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OBSTACLE PROBLEM WITH A DEGENERATE FORCE TERM

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We study the regularity of the free boundary at its intersection with the line $\{x_1 = 0\}$ in the obstacle problem

$$\triangle u = |x_1| \chi_{\{u>0\}} \quad \text{in } D,$$

where $D \subset \mathbb{R}^2$ is a bounded domain such that $D \cap \{x_1 = 0\} \neq \varnothing$.

We obtain a uniform $C^{1,1}$ bound on cubic blowups; we find all homogeneous global solutions; we prove the uniqueness of the blowup limit; we prove the convergence of the free boundary to the free boundary of the blowup limit; at the points with lowest Weiss balanced energy, we prove the convergence of the normal of the free boundary to the normal of the free boundary of the blowup limit and that locally the free boundary is a graph; and, finally, for a particular case we prove that the free boundary is not $C^{1,\alpha}$ regular near to a degenerate point for any $0 < \alpha < 1$.

1. Introduction

Let $D \subset \mathbb{R}^2$ be a bounded domain such that $D \cap \{x_1 = 0\} \neq \varnothing$. Let $g \in H^1(D)$ such that $g \geq 0$ on ∂D. Let $u \in H^1(D)$ be the unique minimiser of the functional

$$\int_D (|\nabla u|^2 + 2|x_1|u)\, dx \tag{1-1}$$

in the admissible set of functions

$$\{u \geq 0 \text{ a.e. in } D \text{ and } u = g \text{ on } \partial D\}.$$

For the existence and uniqueness of the minimiser u one may refer to [Kinderlehrer and Stampacchia 1980].

It is known (see [Petrosyan et al. 2012]) that $u \in C^{1,1}_{\text{loc}}(D)$ and

$$\triangle u = |x_1| \chi_{\{u>0\}} \quad \text{in } D \tag{1-2}$$

in the sense of distributions.

Let us denote by Ω the noncoincidence set and by Γ the free boundary, i.e.,

$$\Omega = \{x \in D \mid u(x) > 0\} \quad \text{and} \quad \Gamma = D \cap \partial\Omega.$$

Let us consider two examples. Set $D = (-1, 1)^2$. For the first example we take $g(x) = \frac{1}{16}(x_1 + x_2)^+$ and for the second example we take $g(x) = x_1^+(c - |x_2|)^+$, where $c \approx 0.42559$. The noncoincidence set and the free boundary are depicted in Figure 1 for both examples.

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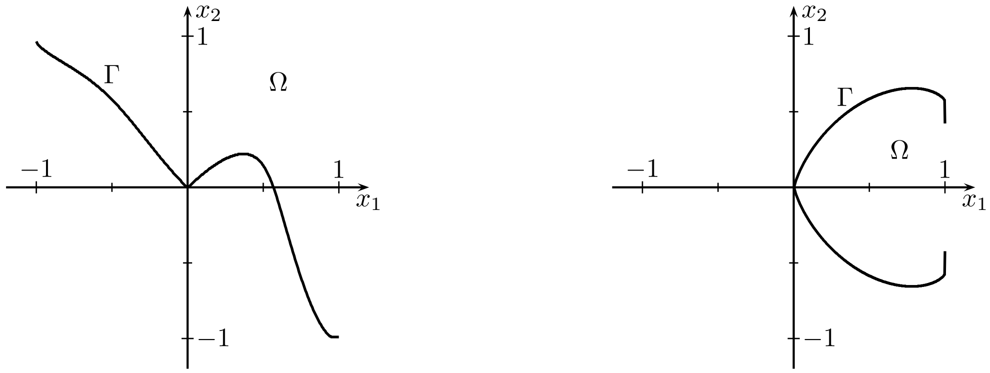
Keywords: free boundary, obstacle problem, degenerate, blowup, regularity.



(a) $g(x) = \frac{1}{16}(x_1 + x_2)^+$.

(b) $g(x) = x_1^+(c - |x_2|)^+$ with $c \approx 0.42559$.

Figure 1. Ω and Γ in the examples.

In the case of the nondegenerate obstacle problem, i.e., when instead of $|x_1|$ we have f satisfying $f \geq c$ in D for some $c > 0$, the Lipschitz and C^1 regularity of the free boundary was proved for the first time in [Caffarelli 1977]. A good reference for nondegenerate obstacle problems is [Caffarelli 1998] and a good reference for obstacle-type problems is [Petrosyan et al. 2012].

In [Yeressian 2015], for a class of degenerate obstacle problems the optimal nondegeneracy of the solution is obtained. The proof of the optimal nondegeneracy is based on specially constructed comparison functions using harmonic polynomials. In this paper the nondegeneracy result in [Yeressian 2015] will be used numerous times.

Our approach to prove the regularity of the free boundaries is based on some directional monotonicity properties satisfied by the solutions. This method is based on the proof of C^1 regularity in [Petrosyan et al. 2012] and is closely related to [Alt 1977].

We use Hopf's lemma to prove the irregularity of the free boundary in a particular case which corresponds to the free boundary near to the origin in the example depicted in Figure 1(b). A related irregularity result has been proved in [Shahgholian et al. 2007], where the authors considered a two-phase membrane problem and in higher dimensions they proved that the free boundary is, in a neighbourhood of each branch point, the union of two C^1-graphs, but in general these graphs are not $C^{1,\mathrm{Dini}}$ ($C^{1,\mathrm{Dini}}$ includes all $C^{1,\alpha}$ for $0 < \alpha < 1$).

Studying obstacle problems with a degenerate force term reveals rather unexpected behaviour of the solution, such as the fact that the free boundary usually forms a certain angle at its intersections with the line $\{x_1 = 0\}$, where the force term is degenerate.

In the problem of the free boundary near contact points with the fixed boundary — see [Shahgholian and Uraltseva 2003] — where the solution satisfies a homogeneous Dirichlet boundary condition, a similar strong influence of the data of the problem on the structure of the free boundary has been observed.

Varvaruca and Weiss [2011; 2012; 2014] have studied 2-dimensional or axisymmetric, 3-dimensional, inviscid, incompressible fluids acted on by gravity and with a free surface. These problems are in the class of Bernoulli free boundary problems, but the degeneracies in the force terms give rise to similar situations as encountered in this paper and has been a motivation for considering the problem in this paper.

This paper is structured as follows. In Section 2, the main results of this paper are presented. In Section 3, we prove uniform $C^{1,1}$ bounds on cubic blowups. In Section 4, using the Weiss balanced energy we prove the homogeneity of the blowup limits. In Section 5, we classify all possible homogeneous global solutions. In Section 6, using the fact that possible blowup limits form a discrete set we prove the uniqueness of the blowup limits. In Section 7, using a lower bound for homogeneous global solutions and the optimal nondegeneracy result in [Yeressian 2015] we prove the convergence of the free boundary to the free boundary of the blowup limit. In Section 8, we prove the convergence of the normal of the free boundary to the normal of the free boundary of the blowup limit at regular points. In Section 9, we prove that in a neighbourhood of a regular point the free boundary can be given as a graph. In Section 10, we prove that under some assumptions the free boundary near to a degenerate point is either flat or not $C^{1,\alpha}$ for any $0 < \alpha < 1$. In Section 11, we conclude this paper with a discussion about further directions of research on obstacle problems with degenerate forces.

2. Main results

Let us define a cubic blowup of u as follows:

Definition 1. Let $B_{r_0} \subset D$, then we define, for $0 < r < r_0$,

$$u_r(x) = \frac{u(rx)}{r^3} \quad \text{for } x \in B_1$$

and call u_r the (cubic) blowup of u at 0.

In the following theorem we prove that for $r > 0$ the family u_r is uniformly bounded in $C^{1,1}(B_1)$.

Theorem 2 (uniform $C^{1,1}$ bounds on blowups). *There exists a $C > 0$ such that, if u is a solution in D, $r_0 > 0$, $B_{r_0} \subset D$ and $0 \in \Gamma$, then we have the estimate*

$$\|u_r\|_{C^{1,1}(B_1)} \leq C \tag{2-1}$$

for $0 < r < \frac{1}{6} r_0$.

The proof of this theorem is based on the optimal growth result proved in [Yeressian 2015].

From the uniform bound (2-1) it follows that, for any sequence r_j such that $r_j \to 0$, there exists a subsequence r_{j_k} and $v \in C^{1,1}(B_1)$ such that $u_{j_k} \to v$ in $C^1(B_1)$.

Let us consider for $u \in H^1(B_r)$ the Weiss balanced energy

$$W(r,u) = \frac{1}{r^6} \int_{B_r} (|\nabla u|^2 + 2|x_1|u)\, dx - \frac{3}{r^7} \int_{\partial B_r} u^2 \, d\sigma(x). \tag{2-2}$$

The Weiss balanced energy [1998; 1999] was introduced to study the free boundary in the nondegenerate obstacle problem. The energy in (2-2) has been adapted to the first-order homogeneity of the force term $|x_1|$. For the Weiss balanced energy for different homogeneities, one may refer to [Petrosyan et al. 2012].

As we will see, for u a solution in D with $0 \in D$, by a monotonicity result for the Weiss balanced energy, the right limit $W(+0,u)$ exists but might be $-\infty$. If $0 \in \Gamma$ then $W(+0,u) > -\infty$.

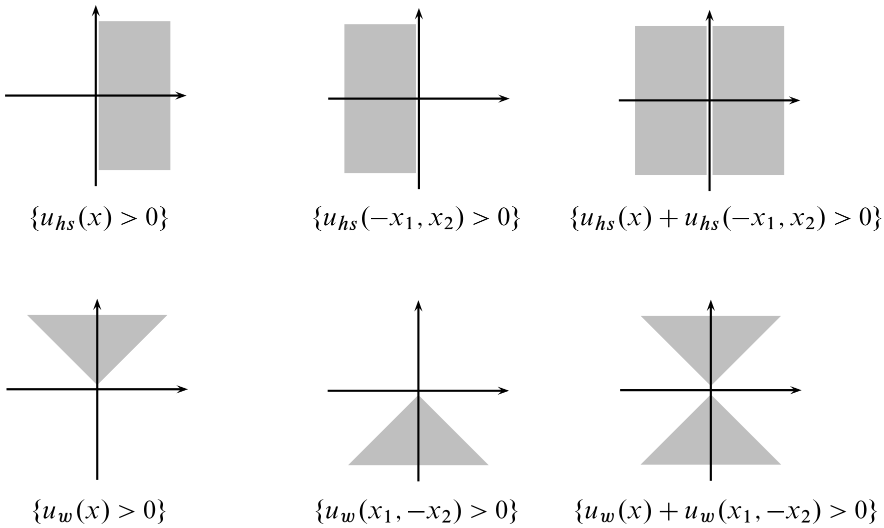


Figure 2. The only possible noncoincidence sets of nontrivial homogeneous global solutions.

Definition 3. Let u be a solution in D, $0 \in D$ and $0 \in \Gamma$. Then we call $v \in C^{1,1}(B_1)$ a blowup limit if there exists $r_j \to 0$ such that $u_{r_j} \to v$ in $C^1(B_1)$.

Using the Weiss balanced energy, if v is a blowup limit at 0 then v is a third-order homogeneous global solution and $W(+0, u) = W(1, v)$.

So we are led to find all the solutions of the obstacle problem

$$\begin{cases} \triangle u = |x_1| \chi_{\{u>0\}} \text{ in } \mathbb{R}^2, \\ u \text{ third-order homogeneous.} \end{cases} \tag{2-3}$$

Clearly $u = 0$ is a trivial solution of (2-3).

Let us define

$$u_{hs}(x) = \tfrac{1}{6}(x_1^+)^3 \quad \text{and} \quad u_w(x) = \left(\tfrac{1}{6}|x_1|^3 + \tfrac{1}{12}x_2^3 - \tfrac{1}{4}x_1^2 x_2\right)\chi_{\{x_2>|x_1|\}}. \tag{2-4}$$

Theorem 4 (classification of homogeneous global solutions). *The only nontrivial solutions of* (2-3) *are* u_w, $u_w(x_1, -x_2)$, $u_w + u_w(x_1, -x_2)$, u_{hs}, $u_{hs}(-x_1, x_2)$ *and* $u_{hs} + u_{hs}(-x_1, x_2)$.

To prove Theorem 4 we first find all the solutions of the corresponding no-sign obstacle problem and then among these solutions we find the nonnegative ones.

All possible noncoincidence sets of nontrivial homogeneous global solutions, i.e., the noncoincidence sets of the nontrivial solutions of (2-3), are depicted in Figure 2.

It is easy to see that $W(1, u_w) = W(1, u_w(x_1, -x_2))$, $W(1, u_w + u_w(x_1, -x_2)) = 2W(1, u_w)$, $W(1, u_{hs}) = W(1, u_{hs}(-x_1, x_2))$, $W(1, u_{hs} + u_{hs}(-x_1, x_2)) = 2W(1, u_{hs})$ and, by direct computation, we see that $0 < W(1, u_w)$ and

$$2W(1, u_w) < W(1, u_{hs}).$$

So we have the following four possible energy levels together with the order between them:

$$W(1,u_w) < 2W(1,u_w) < W(1,u_{hs}) < 2W(1,u_{hs}).$$

Let us define, for $y \in \Gamma \cap \{x_1 = 0\}$ and $r > 0$,

$$W(r, y, u) = W(r, u(\cdot + y)). \tag{2-5}$$

Based on the four possible values of $W(+0, x, u)$ (the value 0 is excluded by the nondegeneracy) for $x \in \Gamma \cap \{x_1 = 0\}$, the points of $\Gamma \cap \{x_1 = 0\}$ get classified in four types.

Definition 5. We call $y \in \Gamma \cap \{x_1 = 0\}$ a degenerate free boundary point if there exists $r_j \to 0$ such that $u(\cdot + y)_{r_j} \to u_{hs}$ or $u(\cdot + y)_{r_j}(x) \to u_{hs}(-x_1, x_2)$ in $C^1(B_1)$.

We use this name for points where a blowup limit is u_{hs} or $u_{hs}(-x_1, x_2)$ by following the naming for similar points in the problem studied in [Varvaruca and Weiss 2011].

In the example depicted in Figure 1(b), the origin is a degenerate free boundary point with u_{hs} as a blowup limit.

By our uniform bounds on the blowups it follows that 0 is degenerate if and only if $W(+0, u) = W(1, u_{hs})$.

Definition 6. We call $y \in \Gamma \cap \{x_1 = 0\}$ a regular free boundary point if there exists $r_j \to 0$ such that $u(\cdot + y)_{r_j} \to u_w$ or $u(\cdot + y)_{r_j}(x) \to u_w(x_1, -x_2)$ in $C^1(B_1)$.

In the example depicted in Figure 1(a) a point close to the origin is a regular free boundary point with u_w as a blowup limit.

By our uniform bounds on the blowups it follows that 0 is regular if and only if $W(+0, u) = W(1, u_w)$, i.e., it has the lowest Weiss balanced energy.

Theorem 7 (uniqueness of blowup limits). *If u is a solution in D, $0 \in D$ and $0 \in \Gamma$ then the blowup limit at the origin is unique.*

Let us define, for $\delta > 0$ and $k = 0$, 1,

$$\sigma_k(\delta) = \sup_{0<r\leq\delta} \|u_r - u_0\|_{C^k(B_1)}, \tag{2-6}$$

where u_0 is the unique blowup limit.

