We verified above that there exists a nonempty open interval $\tilde{I}$ such that

$$\lim_{\epsilon \downarrow 0} \int \left(\prod_{j=1}^{k} \sigma_{t_j}^{\epsilon}(x^{i+1} - x^i)\, d\mu(x^i) \right) d\mu(x^{k+1})$$

is bounded above and below for $t_1, \dots, t_k \in \tilde{I}$. The upper bound appears in (2-3) and the lower bound appears in (2-21).

From here onward, we fix $t_1, \dots, t_k \in \tilde{I}$ and set $\vec{t} = (t_1, \dots, t_k)$. We now define a nonnegative Borel measure on the set of k-chains with the gaps $\vec{t}$. Let $\Lambda_{\vec{t}}^k$ denote a nonnegative Borel measure, defined as

$$\Lambda_{\vec{t}}^k(A) = \lim_{\epsilon \downarrow 0} \int_A \left(\prod_{j=1}^{k} \sigma_{t_j}^{\epsilon}(x^{i+1} - x^i)\, d\mu(x^i) \right) d\mu(x^{k+1}),$$

where $A \subset E \times \cdots \times E$, the $(k+1)$-fold product of the set E.

It follows that $\Lambda_{\vec{t}}^k$ is a finite measure which is not identically zero:

$$0 < \Lambda_{\vec{t}}^k(E \times \cdots \times E). \tag{3-1}$$

The strategy we use to demonstrate the existence of nondegenerate k-chains in E is as follows: We first show that $\Lambda_{\vec{t}}^k$ has support contained in the set of k-chains. This is accomplished by showing that the measure has support contained in all "approximate" k-chains. We then show that the measure of the set of degenerate chains is zero. It follows, since the $\Lambda_{\vec{t}}^k$-measure of the set of k-chains is positive and the $\Lambda_{\vec{t}}^{k}$-measure of the set of degenerate k-chains is zero, that the set of nondegenerate k-chains in E is nonempty.

For each test entry $n \in \mathbb{Z}^{+}$, define the sets of $(1/n)$-approximate k-chains and the set of exact k-chains as

$$A_{n,k} = \left\{(x^1, \dots, x^{k+1}) \in E \times \cdots \times E : t_i - \frac{1}{n} \le |x^{i+1} - x^i| \le t_i + \frac{1}{n} \text{ for each } i = 1, \dots, k\right\}$$

and

$$A_k = \{(x^1, \dots, x^{k+1}) \in E \times \cdots \times E : |x^{i+1} - x^i| = t_i \text{ for each } i = 1, \dots, k\}.$$

Observe that

$$\bigcap_n A_{n,k} = A_k.$$

We now observe that the support of Λ_{t}^{k} is contained in the set of all approximate chains. This follows immediately from the observation that

$$\Lambda_{\vec{t}}^{k}(A_{n,k}^{c}) = 0$$

for each $n \in \mathbb{Z}^{+}$, where $A_{n,k}^{c}$ denotes the complement of the set $A_{n,k}$ in $E \times \cdots \times E$.

Next, we observe that the support of $\Lambda_{\vec{t}}^{k}$ is contained in the set of exact chains. Indeed, it follows from the previous equation that

$$\Lambda_{\vec{t}}^{k}\Big(\bigcup_n A_{n,k}^{c}\Big) \le \sum_n \Lambda_{\vec{t}}^{k}(A_{n,k}^{c}) = 0.$$

Recalling (3-1), we conclude that

$$0 < \Lambda_{\vec{t}}^{k}(E \times \cdots \times E) = \Lambda_{\vec{t}}^{k}\Big(\bigcup_n A_{n,k}^{c}\Big) + \Lambda_{\vec{t}}^{k}\Big(\bigcap_n A_{n,k}\Big), \tag{3-2}$$

and so

$$\Lambda_{\vec{t}}^{k}(A_k) = \Lambda_{\vec{t}}^{k}\Big(\bigcap_n A_{n,k}\Big) > 0.$$

Since $t_1, \dots, t_k \in \tilde{I}$ were chosen arbitrarily, we have shown that $\Lambda_{\vec{t}}^{k}(A_k) > 0$ whenever $\vec{t} = (t_1, \dots, t_k)$ and $t_i \in \tilde{I}$.

We now verify that the set of degenerate chains has $\Lambda_{\vec{t}}^{k}$-measure zero.

Lemma 3.1. *Let*

$$D_k = \{(x^1, \dots, x^{k+1}) \in E \times \cdots \times E : x^i = x^j \text{ for some } i \ne j\}.$$

Then

$$\Lambda_{\vec{t}}^{k}(D_k) = 0.$$

To prove the lemma, we first investigate the quantity

$$\int_{D_k} \Big(\prod_{j=1}^{k} \sigma_{t_j}^{\epsilon}(x^{i+1} - x^i)\, d\mu(x^i)\Big)\, d\mu(x^{k+1}).$$

By the definition of D_k, we can bound the quantity above by

$$\sum_{1\le m<n\le k+1} \int_{\{(x^1,\ldots,x^{k+1}):x^m=x^n\}} \left(\prod_{j=1}^{k} \sigma_{t_j}^{\epsilon}(x^{i+1}-x^i)\, d\mu(x^i) \right) d\mu(x^{k+1}).$$

We can rewrite the integral as

$$\int_{(\mathbb{R}^d)^k} \int_{\{x:x=x^m\}} \left(\prod_{j=1}^{k} \sigma_{t_j}^{\epsilon}(x^{i+1}-x^i) \right) d\mu(x^n)\, d\mu(x^1)\cdots d\mu(x^{n-1})\, d\mu(x^{n+1})\cdots d\mu(x^{k+1}).$$

Since the inside integral is taken over a region of measure 0, this whole integral must be 0. This holds for every choice of m and n, and thus the entire sum must be 0. This completes the proof of the lemma.

In conclusion, we have shown that the set of exact k-chains has positive measure — $\Lambda_{\vec{t}}^{k}(A_k) > 0$ — and that the set of degenerate chains has zero measure — $\Lambda_{\vec{t}}^{k}(D_k) = 0$. It follows that $A_k \neq D_k$ and $A_k \neq \varnothing$. In other words, there exists a nonempty open interval $\tilde{I}$ and *distinct* elements $x^1, \ldots, x^{k+1} \in E$ such that $|x^{i+1} - x^i| = t^i$ for each $i \in \{1, \ldots, k\}$.

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