Theorem 8 (convergence of the free boundary). *There exists $C_1 > 0$ and $C_2 > 0$ such that if u is a solution in D, $0 \in D$ and $0 \in \Gamma$ then, for $x \in \Gamma$ close enough to the origin, if $W(+0, u) \in \{W(1, u_w), 2W(1, u_w)\}$ then we have*

$$d(x, \Gamma_{u_0}) \leq C_1 \big(\sigma_0(C_2|x|)\big)^{1/2} |x|, \tag{2-7}$$

where Γ_{u_0} is the free boundary of the unique blowup limit, and, if $W(+0, u) \in \{W(1, u_{hs}), 2W(1, u_{hs})\}$, then

$$|x_1| \leq C_1 \big(\sigma_0(C_2|x|)\big)^{1/3} |x|. \tag{2-8}$$

The proof of this theorem is based on a lower bound for the nontrivial homogeneous global solutions and the nondegeneracy result proved in [Yeressian 2015].

From Theorem 8, it follows that all points of $\Gamma\cap\{x_1=0\}\cap\{W(+0,x,u)\in\{W(1,u_w),2W(1,u_w)\}\}$ are isolated points of $\Gamma\cap\{x_1=0\}$ (in the topology of $\{x_1=0\}$), in particular.

Theorem 9 (convergence of normals and the free boundary as a graph at regular points). *There exists $C_1>0$ and $C_2>0$ such that if u is a solution in D, $0\in D$ and $0\in\Gamma$ is a regular free boundary point with blowup limit u_w then there exists $\epsilon>0$ and*

$$\gamma\in C\big(-\tfrac{1}{4}\epsilon,\tfrac{1}{4}\epsilon\big)\cap C^1\big(\big(-\tfrac{1}{4}\epsilon,\tfrac{1}{4}\epsilon\big)\backslash\{0\}\big)$$

such that

$$\begin{aligned}\Gamma\cap\big\{|x_1|<\tfrac{1}{4}\epsilon\big\}\cap B_\epsilon&=\big\{(x_1,\gamma(x_1))\;\big|\;x_1\in\big(-\tfrac{1}{4}\epsilon,\tfrac{1}{4}\epsilon\big)\big\},\\ \big|\gamma(x_1)-|x_1|\big|&\le C_1\big(\sigma_0(C_2|x_1|)\big)^{1/2}|x_1| &&\textit{for } x_1\in\big(-\tfrac{1}{4}\epsilon,\tfrac{1}{4}\epsilon\big),\\ \left|\gamma'(x_1)-\frac{x_1}{|x_1|}\right|&\le C_1\big(\sigma_1(C_2|x_1|)\big)^{1/2} &&\textit{for } x_1\in\big(-\tfrac{1}{4}\epsilon,\tfrac{1}{4}\epsilon\big)\backslash\{0\}.\end{aligned}$$

The proof of this theorem is mainly based on a directional monotonicity result proved in Lemma 37. There we prove that $\partial_\nu u\ge 0$ in $B_r(x)$ for $x\in\Gamma\cap\{x_1>0\}\cap\partial B_{1/4}$ if, for a given $\nu\in\partial B_1$ with $\nu\cdot\nu_w>0$, r is small enough and u is close enough to u_w in $C^1(B_1)$. The vector ν_w is the normal to the free boundary of u_w in the half-plane $\{x_1>0\}$, pointing into the noncoincidence set of u_w. This directional monotonicity result establishes the convergence of the normal of the free boundary to the normal of the free boundary of the blowup limit.

As we will see, from Theorem 9 it follows that, in the case when the origin is a regular point but with $u_w(x_1,-x_2)$ as blowup limit, and in the case when $W(+0,u)=2W(1,u_w)$, the free boundary is a graph or the union of two graphs, respectively.

In the following theorem, in particular cases we show that the free boundary near to a degenerate point is not $C^{1,\alpha}$ smooth.

Theorem 10 (an irregularity result at degenerate points). *Let u be a solution in D with $0\in D$. Suppose also that there exists $\delta>0$ such that $B_\delta\subset D$, $\partial_{x_2}u\le 0$ in $B_\delta\cap\{x_1>0,x_2>0\}$, there exists $\rho\in C\big(\big[0,\tfrac{1}{2}\delta\big)\big)\cap C^1\big(\big[0,\tfrac{1}{2}\delta\big)\big)$ such that $\rho(0)=\rho'(+0)=0$, $\rho\ge 0$ in $\big(0,\tfrac{1}{2}\delta\big)$, ρ is convex and*

$$\Omega\cap B_\delta\cap\big\{x_1>0,\;0<x_2<\tfrac{1}{2}\delta\big\}=B_\delta\cap\big\{0<x_2<\tfrac{1}{2}\delta,\;\rho(x_2)<x_1\big\}.$$

Then either $\rho=0$ and $u=u_{hs}$ in $\Omega\cap B_\delta\cap\big\{x_1>0,\;0<x_2<\tfrac{1}{2}\delta\big\}$ or the free boundary part $\Gamma\cap\{x_1>0\}$ is not $C^{1,\alpha}$ regular at 0 for any $0<\alpha<1$.

Let us note that the conditions in this theorem correspond to the example depicted in Figure 1(b).

The proof of this theorem relies on considering the nonnegative function $v=-\partial_{x_2}u$ and using the quantitative Hopf lemma (see [Han and Lin 2011]).

3. Uniform bounds on blowups

The following theorem is a consequence of the Harnack inequality and is a special case of the optimal growth theorem in [Yeressian 2015].

Theorem 11. *There exists a $C > 0$ such that if $B_r(y) \subset D$ then we have*

$$u(x) \leq C(u(y) + r^2(r + |y_1|)) \quad \textit{for } x \in B_{r/2}(y).$$

Based on this optimal growth estimate, in the following theorem we prove an estimate on the growth of the solution near the free boundary.

Lemma 12. *There exists a $C > 0$ such that if u is a solution in D, $y \in \Omega$, $d = d(y, \Gamma)$ and $B_{5d}(y) \subset D$ then*

$$u(x) \leq Cd^2(d + |y_1|) \quad \textit{for } x \in B_d(y). \tag{3-1}$$

Proof. Let $z \in \Gamma$ be such that $d = |y - z|$. We have, for $r = 4d$,

$$B_r(z) = B_{4d}(z) \subset B_{4d+|y-z|}(y) = B_{5d}(y) \subset D.$$

By Theorem 11 we have that, because $z \in \Gamma$ and $B_r(z) \subset D$,

$$u(x) \leq C_1 r^2(r + |z_1|) \quad \text{for } x \in B_{r/2}(z). \tag{3-2}$$

We have

$$B_d(y) \subset B_{d+|y-z|}(z) = B_{2d}(z) = B_{r/2}(z). \tag{3-3}$$

By (3-2) and (3-3) we obtain

$$\begin{aligned} u(x) \leq C_1 r^2(r + |z_1|) = C_1(4d)^2(4d + |z_1|) &\leq C_2 d^2(d + |z_1|) \\ &\leq C_2 d^2(d + |z_1 - y_1| + |y_1|) \\ &\leq C_2 d^2(2d + |y_1|) \leq C_3 d^2(d + |y_1|) \quad \text{for } x \in B_d(y), \end{aligned}$$

which proves the lemma. □

Let us define

$$\psi(t) = \tfrac{1}{6}|t|^3 \quad \text{for } t \in \mathbb{R} \tag{3-4}$$

and, for $t_0 \in \mathbb{R}$,

$$w_{t_0}(t) = \psi(t) - \psi(t_0) - \psi'(t_0)(t - t_0) \quad \text{for } t \in \mathbb{R}.$$

Then there exists $C > 0$ such that for $t, t_0 \in \mathbb{R}$ we have

$$w_{t_0}(t) \leq C|t - t_0|^2(|t_0| + |t - t_0|). \tag{3-5}$$

Proof of Theorem 2. We have

$$\|u_r\|_{L^\infty(B_1)} = \frac{1}{r^3}\|u\|_{L^\infty(B_r)}, \quad \|\nabla u_r\|_{L^\infty(B_1)} = \frac{1}{r^2}\|\nabla u\|_{L^\infty(B_r)}, \quad [\nabla u_r]_{C^{0,1}(B_1)} = \frac{1}{r}[\nabla u]_{C^{0,1}(B_r)}.$$

So, if we prove that for some $C > 0$ we have

$$\|u\|_{L^\infty(B_r)} \le Cr^3, \tag{3-6}$$

$$\|\nabla u\|_{L^\infty(B_r)} \le Cr^2, \tag{3-7}$$

$$[\nabla u]_{C^{0,1}(B_r)} \le Cr, \tag{3-8}$$

then the lemma is proved.

There exists $C > 0$ such that for v a harmonic function in B_1 we have

$$|\nabla v(0)| \le C\|v\|_{L^\infty(B_1)} \quad \text{and} \quad [\nabla v]_{C^{0,1}(B_{1/2})} \le C\|v\|_{L^\infty(B_1)}.$$

By scaling we obtain that for v harmonic in B_η we have

$$|\nabla v(0)| \le \frac{C}{\eta}\|v\|_{L^\infty(B_\eta)} \tag{3-9}$$

$$[\nabla v]_{C^{0,1}(B_{\eta/2})} \le \frac{C}{\eta^2}\|v\|_{L^\infty(B_\eta)}. \tag{3-10}$$

For $x \in \Omega$ let $d = d(x, \Gamma)$; then we have

$$B_{5d}(x) \subset B_{5d+|x|} \subset B_{5|x|+|x|} = B_{6|x|},$$

so if $x \in B_{(1/6)r_0}$ then $B_{5d}(x) \subset D$.

Now, by Lemma 12, we obtain that for $x \in B_{(1/6)r_0}$ we have

$$\|u\|_{L^\infty(B_d(x))} \le Cd^2(d + |x_1|). \tag{3-11}$$

Let $0 < r < \frac{1}{6}r_0$.

To prove (3-6), we compute, for $x \in B_r$,

$$|u(x)| \le \|u\|_{L^\infty(B_d(x))} \le Cd^2(d + |x_1|) \le C|x|^2(|x| + |x_1|) = 2C|x|^3 \le 2Cr^3.$$

To prove (3-7), using $w'_{x_1}(x_1) = 0$, (3-9), (3-11) and (3-5), we compute, for $x \in B_r$,

$$\begin{aligned} |\nabla u(x)| = |\nabla(u - w_{x_1})(x)| &\le \frac{C_1}{d}\|u - w_{x_1}\|_{L^\infty(B_d(x))} \\ &\le \frac{C_1}{d}\|u\|_{L^\infty(B_d(x))} + \frac{C_1}{d}\|w_{x_1}\|_{L^\infty(B_d(x))} \\ &\le C_2 d(d + |x_1|) + C_3 d(d + |x_1|) = C_4 d(d + |x_1|). \end{aligned} \tag{3-12}$$

From (3-12) it follows that

$$|\nabla u(x)| \le 2C_4|x|^2 \le 2C_4 r^2. \tag{3-13}$$

It remains to prove (3-8). We should show that

$$|\nabla u(x) - \nabla u(y)| \le Cr|x - y| \quad \text{for all } x, y \in B_r.$$

Fix x, $y \in B_r$. In the case $B_{|x-y|}\bigl(\frac{1}{2}(x + y)\bigr) \subset \Omega$ let us denote $z = \frac{1}{2}(x + y)$. We have $d = d(z, \Gamma) \ge |x - y|$.

By (3-10) and (3-11), we compute

$$
\begin{aligned}
\frac{|\nabla u(x)-\nabla u(y)|}{|x-y|} &\le [\nabla u]_{C^{0,1}(B_{|x-y|/2}(z))} \\
&\le [\nabla u]_{C^{0,1}(B_{d/2}(z))} \\
&\le [\nabla(u-w_{z_1})]_{C^{0,1}(B_{d/2}(z))} + [w'_{z_1}]_{C^{0,1}(B_{d/2}(z))} \\
&\le \frac{C_1}{d^2}\|u-w_{z_1}\|_{L^\infty(B_d(z))} + [w_{z_1}]_{C^2(B_{d/2}(z))} \\
&\le \frac{C_1}{d^2}\|u\|_{L^\infty(B_d(z))} + \frac{C_1}{d^2}\|w_{z_1}\|_{L^\infty(B_d(z))} + [\psi]_{C^2(B_{d/2}(z))} \\
&\le \frac{C_1}{d^2}C_2 d^2(d+|z_1|) + \frac{C_1}{d^2}C_3 d^2(d+|z_1|) + C_4(d+|z_1|) \\
&= C_5(d+|z_1|) \\
&\le 2C_5 r.
\end{aligned}
$$

In the case $B_{|x-y|}\big(\frac{1}{2}(x+y)\big)\cap\Omega^c \neq \varnothing$, by (3-12) we compute

$$
\begin{aligned}
|\nabla u(x)-\nabla u(y)| &\le |\nabla u(x)| + |\nabla u(y)| \\
&\le Cd(x,\Gamma)(d(x,\Gamma)+|x_1|) + Cd(y,\Gamma)(d(y,\Gamma)+|y_1|) \\
&\le \tfrac{3}{2}C|x-y|(d(x,\Gamma)+|x_1|) + \tfrac{3}{2}C|x-y|(d(y,\Gamma)+|y_1|) \\
&\le C_1 r|x-y|
\end{aligned}
$$

and this finishes the proof of the theorem. □

4. Homogeneity of blowup limits

Most of the results in this section are well known; one may refer to [Petrosyan et al. 2012; Weiss 1998; 1999]. But for the sake of completeness we include the proofs.

The Weiss balanced energy $W(r,u)$ is defined in (2-2).

Lemma 13. *For $r, s > 0$ and $u \in H^1(B_{rs})$, we have $W(rs,u) = W(s,u_r)$.*

For $u \in H^1(B_{r_0})$, $W(r,u)$ as a function of $0 < r < r_0$ is locally bounded and absolutely continuous.

For u a solution in B_{r_0} and $0 < r < r_0$, we have

$$\frac{d}{dr}W(r,u) = 2r\int_{\partial B_1}(\partial_r u_r)^2\,d\sigma(x). \tag{4-1}$$

For u a third-order homogeneous solution in B_1, we have

$$W(1,u) = \int_{B_1}|x_1|u\,dx. \tag{4-2}$$

Proof. Let $r, s > 0$ and $u \in H^1(B_{rs})$. We compute

$$\begin{aligned} W(rs,u) &= \frac{1}{(rs)^6}\int_{B_{rs}} (|\nabla u|^2 + 2|x_1|u)\,dx - \frac{3}{(rs)^7}\int_{\partial B_{rs}} u^2\,d\sigma(x) \\ &= \frac{1}{s^6}\frac{1}{r^4}\int_{B_s} \big(|\nabla u(rx)|^2 + 2r|x_1|u(rx)\big)\,dx - \frac{3}{s^7}\frac{1}{r^6}\int_{\partial B_s} u^2(rx)\,d\sigma(x) \\ &= \frac{1}{s^6}\int_{B_s}\big(|\nabla u_r(x)|^2 + 2|x_1|u_r\big)\,dx - \frac{3}{s^7}\int_{\partial B_s} u_r^2\,d\sigma(x) = W(s,u_r), \end{aligned}$$

which proves the first claim.

Let $u \in H^1(B_{r_0})$; then, for $0 < r < r_0$, by direct computation using polar coordinates we have

$$\int_{\partial B_r} u^2\,d\sigma(x) = -2r\int_{B_{r_0}\setminus B_r} \frac{1}{|x|^2} u(x)\nabla u(x)\cdot x\,dx + \frac{r}{r_0}\int_{\partial B_{r_0}} u^2(x)\,d\sigma(x). \tag{4-3}$$

The equation (4-3) together with the fact that if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ then $\int_{B_r} f\,dx$ as a function of r is bounded and absolutely continuous proves the second claim.

Let u be a solution in B_{r_0}, then we have (see [Petrosyan et al. 2012]) $u \in C^{1,1}_{\mathrm{loc}}(B_{r_0})$. Let $0 < r < r_0$, then we compute

$$\begin{aligned} &\frac{1}{2}\frac{d}{dr}W(r,u) \\ &\quad= \frac{1}{2}\frac{d}{dr}W(1,u_r) \\ &\quad= \frac{1}{2}\bigg(\int_{B_1}\big(2\nabla u_r(x)\cdot\nabla\partial_r u_r(x) + 2|x_1|\partial_r u_r\big)\,dx - 6\int_{\partial B_1} u_r\partial_r u_r\,d\sigma(x)\bigg) \\ &\quad= \int_{B_1}\big(\nabla u_r(x)\cdot\nabla\partial_r u_r(x) + |x_1|\partial_r u_r\big)\,dx - 3\int_{\partial B_1} u_r\partial_r u_r\,d\sigma(x) \\ &\quad= \int_{B_1}\big(-\Delta u_r(x)\partial_r u_r(x) + |x_1|\partial_r u_r\big)\,dx + \int_{\partial B_1}\partial_\nu u_r(x)\partial_r u_r(x)\,d\sigma(x) - 3\int_{\partial B_1} u_r\partial_r u_r\,d\sigma(x) \\ &\quad= \int_{\partial B_1}\big(\partial_\nu u_r(x) - 3u_r\big)\partial_r u_r\,d\sigma(x). \end{aligned}$$

It is easy to see that on ∂B_1 we have

$$\partial_\nu u_r(x) - 3u_r = r\,\partial_r u_r,$$

which proves the third claim.

Let u be a solution in B_1. We compute

$$\begin{aligned} W(1,u) &= \int_{B_1}\big(|\nabla u(x)|^2 + 2|x_1|u\big)\,dx - 3\int_{\partial B_1} u^2\,d\sigma(x) \\ &= \int_{B_1}(-\Delta u(x))u(x)\,dx + \int_{\partial B_1}\partial_\nu u(x)u(x)\,d\sigma(x) + \int_{B_1} 2|x_1|u\,dx - 3\int_{\partial B_1} u^2\,d\sigma(x) \\ &= \int_{\partial B_1}\partial_\nu u(x)u(x)\,d\sigma(x) + \int_{B_1}|x_1|u\,dx - 3\int_{\partial B_1} u^2\,d\sigma(x) \end{aligned}$$

$$= \int_{B_1} |x_1| u\, dx + \int_{\partial B_1} (\partial_\nu u - 3u) u\, d\sigma(x).$$

For a third-order homogeneous function we have $\partial_\nu u = 3u$; thus the last integral is null and this proves the last claim. □

If u is a solution in B_{r_0} for some $r_0 > 0$ then, by (4-1), $W(r,u)$ is nondecreasing in $0 < r < r_0$; thus the limit $\lim_{r\to 0, r>0} W(r,u) = W(+0,u)$ exists but might be $-\infty$. If $0 \in \Gamma$ then by Theorem 2 we have $\|u_r\|_{L^\infty(B_1)} \le C$ for small enough $0 < r$ and from this it follows that

$$-\frac{1}{r^7}\int_{\partial B_r} u^2\, d\sigma(x) = -\int_{\partial B_1} u_r^2\, d\sigma(x) \ge -c_1;$$

thus $W(r,u) \ge -3c_1$ and $W(+0,u) \ge -3c_1 > -\infty$.

For $y \in \Gamma \cap \{x_1 = 0\}$ and $r > 0$, $W(r,y,u)$ is defined in (2-5).

Lemma 14. *$W(+0,x,u)$ is an upper-semicontinuous function of $x \in \Gamma \cap \{x_1 = 0\}$.*

Proof. For $x \in \Gamma \cap \{x_1 = 0\}$, by the monotonicity of $W(r,x,u)$ as a function of $r > 0$ and its continuity as a function of x it follows that $W(+0,x,u) = \lim_{r\to 0, r>0} W(r,x,u)$ is upper-semicontinuous in $\Gamma \cap \{x_1 = 0\}$. □

Assume v is a third-order homogeneous function in B_1, i.e., $v(0) = 0$ and $v(x) = v(x/(2|x|))(2|x|)^3$ for all $x \in B_1\backslash\{0\}$. Then we might extend v as a third-order homogeneous function in $\mathbb{R}^2$ as $v(x) = v(x/(2|x|))(2|x|)^3$ for all $x \in B_1^c$. Let us note that the term on the right-hand side is well defined because for $x \in B_1^c$ we have $x/(2|x|) \in B_1$. From this definition of extension it follows that $v(rx) = r^3 v(x)$ for all $x \in \mathbb{R}^2$ and $r \ge 0$.

The following theorem is a special case of the main theorem in [Yeressian 2015].

Theorem 15. *There exists a $c > 0$ such that if u is a solution in D, $y \in \Omega$ and $B_r(y) \Subset D$ then we have*

$$\sup_{\Omega\cap\partial B_r(y)} u \ge u(y) + cr^2(r + |y_1|).$$

A blowup limit is defined in Definition 3.

Lemma 16. *Let v be a blowup limit. Then v is a third-order nontrivial homogeneous solution in B_1, the third-order homogeneous extension of v in $\mathbb{R}^2$ is a global solution, and $W(+0,u) = W(r,v)$ for $r > 0$.*

Proof. Assume $v \in C^{1,1}(B_1)$ is a blowup limit and $u_{r_j} \to v$ in $C^1(B_1)$.

From $u_{r_j} \ge 0$ in B_1 it follows that $v \ge 0$ in B_1. By the convergence $u_{r_j} \to v$ in $C^1(B_1)$ it follows that $\triangle u_{r_j} \to \triangle v$ in $H^{-1}(B_1)$ and in particular as distributions. Also $\chi_{\{u_{r_j}>0\}} \to \chi_{\{v>0\}}$ in $L^1(B_1)$ and thus $|x_1|\chi_{\{u_{r_j}>0\}} \to |x_1|\chi_{\{v>0\}}$ as distributions. Now (1-2) holds for u_{r_j} in B_1, so passing to the limit as $j\to\infty$ we obtain that v satisfies (1-2) in B_1. This together with $v \ge 0$ in B_1 proves that v is a solution to the obstacle problem in B_1.

For $0 < s < 1$ we compute

$$W(+0,u) = \lim_{j\to\infty} W(sr_j,u) = \lim_{j\to\infty} W(s,u_{r_j}) = W(s,v). \tag{4-4}$$

Thus $W(s, v)$ is independent of $0 < s < 1$.

Now, by (4-1), we obtain that for $0 < s < 1$

$$0 = \frac{d}{ds} W(s, v) = 2s \int_{\partial B_1} (\partial_s v_s)^2 \, d\sigma(x).$$

From here it follows that $\nabla v \cdot x - 3v = 0$ in B_1 and hence v is third-order homogeneous in B_1.

Now let us prove that v is not 0 in B_1, i.e., v is nontrivial.

Let $\delta > 0$ and $B_\delta \subset D$. Let $0 < r < \delta$ and $y \in B_{r/2} \cap \Omega$; then we have

$$B_{r/4}(y) \subset B_{r/4+|y|} \subset B_{r/4+r/2} = B_{3r/4} \Subset D,$$

thus by Theorem 15 we have

$$\sup_{\partial B_{r/4}(y)} u \geq u(y) + c\left(\tfrac{1}{4} r\right)^3.$$

We compute

$$\partial B_{r/4}(y) \subset B_{r/2}(y) \subset B_{r/2+|y|} \subset B_r,$$

so we have

$$\sup_{B_r} u \geq \sup_{\partial B_{r/4}(y)} u \geq u(y) + c\left(\tfrac{1}{4} r\right)^3 \geq \tfrac{1}{4^3} c r^3$$

and thus

$$\sup_{B_1} u_r \geq \tfrac{1}{4^3} c.$$

From this inequality, taking $r = r_j \to 0$, we obtain that v is not identically 0 in B_1.

Let us again denote by v the extension of v in $\mathbb{R}^2$. Then it is easy to see that, because v is a solution in B_1 and $v(rx) = r^3 v(x)$ for $x \in \mathbb{R}^2$ and $r \geq 0$, v is a solution in $\mathbb{R}^2$, i.e., a global solution.

By third-order homogeneity of v we have $W(r, v) = W\left(\tfrac{1}{2}, v\right)$ for $r > 0$ and this together with (4-4) proves the last claim of the lemma. □

5. Homogeneous global solutions

In this section we classify all possible solutions of the problem (2-3). The solutions of (2-3) form the subset of nonnegative solutions of the following no-sign obstacle problem (see [Petrosyan et al. 2012] for more on no-sign obstacle problems)

$$\begin{cases} \triangle u = |x_1| \chi_{\Omega(u)} & \text{in } \mathbb{R}^2, \\ \Omega(u) = \{u = |\nabla u| = 0\}^c, \\ u \ \text{third-order homogeneous.} \end{cases} \tag{5-1}$$

We first classify the nontrivial solutions of (5-1) and then find the subset of nonnegative and nontrivial solutions of (5-1), and thus obtain the classification of the nontrivial solutions of the problem (2-3).

In the rest of this section we always assume that $u \neq 0$ in $\mathbb{R}^2$, i.e., we discuss only the nontrivial solutions, so $\Omega \neq \varnothing$.

In both problems, by homogeneity, the set Ω is an open cone in $\mathbb{R}^2\backslash\{0\}$, i.e., for $x \in \Omega$ and $r > 0$ we have $rx \in \Omega$.

Either Ω is equal to $\mathbb{R}^2\backslash\{0\}$ or it is at most a countable union of disjoint connected open cones in $\mathbb{R}^2\backslash\{0\}$.

To classify the solutions in both problems we first establish if there exists a solution with $\Omega = \mathbb{R}^2\backslash\{0\}$. Then we find all the connected cones Ω not equal to $\mathbb{R}^2\backslash\{0\}$ for which there exists a corresponding solution.

Let us define

$$U(\theta) = u(e^{i\theta}) - \tfrac{1}{3} i\, \partial_\theta u(e^{i\theta}).$$

Lemma 17. *If u is a third-order homogeneous function in a connected open cone $\Omega \subset \mathbb{R}^2$ such that $\Delta u = |x_1|$ then there exists $a \in \mathbb{C}$ such that*

$$U(\theta) = \tfrac{1}{6}|\cos\theta| \cos(\theta) e^{i\theta} + \bar{a} e^{3i\theta} \tag{5-2}$$

for all $e^{i\theta} \in \Omega$ (in the rest of this section we identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$).

Proof. Let us write $v(x) = u(x) - \psi(x_1)$ with ψ as defined in (3-4); then v is a third-order homogeneous harmonic function in the connected open cone $\Omega \subset \mathbb{R}^2$. Thus there exists $a \in \mathbb{C}$ such that

$$v(x) = \Re(\bar{a}(x_1 + i x_2)^3) \quad \text{for all } x \in \Omega.$$

So we have

$$u(e^{i\theta}) = \tfrac{1}{6}|\cos\theta|^3 + \Re(\bar{a} e^{3i\theta}) \tag{5-3}$$

for all $e^{i\theta} \in \Omega$.

Differentiating (5-3) with respect to θ we obtain the desired equation. □

By the homogeneity of u it follows that

$$\{x \in \bar{\Omega} \mid u(x) = |\nabla u(x)| = 0\} = \{re^{i\theta} \in \bar{\Omega} \mid U(\theta) = 0,\ r > 0\}.$$

If $\Omega = \mathbb{R}^2\backslash\{0\}$ then, for u to be a solution to (5-1), U should be a periodic function with period 2π such that $U(\theta) \neq 0$ for all $\theta \in \mathbb{R}$ and if, in addition, u is a solution to (2-3) then we should have $\Re U(\theta) > 0$ for all $\theta \in \mathbb{R}$.

In the case that Ω is an open connected cone not equal to $\mathbb{R}^2\backslash\{0\}$, there exist $\theta_1, \theta_2 \in \mathbb{R}$ such that $\theta_1 < \theta_2 \leq \theta_1 + 2\pi$ and $\Omega = \{re^{i\theta} \mid r > 0,\ \theta_1 < \theta < \theta_2\}$. In this case, if u is a solution to (5-1) with $\Omega = \Omega(u)$, then U should satisfy $U(\theta_1) = U(\theta_2) = 0$ and $U(\theta) \neq 0$ for $\theta_1 < \theta < \theta_2$. If, in addition, u is a solution to (2-3) then we should have $\Re U(\theta) > 0$ for $\theta_1 < \theta < \theta_2$.

Let us define

$$V(\theta) = |\cos\theta| \cos(\theta) e^{2i\theta}. \tag{5-4}$$

It follows that

$$6e^{3i\theta} \bar{U}(\theta) = V(\theta) + 6a. \tag{5-5}$$

Lemma 18. *The function u is a solution of (5-1) with $\Omega = \mathbb{R}^2\backslash\{0\}$ if and only if $-6a \notin V(\mathbb{R})$.*

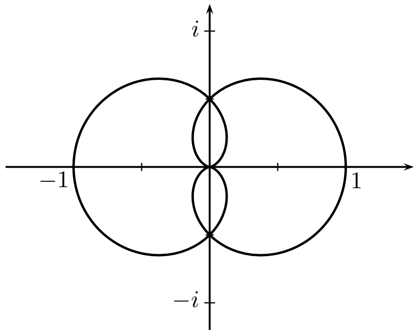


Figure 3. The set $V(\mathbb{R})$.

Proof. The function u is a solution of (5-1) with $\Omega = \mathbb{R}^2 \backslash \{0\}$ if and only if U is 2π-periodic and $U(\theta) \neq 0$ for all $\theta \in \mathbb{R}$.

From (5-2) it follows that U is 2π-periodic and, by (5-5), it is clear that $U(\theta) \neq 0$ for all $\theta \in \mathbb{R}$ if and only if $-6a \notin V(\mathbb{R})$. □

From the definition of V in (5-4) it is clear that $B_1^c \subset (V(\mathbb{R}))^c$, so by Lemma 18 it follows that there are many solutions of (5-1) with $\Omega = \mathbb{R}^2 \backslash \{0\}$.

Let us note that for a connected cone specified by θ_1 and θ_2, the solution with such a cone is unique. This follows from the fact that, because $U(\theta_1) = 0$, by (5-2) a is uniquely obtained and for this value of a the solution u is uniquely given by (5-3). Based on this observation, in the following we do not distinguish between a connected cone and the corresponding solution.

Lemma 19. *The function u is a solution of (5-1) with a connected open cone $\Omega \neq \mathbb{R}^2 \backslash \{0\}$ if and only if one of the following cases hold*:

(i) $\theta_1 \notin \mathbb{Z}\pi + \{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$ *and* $\theta_2 = \theta_1 + 2\pi$.

(ii) $\theta_1 \in \mathbb{Z}\pi + \frac{\pi}{2}$ *and* $\theta_2 = \theta_1 + \pi$.

(iii) $\theta_1 \in \mathbb{Z}\pi + \frac{\pi}{4}$ *and* $\theta_2 = \theta_1 + \frac{\pi}{2}$.

(iv) $\theta_1 \in \mathbb{Z}\pi + \frac{3\pi}{4}$ *and* $\theta_2 = \theta_1 + \frac{3\pi}{2}$.

Proof. Let us remember that we should have $\theta_1, \theta_2 \in \mathbb{R}$, $\theta_1 < \theta_2 \leq \theta_1 + 2\pi$, $U(\theta_1) = U(\theta_2) = 0$ and $U(\theta) \neq 0$ for $\theta_1 < \theta < \theta_2$. It is possible to find all such θ_1 and θ_2 by algebraic computations, but for ease of presentation we resort to geometric arguments.

By (5-5), $U(\theta) = 0$ if and only if $-6a = V(\theta)$, hence we should have $\theta_1, \theta_2 \in \mathbb{R}$, $\theta_1 < \theta_2 \leq \theta_1 + 2\pi$, $V(\theta_1) = V(\theta_2)$ and $V(\theta) \neq V(\theta_1)$ for $\theta_1 < \theta < \theta_2$. Thus we should find the smallest closed loops in the range graph of V. The range graph of V, i.e., the set $V(\mathbb{R})$ is depicted in Figure 3.

Then we have the following four cases:

(i) $-6a = V(\theta_1) \in V(\mathbb{R}) \backslash \{0, \pm\frac{i}{2}\}$ with $\theta_1 \in \mathbb{R} \backslash (\mathbb{Z}\pi + \{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\})$ and the smallest loop is when $\theta_2 = \theta_1 + 2\pi$.

(ii) $-6a = V(\theta_1) = 0$ with $\theta_1 \in \mathbb{Z}\pi + \frac{\pi}{2}$ and the smallest loop is when $\theta_2 = \theta_1 + \pi$.

(iii) $-6a = V(\theta_1) \in \{\pm\frac{i}{2}\}$ with $\theta_1 \in \mathbb{Z}\pi + \frac{\pi}{4}$ and the smallest loop is when $\theta_2 = \theta_1 + \frac{\pi}{2}$.

(iv) $-6a = V(\theta_1) \in \{\pm\frac{i}{2}\}$ with $\theta_1 \in \mathbb{Z}\pi + \frac{3\pi}{4}$ and the smallest loop is when $\theta_2 = \theta_1 + \frac{3\pi}{2}$. □

There is some redundancy in the solutions specified in the previous lemma. In the following lemma we prove that if for two solutions the corresponding connected cones are rotations of each other by a multiple of π then the corresponding solutions are also rotated by the same angle.

Lemma 20. *Let $a, a' \in \mathbb{C}$ and let U, U' be the corresponding functions. If $n \in \mathbb{Z}$ and $\theta_0 \in \mathbb{R}$ are such that $U'(\theta_0 + n\pi) = U(\theta_0)$ then $U'(\theta + n\pi) = U(\theta)$ for all $\theta \in \mathbb{R}$.*

Proof. For any $n \in \mathbb{Z}$ and $\theta \in \mathbb{R}$ we have

$$\begin{aligned} U'(\theta + n\pi) &= \overline{a'}e^{3i(\theta+n\pi)} + \tfrac{1}{6}|\cos(\theta + n\pi)|\cos(\theta + n\pi)e^{i(\theta+n\pi)} \\ &= (-1)^n\overline{a'}e^{3i\theta} + \tfrac{1}{6}|\cos\theta|\cos(\theta)e^{i\theta} = ((-1)^n\overline{a'} - \overline{a})e^{3i\theta} + U(\theta), \end{aligned}$$

from which the lemma follows because if $U'(\theta_0 + n\pi) = U(\theta_0)$ for some θ_0 then $(-1)^n\overline{a'} - \overline{a} = 0$, from which in turn it follows that $U'(\theta + n\pi) = U(\theta)$ for all θ. □

Corollary 21. *Let u and u' be solutions of (5-1) with $\Omega(u) = \{re^{i\theta} \mid \theta_1 < \theta < \theta_2,\ r > 0\}$ and $\Omega(u') = \{re^{i\theta} \mid \theta_1' < \theta < \theta_2',\ r > 0\}$, where $\theta_1 < \theta_2 \le \theta_1 + 2\pi$ and $\theta_1' < \theta_2' \le \theta_1' + 2\pi$. If there exists $n \in \mathbb{Z}$ such that $\theta_1' = \theta_1 + n\pi$ and $\theta_2' = \theta_2 + n\pi$, then $u'(e^{i(\theta+n\pi)}) = u(e^{i\theta})$ for $\theta_1 < \theta < \theta_2$.*

Proof. Let $U(\theta)$ correspond to $u(x)$ and $U'(\theta)$ to $u'(x)$. Then $U(\theta_1) = 0$ and $U'(\theta_1') = 0$. Thus $U(\theta_1) = U'(\theta_1') = U'(\theta_1 + n\pi)$. Now by Lemma 20 the corollary is proved. □

By this corollary we are able to remove some of the redundancies in Lemma 19, as stated in the following corollary:

Corollary 22. *The function u is a solution of (5-1) with a connected open cone $\Omega \neq \mathbb{R}^2\setminus\{0\}$ if and only if one of the following cases hold*:

(i) $\theta_1 \in [0, 2\pi)\setminus\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}\}$ *and* $\theta_2 = \theta_1 + 2\pi$: *the solutions corresponding to θ_1 in $[\pi, 2\pi)\setminus\{\frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}\}$ are equal to the solutions corresponding to $\theta_1 \in [0, \pi)\setminus\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$ rotated by π, respectively.*

(ii) $\theta_1 \in \{\frac{\pi}{2}, \frac{3\pi}{2}\}$ *and* $\theta_2 = \theta_1 + \pi$: *the solution corresponding to $\theta_1 = \frac{3\pi}{2}$ is equal to the solution corresponding to $\theta_1 = \frac{\pi}{2}$ rotated by π,*

(iii) $\theta_1 \in \{\frac{\pi}{4}, \frac{5\pi}{4}\}$ *and* $\theta_2 = \theta_1 + \frac{\pi}{2}$: *the solution corresponding to $\theta_1 = \frac{5\pi}{4}$ is equal to the solution corresponding to $\theta_1 = \frac{\pi}{4}$ rotated by π.*

(iv) $\theta_1 \in \{\frac{3\pi}{4}, \frac{7\pi}{4}\}$ *and* $\theta_2 = \theta_1 + \frac{3\pi}{2}$: *the solution corresponding to $\theta_1 = \frac{7\pi}{4}$ is equal to the solution corresponding to $\theta_1 = \frac{3\pi}{4}$ rotated by π.*

By Lemma 18 we have obtained the solutions of (5-1) with $\Omega = \mathbb{R}^2\setminus\{0\}$ and by Corollary 22 we have obtained all the solutions of (5-1) with a connected open cone $\Omega \neq \mathbb{R}^2\setminus\{0\}$. Now we turn to finding the nonnegative solutions among these solutions.

To check the nonnegativity of a solution u, in the following lemma we write $u(e^{i\theta})$ in a closed form.

Lemma 23. *Let $\theta_1 < \theta_2 \le \theta_1 + 2\pi$ and let u be a solution to (5-1) in the cone corresponding to θ_1 and θ_2. Then we have*

$$6u(e^{i\theta}) = |\cos\theta|^3 - |\cos\theta_1|\cos(\theta_1)\cos(3\theta - 2\theta_1). \tag{5-6}$$

Proof. Because $U(\theta_1) = 0$, by (5-5) we have $6\overline{a} = -\overline{V}(\theta_1)$.

Now, by (5-3) we compute

$$\begin{aligned}
6u(e^{i\theta}) = |\cos\theta|^3 + \Re(6\overline{a}e^{3i\theta}) &= |\cos\theta|^3 - \Re(\overline{V}(\theta_1)e^{3i\theta}) \\
&= |\cos\theta|^3 - \Re(|\cos\theta_1|\cos(\theta_1)e^{-2i\theta_1}e^{3i\theta}) \\
&= |\cos\theta|^3 - \Re(|\cos\theta_1|\cos(\theta_1)e^{(3\theta-2\theta_1)i}) \\
&= |\cos\theta|^3 - |\cos\theta_1|\cos(\theta_1)\Re(e^{(3\theta-2\theta_1)i}) \\
&= |\cos\theta|^3 - |\cos\theta_1|\cos(\theta_1)\cos(3\theta-2\theta_1),
\end{aligned}$$

which proves (5-6). ☐

Lemma 24. *There exists no solution to the problem (2-3) with $\Omega = \{u > 0\} = \mathbb{R}^2\setminus\{0\}$.*

Proof. On the line segments $\{x_1 = 0\}\setminus\{0\}$, i.e., for $\theta = \pm\frac{\pi}{2}$, we have

$$\begin{aligned}
6u(e^{\pm i\pi/2}) &= \left|\cos\left(\pm\tfrac{\pi}{2}\right)\right|^3 - |\cos\theta_1|\cos(\theta_1)\cos\left(\pm\tfrac{3\pi}{2} - 2\theta_1\right) \\
&= -|\cos\theta_1|\cos(\theta_1)\cos\left(\pm\tfrac{3\pi}{2} - 2\theta_1\right) \\
&= \pm|\cos\theta_1|\cos(\theta_1)\sin(2\theta_1).
\end{aligned} \tag{5-7}$$

If $|\cos\theta_1|\cos(\theta_1)\sin(2\theta_1) = 0$ then $u(e^{\pm i\frac{\pi}{2}}) = 0$, which is in contradiction with $\Omega = \{u > 0\} = \mathbb{R}^2\setminus\{0\}$. If $|\cos\theta_1|\cos(\theta_1)\sin(2\theta_1) \ne 0$ then we can choose $\theta = \frac{\pi}{2}$ or $\theta = -\frac{\pi}{2}$ and obtain $u(e^{i\theta}) < 0$, which is again in contradiction with $\Omega = \{u > 0\} = \mathbb{R}^2\setminus\{0\}$. ☐

Lemma 25. *The function u is a solution of (2-3) with a connected open cone $\Omega \ne \mathbb{R}^2\setminus\{0\}$ if and only if one of the following cases hold*:

(i) *$\theta_1 \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$ and $\theta_2 = \theta_1 + \pi$: the solution corresponding to $\theta_1 = \frac{3\pi}{2}$ is equal to the solution corresponding to $\theta_1 = \frac{\pi}{2}$ rotated by π.*

(ii) *$\theta_1 \in \left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}$ and $\theta_2 = \theta_1 + \frac{\pi}{2}$: the solution corresponding to $\theta_1 = \frac{5\pi}{4}$ is equal to the solution corresponding to $\theta_1 = \frac{\pi}{4}$ rotated by π.*

Proof. We first show that the solutions given in parts (i) and (iv) of Corollary 22 are not nonnegative and then we show that the solutions given in parts (ii) and (iii) are nonnegative.

To prove the failure of nonnegativity of solutions given in part (i) of Corollary 22 we need only to consider $\theta_1 \in [0,\pi)\setminus\left\{\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\right\}$ with $\theta_2 = \theta_1 + 2\pi$ and, to prove the failure of nonnegativity of solutions given in part (iv), we need only to consider $\theta_1 = \frac{3\pi}{4}$ with $\theta_2 = \theta_1 + \frac{3\pi}{2}$.

For all these cases let us consider $\theta = \frac{3\pi}{2}$, then $\theta_1 < \theta < \theta_2$ and, by a similar computation as in (5-7), we obtain that

$$6u(e^{3\pi i/2}) = -|\cos\theta_1|\cos(\theta_1)\sin(2\theta_1).$$

Because for $\theta_1 \in [0, \pi)$ we have

$$|\cos\theta_1| \cos(\theta_1) \sin(2\theta_1) = 2|\cos\theta_1| \cos^2(\theta_1) \sin\theta_1 \geq 0,$$

this proves that the respective solutions take a nonpositive value at $\theta = \frac{3\pi}{2}$. If $u(e^{3\pi i/2}) < 0$ then u is not nonnegative. If $u(e^{3\pi i/2}) = 0$ and u was nonnegative then we would have $\partial_\theta u(e^{3\pi i/2}) = 0$, which is in contradiction with the connectedness of Ω.

To prove that the solutions given in Corollary 22(ii) are solutions of (2-3), we need only to consider the case when $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = \theta_1 + \pi$. We compute

$$6u(e^{i\theta}) = |\cos\theta|^3 - \left|\cos\tfrac{\pi}{2}\right| \cos\left(\tfrac{\pi}{2}\right) \cos\left(3\theta - 2\left(\tfrac{\pi}{2}\right)\right) = |\cos\theta|^3 \tag{5-8}$$

and, because $|\cos\theta| > 0$ for $\frac{\pi}{2} < \theta < \frac{3\pi}{2}$, we obtain that u is a solution of (2-3).

To prove that the solutions given in Corollary 22(iii) are solutions of (2-3), we need only to consider the case when $\theta_1 = \frac{\pi}{4}$ and $\theta_2 = \theta_1 + \frac{\pi}{2}$. We compute

$$6u(e^{i\theta}) = |\cos\theta|^3 - \left|\cos\tfrac{\pi}{4}\right| \cos\left(\tfrac{\pi}{4}\right) \cos\left(3\theta - \tfrac{\pi}{2}\right) = |\cos\theta|^3 - \tfrac{1}{2}\cos\left(3\theta - \tfrac{\pi}{2}\right) = |\cos\theta|^3 - \tfrac{1}{2}\sin(3\theta). \tag{5-9}$$

Let $\theta = \frac{\pi}{2} + \gamma$ for $-\frac{\pi}{4} < \gamma < \frac{\pi}{4}$; then

$$6u(e^{i(\pi/2+\gamma)}) = \left|\cos\left(\tfrac{\pi}{2} + \gamma\right)\right|^3 - \tfrac{1}{2}\sin\left(3\left(\tfrac{\pi}{2} + \gamma\right)\right) = |\sin\gamma|^3 + \tfrac{1}{2}\cos(3\gamma).$$

It follows that $6u(e^{i(\pi/2+\gamma)}) = 6u(e^{i(\pi/2-\gamma)})$, so we need only to consider $0 \leq \gamma < \frac{\pi}{4}$. For $0 \leq \gamma < \frac{\pi}{4}$ we have $\sin\gamma \geq 0$, thus

$$6u(e^{i(\pi/2+\gamma)}) = \sin^3\gamma + \tfrac{1}{2}\cos(3\gamma) = \tfrac{1}{2}\cos^3(\gamma)(\tan\gamma - 1)^2(2\tan\gamma + 1) > 0;$$

therefore we obtain that u is a solution of (2-3). □

Lemma 26. *In the original variable $x \in \mathbb{R}^2$, the only solutions of (2-3) with a connected open cone $\Omega \neq \mathbb{R}^2 \setminus \{0\}$ are the following four solutions together with their noncoincidence cone Ω and their free boundary Γ:*

$$\begin{aligned}
u(x) &= u_{hs}(x), & \Omega &= \{x_1 > 0\}, & \Gamma &= \{x_1 = 0\};\\
u(x) &= u_{hs}(-x_1, x_2), & \Omega &= \{x_1 < 0\}, & \Gamma &= \{x_1 = 0\};\\
u(x) &= u_w(x), & \Omega &= \{x_2 > |x_1|\}, & \Gamma &= \{x_2 = |x_1|\};\\
u(x) &= u_w(x_1, -x_2), & \Omega &= \{x_2 < -|x_1|\}, & \Gamma &= \{x_2 = -|x_1|\}.
\end{aligned}$$

Proof. We compute the solutions given in Lemma 25 in the original variable.

For solutions given in Lemma 25(i), we only consider the case when $\theta_1 = \frac{\pi}{2}$ and $\theta_2 = \theta_1 + \pi$. We have

$$\left\{x = re^{i\theta} \;\middle|\; r > 0,\ \tfrac{\pi}{2} < \theta < \tfrac{3\pi}{2}\right\} = \{x_1 < 0\}.$$

Now, for $x = re^{i\theta} \in \{x_1 < 0\}$, using the computation in (5-8) we compute

$$6u(x) = 6u(re^{i\theta}) = 6r^3 u(e^{i\theta}) = r^3|\cos\theta|^3 = r^3|x_1/r|^3 = |x_1|^3 = (x_1^-)^3.$$

For solutions given in Lemma 25(ii) we only consider the case when $\theta_1 = \frac{\pi}{4}$ and $\theta_2 = \theta_1 + \frac{\pi}{2}$. We have

$$\{x = re^{i\theta} \mid r > 0,\ \tfrac{\pi}{4} < \theta < \tfrac{3\pi}{4}\} = \{x_2 > |x_1|\}.$$

Now, for $x = re^{i\theta} \in \{x_2 > |x_1|\}$, using the computation in (5-9) we compute

$$\begin{aligned} 6u(x) = 6u(re^{i\theta}) = 6r^3u(e^{i\theta}) &= r^3\big(|\cos\theta|^3 - \tfrac{1}{2}\sin(3\theta)\big) \\ &= r^3\big(|\cos\theta|^3 - \tfrac{1}{2}(3\cos^2(\theta)\sin\theta - \sin^3\theta)\big) \\ &= r^3\big(|x_1/r|^3 - \tfrac{1}{2}(3(x_1/r)^2x_2/r - (x_2/r)^3)\big) \\ &= |x_1|^3 - \tfrac{1}{2}(3x_1^2x_2 - x_2^3), \end{aligned}$$

which completes the proof of the lemma. □

Proof of Theorem 4. By Lemma 24 there exists no solution to the problem (2-3) with $\Omega = \{u > 0\} = \mathbb{R}^2 \backslash \{0\}$.

So we are left only with solutions whose noncoincidence open cone Ω is a countable union of disjoint connected open cones. But, considering the only possible connected open cones as noncoincidence sets enumerated in Lemma 26, we come to the conclusion that, except for the solutions with connected cones, there exist two additional solutions, $u_w + u_w(x_1, -x_2)$ and $u_{hs} + u_{hs}(-x_1, x_2)$, each a combination of two solutions with connected open cones. □

Lemma 27. *We have*

$$W(1, u_{hs}) = \tfrac{\pi}{96} \quad \textit{and} \quad W(1, u_w) = \tfrac{1}{192}\big(\pi - \tfrac{8}{3}\big).$$

Proof. For any solution of (2-3) with connected open cone, we have, using (4-2),

$$\begin{aligned} W(1,u) = \int_{B_1} |x_1|u\,dx = \int_0^1 \int_{\partial B_r} |x_1|u\,d\sigma(x)\,dr &= \int_0^1 \int_{\partial B_1} |ry_1|u(ry)r\,d\sigma(y)\,dr \\ &= \int_0^1 r^5\,dr \int_{\partial B_1} |y_1|u(y)\,d\sigma(y) \\ &= \frac{1}{6}\int_{\theta_1}^{\theta_2} |\cos\theta|u(e^{i\theta})\,d\theta. \end{aligned}$$

For the half-space solution u_{hs}, we compute, using (5-8),

$$W(1, u_{hs}) = \frac{1}{36}\int_{\pi/2}^{3\pi/2} |\cos\theta|^4\,d\theta = \frac{1}{18}\int_0^{\pi/2} \cos^4\theta\,d\theta = \frac{\pi}{96}.$$

For the wedge solution u_w, we compute, using (5-9),

$$\begin{aligned} W(1, u_w) &= \frac{1}{36}\int_{\pi/4}^{3\pi/4} \big(|\cos\theta|^4 - \tfrac{1}{2}|\cos\theta|\sin(3\theta)\big)\,d\theta \\ &= \frac{1}{18}\int_{\pi/4}^{\pi/2} \cos^4\theta\,d\theta - \frac{1}{36}\int_{\pi/4}^{\pi/2} \cos(\theta)\sin(3\theta)\,d\theta = \tfrac{1}{192}\big(\pi - \tfrac{8}{3}\big), \end{aligned}$$

which completes the proof of the lemma. □

Corollary 28. *We have*

$$
\begin{aligned}
0 < W(1,u_w) = W(1,u_w(x_1,-x_2)) < W(1,u_w+u_w(x_1,-x_2)) = 2W(1,u_w)\\
< W(1,u_{hs}) = W(1,u_{hs}(-x_1,x_2)) < W(1,u_{hs}+u_{hs}(-x_1,x_2)) = 2W(1,u_{hs}).
\end{aligned}
$$

Proof. The only inequality that is not clear is the inequality $2W(1,u_w) < W(1,u_{hs})$. But this is verified by the explicit values computed in the previous lemma. ☐

Corollary 29. *The set $\Gamma \cap \{x_1 = 0\}$ might be decomposed into four disjoint sets according to four possible values of the Weiss balanced energy. The closure of the set of points with a given energy w is a subset of the set of points with energy larger than or equal to w.*

Proof. Let $y \in \Gamma \cap \{x_1 = 0\}$; then, by the translation $u(x+y)$, we might assume that $y = 0$. Let $0 < \delta$ be such that $B_\delta \subset D$. Let us consider the family u_r for $0 < r < \frac{1}{6}\delta$. By Theorem 2 this family is uniformly bounded in $C^{1,1}(B_1)$. Thus there exists $r_j \to 0$ and $v \in C^{1,1}(B_1)$ such that $u_{r_j} \to v$ in $C^1(B_1)$. By Lemma 16, v is a nontrivial homogeneous global solution and $W(+0,u) = W(1,v)$. The possible values of $W(1,v)$ are only of the four values given in the previous corollary and this shows that the free boundary points $\Gamma \cap \{x_1 = 0\}$ divide into four disjoint sets depending on the Weiss balanced energy of the blowups at that point.

The last claim follows from the upper semicontinuity of $W(+0,x,u)$ stated in Lemma 14. ☐

For example, from Corollary 29 it follows that the set $\Gamma \cap \{x_1 = 0\} \cap \{W(+0,x,u) = 2W(1,u_{hs})\}$ is closed. Actually, at the end of Section 7 we will show that all points of $\Gamma \cap \{x_1 = 0\} \cap \{W(+0,x,u)$ in $\{W(1,u_w), 2W(1,u_w)\}\}$ are isolated points of $\Gamma \cap \{x_1 = 0\}$.

In the following lemma we obtain a lower bound for the homogeneous global solutions, which will be used in Lemma 32.

Lemma 30. *There exists a $c > 0$ such that for all homogeneous global solutions u we have*

$$
u(x) \geq c d^2(x,\{u=0\})\big(d(x,\{u=0\}) + |x_1|\big) \quad \text{for } x \in \mathbb{R}^2. \tag{5-10}
$$

Proof. It is easy to see that we need to prove (5-10) for the cases when $u = u_w$ or $u = u_{hs}$.

In the case $u = u_{hs}$, for $x_1 \leq 0$ both sides of the inequality (5-10) are 0. For $x_1 > 0$ we have $d(x,\{u_{hs}=0\}) = x_1$, hence

$$
\begin{aligned}
u_{hs}(x) = \tfrac{1}{6}x_1^3 &= \tfrac{1}{6}d^2(x,\{u_{hs}=0\})\big(\tfrac{1}{2}d(x,\{u_{hs}=0\}) + \tfrac{1}{2}x_1\big)\\
&= \tfrac{1}{12}d^2(x,\{u_{hs}=0\})\big(d(x,\{u_{hs}=0\}) + x_1\big)
\end{aligned}
$$

and this proves (5-10) for $u = u_{hs}$.

In the case $u = u_w$, for $x_2 < |x_1|$ both sides of the inequality are 0. Also, by the symmetry $u_w(x_1,x_2) = u_w(-x_1,x_2)$ we need only to consider the case $x_2 > x_1 > 0$.

For $x_2 > x_1 > 0$ it is easy to see that $d(x, \{u_w = 0\}) = \frac{1}{\sqrt{2}}(x_2 - x_1)$, thus for $x_2 > x_1 > 0$ we compute

$$\begin{aligned} u_w(x) = \tfrac{1}{6}x_1^3 + \tfrac{1}{12}x_2^3 - \tfrac{1}{4}x_1^2 x_2 &= \tfrac{1}{12}(x_2 - x_1)^2(2x_1 + x_2) \\ &= \tfrac{1}{12}\big(\sqrt{2}d(x, \{u_w = 0\})\big)^2\big(3x_1 + \sqrt{2}d(x, \{u_w = 0\})\big) \\ &\geq \tfrac{1}{6}\sqrt{2}d^2(x, \{u_w = 0\})\big(d(x, \{u_w = 0\}) + x_1\big), \end{aligned}$$

which proves the desired inequality. ◻

In the next lemma we prove directional monotonicity type inequalities, which will be used in Lemma 37.

Lemma 31. *There exists a $C > 0$ such that $a\partial_\nu u_w - u_w \geq 0$ in $B_1 \cap \{(1+\epsilon)x_1 > x_2 > x_1 > 0\}$ if $\nu = e^{i(3\pi/4+\gamma)}$, $\epsilon > 0$, $-\frac{\pi}{2} < \gamma < \frac{\pi}{2}$ and $C(1/a + 1)\epsilon \leq \cos\gamma$.*

Proof. For $x_2 > x_1 > 0$ we have

$$\begin{aligned} u_w(x) &= \tfrac{1}{6}x_1^3 + \tfrac{1}{12}x_2^3 - \tfrac{1}{4}x_1^2 x_2 = \tfrac{1}{12}(x_2 - x_1)^2(2x_1 + x_2), \\ \partial_{x_1} u_w(x) &= \tfrac{1}{2}x_1^2 - \tfrac{1}{2}x_1 x_2 = -\tfrac{1}{2}(x_2 - x_1)x_1, \\ \partial_{x_2} u_w(x) &= \tfrac{1}{4}x_2^2 - \tfrac{1}{4}x_1^2 = \tfrac{1}{4}(x_2 - x_1)(x_1 + x_2). \end{aligned}$$

Thus we may compute, for $x_2 > x_1 > 0$,

$$\begin{aligned} a\partial_\nu u_w(x) - u_w(x) &= a\Big(\nu_1\big(-\tfrac{1}{2}(x_2 - x_1)x_1\big) + \nu_2\big(\tfrac{1}{4}(x_2 - x_1)(x_1 + x_2)\big)\Big) - \tfrac{1}{12}(x_2 - x_1)^2(2x_1 + x_2) \\ &= \tfrac{1}{2}(x_2 - x_1)\Big(a\big(-\nu_1 x_1 + \nu_2\big(\tfrac{1}{2}(x_1 + x_2)\big)\big) - \tfrac{1}{6}(x_2 - x_1)(2x_1 + x_2)\Big). \end{aligned} \tag{5-11}$$

Thus, to have $a\partial_\nu u_w(x) - u_w(x) \geq 0$ for $x \in \mathbb{R}^2$ satisfying $x_2 > x_1 > 0$ we should have

$$a\big(-\nu_1 x_1 + \nu_2\big(\tfrac{1}{2}(x_1 + x_2)\big)\big) \geq \tfrac{1}{6}(x_2 - x_1)(2x_1 + x_2)$$

and, rearranging this further, we get the equivalent inequality

$$\nu_2 - \nu_1 \geq \frac{1}{2x_1}(x_2 - x_1)\Big(\frac{1}{3a}(2x_1 + x_2) - \nu_2\Big).$$

Now, for $x \in B_1$ we have the bounds $x_1 < 1$ and $x_2 < 1$. Also, if $0 < x_1 < x_2$ then $x_2 - x_1 > 0$. So it is sufficient to have the inequality

$$\nu_2 - \nu_1 \geq \frac{1}{2x_1}(x_2 - x_1)\Big(\frac{1}{a} - \nu_2\Big). \tag{5-12}$$

By $0 < x_1 < x_2 < (1+\epsilon)x_1$ we have $0 < (x_2 - x_1)/x_1 < \epsilon$. Thus, if $1/a - \nu_2 > 0$ then we should have

$$\nu_2 - \nu_1 \geq \frac{\epsilon}{2}\Big(\frac{1}{a} - \nu_2\Big)$$

and if $1/a - \nu_2 \leq 0$ then we should have $\nu_2 - \nu_1 \geq 0$. Because $\nu_2 \geq -1$, for both cases it is sufficient to have

$$\nu_2 - \nu_1 \geq \frac{\epsilon}{2}\Big(\frac{1}{a} + 1\Big). \tag{5-13}$$

We compute

$$\nu_2 - \nu_1 = \sin\big(\tfrac{3\pi}{4} + \gamma\big) - \cos\big(\tfrac{3\pi}{4} + \gamma\big) = \sqrt{2}\cos\gamma. \tag{5-14}$$

From (5-13) and (5-14) it follows that it is sufficient to have

$$\cos\gamma \geq \frac{\sqrt{2}}{4}\Big(\frac{1}{a}+1\Big)\epsilon$$

and, taking $C \geq \frac{\sqrt{2}}{4}$, the second part is also proved. □

6. Uniqueness of blowup limits

Proof of Theorem 7. By Lemma 16 a blowup limit at the origin is a third-order homogeneous global solution.

By Theorem 4 we have six nontrivial homogeneous global solutions. Let us enumerate them by u^i for $i = 1, \dots, 6$.

Assume by contradiction that there exist $r_j \to 0$ and $\tilde{r}_j \to 0$ such that $u_{r_j} \to u^1$ and $u_{\tilde{r}_j} \to u^2$ in $C^1(B_1)$.

There exists $\epsilon > 0$ such that $\|u^i - u^1\|_{C(B_1)} > \epsilon$ for $i = 2, \dots, 6$.

Let us write $f(r) = \|u_r - u^1\|_{C(B_1)}$.

Because u is uniformly continuous in a neighbourhood of 0 we have that $f(r)$ is continuous for small enough $r > 0$. We have also $f(r_j) \to 0$ and $f(\tilde{r}_j) \to \|u^2 - u^1\|_{C(B_1)} > \epsilon$. Thus there exists $\hat{r}_j \to 0$ such that $f(\hat{r}_j) = \frac{1}{2}\epsilon$.

By Theorem 2, $u_{\hat{r}_j}$ is uniformly bounded in $C^{1,1}(B_1)$ for large j. Thus there exists a subsequence j_k such that $u_{\hat{r}_{j_k}}$ converges in C^1. By Lemma 16 the limit of $u_{\hat{r}_{j_k}}$ is a third-order nontrivial homogeneous global solution. This is in contradiction with $f(\hat{r}_{j_k}) = \frac{1}{2}\epsilon$ and the choice of ϵ. □

7. Convergence of the free boundary to the free boundary of the blowup limit

In the following lemma, roughly speaking, we prove two inclusions. First, if u is close to a nontrivial homogeneous global solution u_0 then, for x far from $\{u_0 = 0\}$, we have $u(x) > 0$. Second, if u is close to a solution u_0 then, for x far from $\{u_0 > 0\}$, we have $x \in \{u = 0\}^\circ$.

Lemma 32. *There exists $c > 0$ such that if u_0 is a nontrivial homogeneous global solution and u is a solution in B_1, then we have*

$$\big\{x \in B_1 \;\big|\; c d^2(x, \{u_0 = 0\})\big(d(x, \{u_0 = 0\}) + |x_1|\big) > \|u - u_0\|_{L^\infty(B_1)}\big\} \subset \{u > 0\}; \tag{7-1}$$

here $\{u_0 = 0\} = \{x \in \mathbb{R}^2 \mid u_0(x) = 0\}$ and $\{u > 0\} = \{x \in B_1 \mid u(x) > 0\}$.

If u_0 and u are solutions in B_1 and

$$\|u - u_0\|_{L^\infty(B_1)} < c,$$

then

$$\big\{x \in B_{1/2} \;\big|\; c d^2(x, \{u_0 > 0\})\big(d(x, \{u_0 > 0\}) + |x_1|\big) > \|u - u_0\|_{L^\infty(B_1)}\big\} \subset \{u = 0\}^\circ; \tag{7-2}$$

here $\{u_0 = 0\} = \{x \in B_1 \mid u_0(x) = 0\}$ and $\{u = 0\} = \{x \in B_1 \mid u(x) = 0\}$.

Proof. Assume u_0 is a nontrivial homogeneous global solution and u is a solution in B_1. Using Lemma 30 for $x \in B_1$ we compute

$$\begin{aligned} u(x) = u_0(x) + u(x) - u_0(x) &\geq u_0(x) - \|u-u_0\|_{L^\infty(B_1)} \\ &\geq c_1 d^2(x,\{u_0=0\})\bigl(d(x,\{u_0=0\})+|x_1|\bigr) - \|u-u_0\|_{L^\infty(B_1)}; \end{aligned}$$

here c_1 is the constant in Lemma 30. So, if

$$\|u-u_0\|_{L^\infty(B_1)} < \tfrac{1}{2}c_1 d^2(x,\{u_0=0\})\bigl(d(x,\{u_0=0\})+|x_1|\bigr)$$

then

$$u(x) > \tfrac{1}{2}c_1 d^2(x,\{u_0=0\})\bigl(d(x,\{u_0=0\})+|x_1|\bigr)$$

and this proves (7-1) with $0 < c \leq \frac{1}{2}c_1$.

Assume u_0 and u are solutions in B_1. By Theorem 15 there exists $c_2 > 0$ such that, if $y \in B_1$, $u(y) > 0$ and $B_r(y) \Subset B_1$, then we have

$$\sup_{\{u>0\}\cap\partial B_r(y)} u \geq u(y) + c_2 r^2(r+|y_1|).$$

Thus, if $y \in B_1$, $u(y) > 0$, $B_r(y) \Subset \{u_0=0\}\cap B_1$ and $c_2 r^2(r+|y_1|) > \|u-u_0\|_{L^\infty(B_1)}$, then we have

$$\begin{aligned} 0 = \sup_{\{u>0\}\cap\partial B_r(y)} u_0 &= \sup_{\{u>0\}\cap\partial B_r(y)} (u-(u-u_0)) \\ &\geq \sup_{\{u>0\}\cap\partial B_r(y)} u - \|u-u_0\|_{L^\infty(B_1)} \\ &\geq u(y) + c_2 r^2(r+|y_1|) - \|u-u_0\|_{L^\infty(B_1)} \\ &\geq c_2 r^2(r+|y_1|) - \|u-u_0\|_{L^\infty(B_1)}; \end{aligned}$$

a contradiction. Thus, if $y \in B_1$, $B_r(y) \Subset \{u_0=0\}\cap B_1$ and $c_2 r^2(r+|y_1|) > \|u-u_0\|_{L^\infty(B_1)}$, then $u(y) = 0$.

For $y \in (\{u_0=0\}\cap B_1)^\circ$, setting $r = \frac{1}{2}d\bigl(y,(\{u_0=0\}\cap B_1)^c\bigr)$ it follows that if

$$\tfrac{1}{4}c_2 d^2\bigl(y,(\{u_0=0\}\cap B_1)^c\bigr)\bigl(\tfrac{1}{2}d\bigl(y,(\{u_0=0\}\cap B_1)^c\bigr)+|y_1|\bigr) > \|u-u_0\|_{L^\infty(B_1)}$$

then $u(y) = 0$. This proves that

$$\bigl\{x \in B_1 \bigm| \tfrac{1}{8}c_2 d^2\bigl(x,(\{u_0=0\}\cap B_1)^c\bigr)\bigl(d\bigl(x,(\{u_0=0\}\cap B_1)^c\bigr)+|x_1|\bigr) > \|u-u_0\|_{L^\infty(B_1)}\bigr\} \subset \{u=0\}.$$

By the continuity of $d\bigl(x,(\{u_0=0\}\cap B_1)^c\bigr)$ as a function of x it follows that

$$\begin{aligned} \bigl\{x \in B_1 \bigm| \tfrac{1}{8}c_2 d^2\bigl(x,(\{u_0=0\}\cap B_1)^c\bigr)\bigl(d\bigl(x,(\{u_0=0\}\cap B_1)^c\bigr)+|x_1|\bigr) > \|u-u_0\|_{L^\infty(B_1)}\bigr\} \\ \subset \{u=0\}^\circ. \qquad (7\text{-}3) \end{aligned}$$

Let $x \in B_{1/2}$; then we compute

$$d\bigl(x,(\{u_0=0\}\cap B_1)^c\bigr) = d(x,\{u_0>0\}\cup B_1^c) = \min\bigl(d(x,\{u_0>0\}),d(x,B_1^c)\bigr) \geq \min\bigl(d(x,\{u_0>0\}),\tfrac{1}{2}\bigr),$$

so we have

$$\begin{aligned} d^2\big(x, (\{u_0 = 0\} \cap B_1)^c\big)\big(d\big(x, (\{u_0 = 0\} \cap B_1)^c\big) + |x_1|\big) \\ = \min\Big(d^2(x, \{u_0 > 0\})\big(d(x, \{u_0 > 0\}) + |x_1|\big), \big(\tfrac{1}{2}\big)^2\big(\tfrac{1}{2} + |x_1|\big)\Big) \\ \ge \min\big(d^2(x, \{u_0 > 0\})\big(d(x, \{u_0 > 0\}) + |x_1|\big), \tfrac{1}{8}\big). \end{aligned} \tag{7-4}$$

So, by (7-3) and (7-4), if

$$\|u - u_0\|_{L^\infty(B_1)} < \tfrac{1}{64} c_2$$

then

$$\big\{x \in B_{1/2} \mid \tfrac{1}{8} c_2 d^2(x, \{u_0 > 0\})\big(d(x, \{u_0 > 0\}) + |x_1|\big) > \|u - u_0\|_{L^\infty(B_1)}\big\} \subset \{u = 0\}^\circ \tag{7-5}$$

and, by choosing $0 < c \le \frac{1}{64} c_2$, this finishes the proof of the lemma. □

By the inclusions proved in the previous lemma, in the following lemma we show that for u a solution and u_0 a nontrivial homogeneous global solution, if u is close enough to u_0 then the free boundary of u is in a quantitatively specified neighbourhood of the free boundary of u_0.

Lemma 33. *There exists $c > 0$ such that, if u is a solution in B_1 and u_0 is a nontrivial homogeneous global solution, then if*

$$\|u - u_0\|_{L^\infty(B_1)} < c \tag{7-6}$$

we have

$$\Gamma \cap B_{1/2} \subset \big\{c d^2(x, \Gamma_{u_0})(d(x, \Gamma_{u_0}) + |x_1|) \le \|u - u_0\|_{L^\infty(B_1)}\big\}.$$

Proof. If $u = u_0$ in B_1 then the claim is obvious, so we assume that $u_0 \neq u$ in B_1.

Assume there exists $x \in \Gamma \cap B_{1/2}$ such that

$$c d^2(x, \Gamma_{u_0})(d(x, \Gamma_{u_0}) + |x_1|) > \|u - u_0\|_{L^\infty(B_1)};$$

here $c > 0$ is as in Lemma 32.

Then, because

$$d(x, \Gamma_{u_0}) = \max\big(d(x, \{u_0 = 0\}), d(x, \{u_0 > 0\})\big),$$

we should have either

$$c d^2(x, \{u_0 = 0\})\big(d(x, \{u_0 = 0\}) + |x_1|\big) > \|u - u_0\|_{L^\infty(B_1)} \tag{7-7}$$

or

$$c d^2(x, \{u_0 > 0\})\big(d(x, \{u_0 > 0\}) + |x_1|\big) > \|u - u_0\|_{L^\infty(B_1)}. \tag{7-8}$$

In the case when (7-7) holds then, by (7-1), we obtain that $u(x) > 0$, which is in contradiction with $x \in \Gamma$.

In the case when (7-8) holds then, because also (7-6) holds by (7-2), we obtain that $x \in \{u = 0\}^\circ$, which is in contradiction with $x \in \Gamma$ and this finishes the proof of the lemma. □

Lemma 34. *There exists $c > 0$ such that if u_0 is a nontrivial homogeneous global solution, u is a solution in D, $0 \in D$ and $0 \in \Gamma$ then, for $x \in \Gamma$ such that $B_{4|x|} \subset D$ and*

$$\|u_{4|x|} - u_0\|_{L^\infty(B_1)} < c,$$

we have

$$cd^2(x, \Gamma_{u_0})(d(x, \Gamma_{u_0}) + |x_1|) \leq |x|^3 \|u_{4|x|} - u_0\|_{L^\infty(B_1)}.$$

Proof. Let c be as in Lemma 32.

Let $r > 0$ and assume

$$\|u_r - u_0\|_{L^\infty(B_1)} < c;$$

then, by Lemma 33, we have

$$\Gamma_{u_r} \cap B_{1/2} \subset \{cd^2(y, \Gamma_{u_0})(d(y, \Gamma_{u_0}) + |y_1|) \leq \|u_r - u_0\|_{L^\infty(B_1)}\}.$$

Then, because Γ_{u_0} is a cone and $\Gamma_u \cap B_{r/2} = r(\Gamma_{u_r} \cap B_{1/2})$, we obtain

$$\begin{aligned}
\Gamma_u \cap B_{r/2} &\subset \{ry \in B_{r/2} \mid cd^2(y, \Gamma_{u_0})\big(d(y, \Gamma_{u_0}) + |y_1|\big) \leq \|u_r - u_0\|_{L^\infty(B_1)}\} \\
&= \left\{x \in B_{r/2} \,\middle|\, cd^2\Big(\frac{x}{r}, \Gamma_{u_0}\Big)\Big(d\Big(\frac{x}{r}, \Gamma_{u_0}\Big) + \Big|\frac{x_1}{r}\Big|\Big) \leq \|u_r - u_0\|_{L^\infty(B_1)}\right\} \\
&= \{x \in B_{r/2} \mid cd^2(x, \Gamma_{u_0})(d(x, \Gamma_{u_0}) + |x_1|) \leq r^3 \|u_r - u_0\|_{L^\infty(B_1)}\}.
\end{aligned}$$

For those $x \in \Gamma_u$ such that $B_{4|x|} \subset D$, we may consider $r = 4|x|$.

So, if

$$\|u_{4|x|} - u_0\|_{L^\infty(B_1)} < c$$

then, because $x \in \Gamma_u \cap B_{2|x|}$, we have

$$cd^2(x, \Gamma_{u_0})\big(d(x, \Gamma_{u_0}) + |x_1|\big) \leq 4^3 |x|^3 \|u_{4|x|} - u_0\|_{L^\infty(B_1)}. \qquad \square$$

Proof of Theorem 8. Let us consider the case $W(+0, u) = W(1, u_w)$ with the blowup limit u_w. Then for $x \in \{x_1 > 0,\ x_2 > -x_1\}$ we have $d(x, \Gamma_{u_w}) = \frac{\sqrt{2}}{2}|x_2 - x_1|$ and, for $x \in \{x_1 > 0,\ x_2 \leq -x_1\}$, we have $d(x, \Gamma_{u_w}) = |x| \geq \frac{\sqrt{2}}{2}|x_2 - x_1|$. Thus we compute, for $x_1 > 0$,

$$d(x, \Gamma_{u_w}) + |x_1| \geq \tfrac{1}{2}\sqrt{2}|x_2 - x_1| + |x_1| \geq c_1|x|. \tag{7-9}$$

By symmetry we obtain the same inequality for $x_1 < 0$.

Now, by Lemma 34 we obtain the inequality (2-7). For the remaining cases, when $W(+0, u)$ is in $\{W(1, u_w), 2W(1, u_w)\}$, we can compute similarly.

In the cases when $W(+0, u) \in \{W(1, u_{hs}), 2W(1, u_{hs})\}$ we have $\Gamma_{u_0} = \{x_1 = 0\}$ and $d(x, \Gamma_{u_0}) = |x_1|$, so (2-8) follows immediately from Lemma 34. $\square$

Corollary 35. *Let u be a solution in D; then the points of*

$$\Gamma \cap \{x_1 = 0\} \cap \{W(+0, x, u) \in \{W(1, u_w), 2W(1, u_w)\}\}$$

are isolated points of $\Gamma \cap \{x_1 = 0\}$ (in the topology of $\{x_1 = 0\}$).

Proof. Assume $W(+0,u) \in \{W(1,u_w), 2W(1,u_w)\}$; then, by (2-7), the free boundary should converge to the free boundary of the blowup limit tangentially. But this is not the case if the origin is not an isolated point of $\Gamma \cap \{x_1 = 0\}$. □

8. Convergence of the normal of the free boundary to the normal of the free boundary of the blowup limit at regular points

In the following lemma we prove a nondegeneracy type result for $u - a\partial_\nu u$ far from the degeneracy line $\{x_1 = 0\}$.

Lemma 36. *If u is a solution in D, $y \in \Omega$, $B_r(y) \Subset D \cap \{x_1 \geq \frac{1}{16}\}$ and $u(y) - \frac{1}{32}\partial_\nu u(y) > 0$, then we have*

$$\tfrac{1}{128} r^2 \leq \sup_{\Omega \cap \partial B_r(y)} (u - a\partial_\nu u).$$

Proof. Let y and r be as in the statement of the theorem.

We define, for $a > 0$ and $c > 0$,

$$h(x) = u(x) - a\partial_\nu u(x) - (u(y) - a\partial_\nu u(y)) - c|x-y|^2.$$

We compute

$$\triangle h(x) = |x_1| - a\nu_1 x_1/|x_1| - 4c \geq \tfrac{1}{16} - a - 4c \quad \text{in } \Omega \cap \{x_1 \geq \tfrac{1}{16}\},$$

so if we choose $a = \frac{1}{32}$ and $c = \frac{1}{128}$ then we have

$$\triangle h \geq 0 \quad \text{in } \Omega \cap \{x_1 \geq \tfrac{1}{16}\}. \tag{8-1}$$

Also we have

$$h(y) = 0. \tag{8-2}$$

For $x \in \Gamma$ we have $u(x) - \frac{1}{32}\partial_\nu u(x) = 0$, thus if $u(y) - \frac{1}{32}\partial_\nu u(y) > 0$ then we have

$$h(x) = -\big(u(y) - \tfrac{1}{32}\partial_\nu u(y)\big) - \tfrac{1}{128}|x-y|^2 < 0 \quad \text{on } \Gamma. \tag{8-3}$$

Because $B_r(y) \subset \{x_1 \geq \frac{1}{16}\}$, by (8-1) we have that h is subharmonic in the domain $\Omega \cap B_r(y)$. Applying the maximum principle for the domain $\Omega \cap B_r(y)$ and the subharmonic function h, we have

$$h(y) \leq \sup_{\partial(\Omega \cap B_r(y))} h. \tag{8-4}$$

By (8-2) and (8-4), we obtain

$$0 \leq \sup_{\partial(\Omega \cap B_r(y))} h. \tag{8-5}$$

Because

$$\partial(\Omega \cap B_r(y)) = (\partial\Omega \cap B_r(y)) \cup (\Omega \cap \partial B_r(y)),$$

by (8-3) and (8-5) we obtain

$$0 \leq \sup_{\Omega \cap \partial B_r(y)} h. \tag{8-6}$$

By the definition of h, from (8-6) we get the inequality

$$u(y) - \tfrac{1}{32}\partial_\nu u(y) + \tfrac{1}{128}r^2 \le \sup_{\Omega\cap\partial B_r(y)} \bigl(u - \tfrac{1}{32}\partial_\nu u\bigr) \tag{8-7}$$

and this proves the lemma. □

Let ν_w be the normal to $\Gamma_{u_w} \cap \{x_1 > 0\}$ pointing into $\{u_w > 0\}$, i.e.,

$$\nu_w = \tfrac{1}{\sqrt{2}}(-1, 1).$$

In the following lemma we prove a crucial directional monotonicity result, which will be used in the proof of the convergence of normals.

Lemma 37. *There exists $c > 0$ such that, if u is a solution in B_1, $x_u \in \Gamma_u \cap \partial B_{1/4} \cap \{x_1 > 0\}$, $\nu \in \partial B_1$ and $r > 0$ are such that*

$$\|u - u_w\|^{1/2}_{C^1(B_1)} + r \le c\nu\cdot\nu_w,$$

then

$$\tfrac{1}{32}\partial_\nu u - u \ge 0 \quad \textit{in } \Omega \cap B_r(x_u).$$

Proof. We have

$$\{x_{u_w}\} = \Gamma_{u_w} \cap \partial B_{1/4} \cap \{x_1 > 0\}, \quad \text{where} \quad x_{u_w} = \tfrac{\sqrt{2}}{8}(1, 1).$$

Step 1. In this step we show that there exists $C_1 > 0$ such that

$$|x_u - x_{u_w}| \le C_1\|u - u_w\|^{1/2}_{L^\infty(B_1)}. \tag{8-8}$$

By Lemma 33 there exists $c > 0$ such that if $\|u - u_w\|_{L^\infty(B_1)} < c$ then

$$\Gamma_u \cap B_{1/2} \subset \bigl\{c(d(x, \Gamma_{u_w}))^2(d(x, \Gamma_{u_w}) + |x_1|) \le \|u - u_w\|_{L^\infty(B_1)}\bigr\}. \tag{8-9}$$

We have $x_u \in \Gamma_u \cap \partial B_{1/4} \cap \{x_1 > 0\}$; thus, by (8-9),

$$c(d(x_u, \Gamma_{u_w}))^2(d(x_u, \Gamma_{u_w}) + |x_{u,1}|) \le \|u - u_w\|_{L^\infty(B_1)}. \tag{8-10}$$

As in (7-9) there exists $c_1 > 0$ such that

$$d(x_u, \Gamma_{u_w}) + |x_{u,1}| \ge c_1|x_u| = \tfrac{1}{4}c_1. \tag{8-11}$$

Also, because $x_u \in \partial B_{1/4} \cap \{x_1 > 0\}$ there exists $C_2 > 0$ such that

$$|x_u - x_{u_w}| \le C_2 d(x_u, \Gamma_{u_w}). \tag{8-12}$$

Now, by (8-10), (8-11) and (8-12), it follows that there exists $C_3 > 0$ such that

$$|x_u - x_{u_w}| \le C_3\|u - u_w\|^{1/2}_{L^\infty(B_1)}. \tag{8-13}$$

Step 2. In this step we show that there exists $\delta > 0$ such that if

$$\|u - u_w\|_{L^\infty(B_1)} < \delta \quad \text{and} \quad 0 < r < \tfrac{1}{48}$$

then, for $x \in \Omega \cap B_{1/48}(x_u)$, if $u(x) - \frac{1}{32}\partial_\nu u(x) > 0$ we have

$$\tfrac{1}{128} r^2 \le \sup_{\Omega\cap\partial B_r(x)} \left(u - \tfrac{1}{32}\partial_\nu u\right).$$

By Step 1, if

$$C_3 \|u - u_w\|^{1/2}_{L^\infty(B_1)} < \tfrac{1}{48}$$

then $|x_u - x_{u_w}| < \frac{1}{48}$. Thus $x_{u,1} > x_{u_w,1} - \frac{1}{48}$ and

$$B_{1/48}(x_u) \subset \left\{x_1 > x_{u_w,1} - \tfrac{1}{48} - \tfrac{1}{48}\right\} = \left\{x_1 > x_{u_w,1} - \tfrac{1}{24}\right\}$$

and, for $x \in B_{1/48}(x_u)$, we have

$$B_{1/48}(x) \subset \left\{x_1 > x_{u_w,1} - \tfrac{1}{24} - \tfrac{1}{48}\right\} = \left\{x_1 > x_{u_w,1} - \tfrac{1}{16}\right\} = \left\{x_1 > \tfrac{\sqrt{2}}{8} - \tfrac{1}{16}\right\} \subset \left\{x_1 > \tfrac{1}{8} - \tfrac{1}{16}\right\} = \left\{x_1 > \tfrac{1}{16}\right\}.$$

Now, by Lemma 36, if

$$0 < r < \tfrac{1}{48},$$

$x \in \Omega \cap B_{1/48}(x_u)$ and $u(x) - \frac{1}{32}\partial_\nu u(x) > 0$, then we have

$$\tfrac{1}{128} r^2 \le \sup_{\Omega\cap\partial B_r(x)} \left(u - \tfrac{1}{32}\partial_\nu u\right).$$

Step 3. In this step we show that there exists $C_4 > 0$ such that $\frac{1}{32}\partial_\nu u_w - u_w \ge 0$ in $B_\eta(x_{u_w})$ if $0 < \eta < \frac{1}{16}$, $\nu \in \partial B_1$ and $C_4 \eta \le \nu \cdot \nu_w$.

Assume $x \in B_\eta(x_{u_w})$ with $0 < \eta < \frac{1}{16}$. Then

$$x_1 > x_{u_w,1} - \eta > x_{u_w,1} - \tfrac{1}{16} = \tfrac{\sqrt{2}}{8} - \tfrac{1}{16} > \tfrac{1}{8} - \tfrac{1}{16} = \tfrac{1}{16}$$

and

$$\begin{aligned}\frac{x_2}{x_1} = 1 + \frac{x_2 - x_1}{x_1} \le 1 + \frac{|x_2 - x_1|}{x_1} &< 1 + 16|x_2 - x_1| = 1 + 16\sqrt{2}\, d(x, \{x_2 = x_1\}) \\ &\le 1 + 16\sqrt{2}|x - x_{u_w}| \le 1 + 16\sqrt{2}\eta;\end{aligned}$$

hence by Lemma 31 we have $\frac{1}{32}\partial_\nu u_w(x) - u_w(x) \ge 0$ if $\nu \in \partial B_1$ and

$$C\left(\tfrac{1}{1/32} + 1\right)(16\sqrt{2}\eta) \le \nu \cdot \nu_w$$

with $C > 0$ as in Lemma 31.

Step 4. In this step we show that there exists $\delta_1 > 0$ and $C_5 > 0$ such that, if

$$\|u - u_w\|_{L^\infty(B_1)} < \delta_1, \quad 0 < r < \tfrac{1}{48}, \quad 0 < r_1 < \tfrac{1}{48}, \tag{8-14}$$

$$\nu \in \partial B_1, \quad C_4\left(r + r_1 + C_3 \|u - u_w\|^{1/2}_{L^\infty(B_1)}\right) \le \nu \cdot \nu_w, \tag{8-15}$$

$$C_5 \|u - u_w\|^{1/2}_{C^1(B_1)} < r, \tag{8-16}$$

then

$$u - \tfrac{1}{32}\partial_\nu u \le 0 \quad \text{in } \Omega \cap B_{r_1}(x_u). \tag{8-17}$$

By Step 1 there exists $0 < \delta_1 < \delta$ such that if

$$\|u - u_w\|_{L^\infty(B_1)} < \delta_1 \tag{8-18}$$

then

$$|x_u - x_{u_w}| < \tfrac{1}{48}. \tag{8-19}$$

Let

$$0 < r < \tfrac{1}{48} \quad \text{and} \quad 0 < r_1 < \tfrac{1}{48}. \tag{8-20}$$

Assume now that both (8-18) and (8-20) hold.

We define

$$\eta = r + r_1 + |x_u - x_{u_w}|;$$

then by (8-19) and (8-20) we have

$$0 < \eta < \tfrac{1}{16}. \tag{8-21}$$

By Step 2 for $x \in \Omega \cap B_{r_1}(x_u)$, if $u(x) - \frac{1}{32}\partial_\nu u(x) > 0$ then

$$\tfrac{1}{128} r^2 \le \sup_{\Omega \cap \partial B_r(x)} \left(u - \tfrac{1}{32}\partial_\nu u\right). \tag{8-22}$$

By (8-21) and Step 3 we have $\frac{1}{32}\partial_\nu u_w - u_w \ge 0$ in $B_\eta(x_{u_w})$ if

$$\nu \in \partial B_1 \quad \text{and} \quad C_4 \eta \le \nu \cdot \nu_w. \tag{8-23}$$

Assume now that (8-23) holds.

We have

$$B_r(x) \subset B_{r+|x-x_u|}(x_u) \subset B_{r+|x-x_u|+|x_u-x_{u_w}|}(x_{u_w}) \subset B_{r+r_1+|x_u-x_{u_w}|}(x_{u_w}) \subset B_\eta(x_{u_w}).$$

We compute

$$\begin{aligned} \sup_{\Omega \cap \partial B_r(x)} \left(u - \tfrac{1}{32}\partial_\nu u\right) &\le \sup_{\Omega \cap \partial B_r(x)} \left(u_w - \tfrac{1}{32}\partial_\nu u_w\right) + \sup_{\Omega \cap \partial B_r(x)} \left(u - \tfrac{1}{32}\partial_\nu u - \left(u_w - \tfrac{1}{32}\partial_\nu u_w\right)\right) \\ &\le C_6 \|u - u_w\|_{C^1(B_1)}. \end{aligned}$$

Therefore, by (8-22), if

$$\tfrac{1}{128} r^2 > C_6 \|u - u_w\|_{C^1(B_1)}$$

then

$$u - \tfrac{1}{32}\partial_\nu u \le 0 \quad \text{in } \Omega \cap B_{r_1}(x_u).$$

Step 5. In this step we finish the proof of the lemma.

Choosing

$$r = 2C_5 \|u - u_w\|_{C^1(B_1)}^{1/2},$$

(8-16) holds. Noticing that $\nu \cdot \nu_w \le 1$ we obtain that, by choosing $c > 0$ small enough, if

$$\nu \in \partial B_1, \quad \|u - u_w\|_{C^1(B_1)}^{1/2} + r_1 \le c\nu \cdot \nu_w$$

holds then (8-14) and (8-15) hold and thus, by Step 4, (8-17) holds and this proves the lemma. □

For $0 \le \delta < 1$ let us define the open cone

$$C_\delta = \{x \in \mathbb{R}^2 \mid x \cdot \nu_w > \delta |x|\}.$$

Corollary 38. *If u is a solution in B_1, $x \in \Gamma \cap \partial B_{1/4} \cap \{x_1 > 0\}$, $0 < \delta < 1$ and $r > 0$ are such that*

$$\|u - u_w\|^{1/2}_{C^1(B_1)} + r \le c\delta$$

with $c > 0$ as in Lemma 37, then

$$B_r(x) \cap (x + C_\delta) \subset \{u > 0\} \quad \textit{and} \quad B_r(x) \cap (x - C_\delta) \subset \{u = 0\}. \tag{8-24}$$

Proof. By Lemma 37 and the definition of C_δ we have that, for all $\nu \in C_\delta$,

$$\partial_\nu u \ge 0 \quad \text{in } B_r(x_u). \tag{8-25}$$

From (8-30), because $u \ge 0$,

$$z \in B_r(x) \quad \text{and} \quad u(z) = 0 \quad \Longrightarrow \quad B_r(x) \cap (z - C_\delta) \subset \{u = 0\}. \tag{8-26}$$

In particular, because $u(x) = 0$ we have

$$B_r(x) \cap (x - C_\delta) \subset \{u = 0\}.$$

Now assume there exists $y \in B_r(x) \cap (x + C_\delta)$ such that $u(y) = 0$. By (8-26) we have that $u = 0$ in $B_r(x) \cap (y - C_\delta)$. From $y \in x + C_\delta$ it follows that $x \in y - C_\delta$, thus x is in the interior of $B_r(x) \cap (y - C_\delta)$, where we have shown that $u = 0$ and this contradicts $x \in \Gamma$. □

It is easy to see that, for the cone C'_δ conjugate to the cone C_δ, we have

$$C'_\delta = \{x \in \mathbb{R}^2 \mid x \cdot y \ge 0 \text{ for all } y \in C_\delta\} = \bar{C}_{\sqrt{1-\delta^2}}. \tag{8-27}$$

Theorem 39. *There exists $C_1 > 0$ such that, if u is a solution in D, $0 \in D$ and $0 \in \Gamma$ is a regular point with blowup limit u_w, then there exists $\epsilon > 0$ such that all points of $\Gamma \cap \{x_1 > 0\} \cap B_\epsilon$ are usual (for $x_1 > 0$ the force term is nondegenerate) regular free boundary points and*

$$|n(x) - \nu_w| \le C_1 \|u_{4|x|} - u_w\|^{1/2}_{C^1(B_1)} \tag{8-28}$$

for $x \in \Gamma \cap \{x_1 > 0\} \cap B_\epsilon$, where $n(x)$ is the normal to Γ at x, pointing into Ω.

Proof. If there exists $r > 0$ such that $u = u_w$ in B_r then the claim of the theorem holds trivially. So we might assume that for all $r > 0$ we have $u \ne u_w$ in B_r.

Let $x \in \Gamma \cap \{x_1 > 0\} \cap B_1$. By the uniqueness of the blowup limit and Theorem 2 we have that $u_{4|x|} \to u_w$ in $C^1(B_1)$ as $x \to 0$. Thus there exists $\epsilon > 0$ such that for $|x| < \epsilon$ we have

$$\|u_{4|x|} - u_w\|_{C^1(B_1)} < \left(\frac{c}{2}\right)^2 \tag{8-29}$$

with $c > 0$ as in Lemma 37.

Let $y = \frac{1}{4}x/|x|$. Then $y \in \Gamma_{u_{4|x|}} \cap \partial B_{1/4} \cap \{x_1 > 0\}$. By (8-29), if we choose

$$\delta = \frac{2}{c}\|u_{4|x|} - u_w\|^{1/2}_{C^1(B_1)} \tag{8-30}$$

then $0 < \delta < 1$.

Also let us set

$$r = \|u_{4|x|} - u_w\|^{1/2}_{C^1(B_1)}. \tag{8-31}$$

Then, by (8-30) and (8-31) we have

$$\|u_{4|x|} - u_w\|^{1/2}_{C^1(B_1)} + r = c\delta \tag{8-32}$$

and consequently, by Corollary 38, we have

$$B_r(y) \cap (y + C_\delta) \subset \{u_{4|x|} > 0\} \quad \text{and} \quad B_r(y) \cap (y - C_\delta) \subset \{u_{4|x|} = 0\}. \tag{8-33}$$

From (8-33) it follows that

$$B_{4|x|r}(x) \cap (x + C_\delta) \subset \{u > 0\} \quad \text{and} \quad B_{4|x|r}(x) \cap (x - C_\delta) \subset \{u = 0\}. \tag{8-34}$$

Now, if x is a singular free boundary point then the blowup limit is a nonzero homogeneous quadratic polynomial. But, by (8-34), this polynomial should be equal to 0 in $-C_\delta$, which brings us to contradiction. Thus all points of $\Gamma \cap \{x_1 > 0\} \cap B_\epsilon$ are regular points.

Now assume $|x| < \epsilon$; then, because x is a regular point, Γ has a normal at this point. Let $n(x)$ be the normal to Γ pointing into Ω. From (8-34) it follows that $n(x) \in C'_\delta$. Now, by (8-27), we have

$$n(x) \in \overline{C}_{\sqrt{1-\delta^2}},$$

so

$$n(x) \cdot \nu_w \geq \sqrt{1 - \delta^2}.$$

We compute

$$|n(x) - \nu_w|^2 = 2 - 2n(x) \cdot \nu_w \leq 2 - 2\sqrt{1-\delta^2} = \frac{2\delta^2}{1 + \sqrt{1-\delta^2}} \leq 2\delta^2 \tag{8-35}$$

and (8-28) follows from (8-30) and (8-35). □

9. Free boundary as a graph near regular points

The following two lemmas will be used in Lemma 42.

Lemma 40. *If u is a solution in D, $0 \in D$ and $0 \in \Gamma$ is a regular free boundary point with blowup limit u_w, then there exists an $\epsilon > 0$ such that $u(0,t) > 0$ for $0 < t < \epsilon$ and $(0,t) \in \{u = 0\}^\circ$ for $-\epsilon < t < 0$.*

Proof. Let $x = (0,t) \in B_\epsilon$, $0 < t < \epsilon$, then we compute

$$\begin{aligned} d^2\big(\tfrac{1}{2}x/|x|, \{u_w = 0\}\big)\big(d\big(\tfrac{1}{2}x/|x|, \{u_w = 0\}\big) + \big|\tfrac{1}{2}x_1/|x|\big|\big) &= d^3\big(\tfrac{1}{2}x/|x|, \{u_w = 0\}\big) \\ &= d^3\big(\tfrac{1}{2}e_2, \{u_w = 0\}\big) = \big(\tfrac{\sqrt{2}}{4}\big)^3. \end{aligned} \tag{9-1}$$

For small enough ϵ, if $|x| < \epsilon$ then

$$\|u_{2|x|} - u_w\|_{L^\infty(B_1)} < c\big(\tfrac{\sqrt{2}}{4}\big)^3 \tag{9-2}$$

with c as in Lemma 32. Thus, by (9-1), (9-2) and (7-1), we have $u_{2|x|}\big(\frac{1}{2}x/|x|\big) > 0$, so $u(x) > 0$.

Let $x = (0, t) \in B_\epsilon$, $-\epsilon < t < 0$, then we compute

$$\begin{aligned} d^2\big(\tfrac{1}{4}x/|x|, \{u_w > 0\}\big)\big(d\big(\tfrac{1}{4}x/|x|, \{u_w > 0\}\big) + \big|\tfrac{1}{4}x_1/|x|\big|\big) &= d^3\big(\tfrac{1}{4}x/|x|, \{u_w > 0\}\big) \\ &= d^3\big(-\tfrac{1}{4}e_2, \{u_w > 0\}\big) = \tfrac{1}{4^3}. \end{aligned} \tag{9-3}$$

For small enough ϵ, if $|x| < \epsilon$ then

$$\|u_{4|x|} - u_w\|_{L^\infty(B_1)} < \tfrac{1}{4^3}c. \tag{9-4}$$

Thus, by (9-3), (9-4) and (7-2), we have $x/(4|x|) \in \{u_{4|x|} = 0\}^\circ$, so $x \in \{u = 0\}^\circ$. □

Lemma 41. *If u is a solution in D, $0 \in D$ and $0 \in \Gamma$ is a regular free boundary point with blowup limit u_w, then there exists an $\epsilon > 0$ such that for every $0 < x_1 < \frac{1}{4}\epsilon$ there exists a unique x_2 such that $x = (x_1, x_2) \in \Gamma \cap B_\epsilon$ and, for $(x_1, t) \in B_\epsilon$, we have $u(x_1, t) > 0$ if $t > x_2$ and $(x_1, t) \in \{u = 0\}^\circ$ if $t < x_2$.*

Proof. First we show that there exists $\epsilon > 0$ such that for all $0 < x_1 < \frac{1}{4}\epsilon$ there exists x_2 such that $(x_1, x_2) \in \Gamma \cap B_\epsilon$.

Let $\epsilon > 0$, to be chosen later. Let $0 < x_1 < \frac{1}{4}\epsilon$; then we compute

$$\big|\big(x_1, \tfrac{3}{4}\epsilon\big)\big|^2 < \big(\tfrac{1}{4}\epsilon\big)^2 + \big(\tfrac{3}{4}\epsilon\big)^2 = \tfrac{10}{16}\epsilon^2 < \epsilon^2;$$

thus $\big(x_1, \frac{3}{4}\epsilon\big) \in B_\epsilon$. We compute

$$d\big(\big(x_1/\epsilon, \tfrac{3}{4}\big), \{u_w = 0\}\big) = \tfrac{\sqrt{2}}{2}\big(\tfrac{3}{4} - x_1/\epsilon\big) \geq \tfrac{\sqrt{2}}{2}\big(\tfrac{3}{4} - \tfrac{1}{4}\big) = \tfrac{\sqrt{2}}{4}$$

and

$$d^2\big(\big(x_1/\epsilon, \tfrac{3}{4}\big), \{u_w = 0\}\big)\big(d\big(\big(x_1/\epsilon, \tfrac{3}{4}\big), \{u_w = 0\}\big) + |x_1/\epsilon|\big) \geq d^3\big(\big(x_1/\epsilon, \tfrac{3}{4}\big), \{u_w = 0\}\big) \geq \big(\tfrac{\sqrt{2}}{4}\big)^3.$$

Thus, if ϵ is small enough that

$$\|u_\epsilon - u_w\|_{L^\infty(B_1)} < c\big(\tfrac{\sqrt{2}}{4}\big)^3$$

with c as in Lemma 32, then by (7-1) we obtain that

$$u_\epsilon\big(x_1/\epsilon, \tfrac{3}{4}\big) > 0$$

and therefore

$$u\big(x_1, \tfrac{3}{4}\epsilon\big) > 0. \tag{9-5}$$

Let $0 < x_1 < \frac{1}{4}\epsilon$; then we compute

$$\big|\big(x_1, -\tfrac{1}{4}\epsilon\big)\big|^2 < \big(\tfrac{1}{4}\epsilon\big)^2 + \big(\tfrac{1}{4}\epsilon\big)^2 = \big(\tfrac{\sqrt{2}}{4}\epsilon\big)^2 < \big(\tfrac{1}{2}\epsilon\big)^2,$$

thus $\left(x_1, -\frac{1}{4}\epsilon\right) \in B_{\epsilon/2} \subset B_\epsilon$.

We compute

$$d\left(\left(x_1/\epsilon, -\tfrac{1}{4}\right), \{u_w > 0\}\right) \geq \tfrac{1}{4}$$

and

$$d^2\left(\left(x_1/\epsilon, -\tfrac{1}{4}\right), \{u_w > 0\}\right)\left(d\left(\left(x_1/\epsilon, -\tfrac{1}{4}\right), \{u_w > 0\}\right) + |x_1/\epsilon|\right) \geq \tfrac{1}{4^3}.$$

Thus, if ϵ is small enough that

$$\|u_\epsilon - u_w\|_{L^\infty(B_1)} < \tfrac{1}{4^3}c,$$

then by (7-2) we obtain that

$$\left(x_1/\epsilon, -\tfrac{1}{4}\right) \in \{u_\epsilon = 0\}^\circ$$

and therefore

$$\left(x_1, -\tfrac{1}{4}\epsilon\right) \in \{u = 0\}^\circ. \tag{9-6}$$

From (9-5), (9-6) and the continuity of u it follows that there exists $-\frac{1}{4}\epsilon < x_2 < \frac{3}{4}\epsilon$ such that $(x_1, x_2) \in \Gamma$. This finishes the proof of the existence of x_2.

By Corollary 38 there exists $c > 0$ such that, if $y \in \Gamma_u \cap \{y_1 > 0\}$ and

$$\|u_{4|y|} - u_w\|_{C^1(B_1)} \leq \left(\tfrac{1}{4}c\right)^2,$$

then

$$B_{c|y|}(y) \cap (y + C_{1/2}) \subset \{u > 0\} \quad \text{and} \quad B_{c|y|}(y) \cap (y - C_{1/2}) \subset \{u = 0\}. \tag{9-7}$$

Now let ϵ be small enough that $\sigma_1(4\epsilon) \leq \left(\frac{1}{4}c\right)^2$. Then (9-7) holds for $y \in \Gamma_u \cap B_\epsilon \cap \{y_1 > 0\}$.

Because $x = (x_1, x_2) \in \Gamma_u \cap B_\epsilon \cap \{x_1 > 0\}$, by (9-7) we have

$$B_{c|x|}(x) \cap (x + C_{1/2}) \subset \{u > 0\} \quad \text{and} \quad B_{c|x|}(x) \cap (x - C_{1/2}) \subset \{u = 0\}. \tag{9-8}$$

Assume there exists $(x_1, t) \in B_\epsilon$ such that $t > x_2$ and $u(t, x_2) = 0$. Let t^* be the infimum of such t, i.e.,

$$t^* = \inf\{t > x_2 \mid (x_1, t) \in B_\epsilon \text{ and } u(t, x_2) = 0\}.$$

From the first inclusion in (9-8) we have that $t^* > x_2$. Thus for $x_2 < s < t^*$ we have $u(x_1, s) > 0$, therefore (x_1, t^*) is on the boundary of $\{u > 0\}$. We obtain that $(x_1, t^*) \in \Gamma_u$. But now, because $(x_1, t^*) \in \Gamma_u \cap B_\epsilon \cap \{x_1 > 0\}$, by the second inclusion in (9-7) at the point (x_1, t^*) we come to a contradiction.

Now assume that there exists $(x_1, t) \in B_\epsilon$ such that $t < x_2$ and $(t, x_2) \in \overline{\{u > 0\}}$. Let t^* be the supremum of such t, i.e.,

$$t^* = \sup\{t < x_2 \mid (x_1, t) \in B_\epsilon \cap \overline{\{u > 0\}}\}.$$

From the second inclusion in (9-8) we have that $t^* < x_2$. Thus for $t^* < s < x_2$ we have $(x_1, s) \in \{u = 0\}^\circ$, therefore $(x_1, t^*) \in \Gamma_u$. But now, because $(x_1, t^*) \in \Gamma_u \cap B_\epsilon \cap \{x_1 > 0\}$, by the first inclusion in (9-7) at the point (x_1, t^*) we come to a contradiction. $\square$

In the following lemma we prove that near to regular points the free boundary is a continuous graph.

Lemma 42. *If u is a solution in D, $0 \in D$ and $0 \in \Gamma$ is a regular free boundary point with blowup limit u_w, then there exists an $\epsilon > 0$ and $\gamma \in C\big(\big[0, \frac{1}{4}\epsilon\big)\big)$ such that $\gamma(0) = 0$, we have $(x_1, \gamma(x_1)) \in B_\epsilon$ for $0 < x_1 < \frac{1}{4}\epsilon$, and*

$$\{u = 0\} \cap B_\epsilon \cap \big\{0 \le x_1 < \tfrac{1}{4}\epsilon\big\} = \big\{x \in B_\epsilon \;\big|\; 0 \le x_1 < \tfrac{1}{4}\epsilon,\ x_2 \le \gamma(x_1)\big\}. \tag{9-9}$$

Proof. By Lemma 41 there exists an $\epsilon > 0$ such that, for every $0 < x_1 < \frac{1}{4}\epsilon$, there exists a unique x_2 such that $x = (x_1, x_2) \in \Gamma \cap B_\epsilon$; let us define $\gamma(x_1) = x_2$. Let us also define $\gamma(0) = 0$.

Then, by Lemmas 40 and 41, we have (9-9).

Now let us show that γ is continuous. Assume there exists $0 \le y < \frac{1}{4}\epsilon$ such that γ is discontinuous at y. Then there exists $x_j \to y$ such that $\gamma(x_j) \to z$ and either $z > \gamma(y)$ or $z < \gamma(y)$.

In the case $z > \gamma(y)$ we have $u(y, z) > 0$, which is in contradiction with $u(x_j, \gamma(x_j)) = 0$ and the continuity of u.

In the case $z < \gamma(y)$ we have $(y, z) \in \{u = 0\}^\circ$, which is in contradiction with $(x_j, \gamma(x_j)) \in \Gamma$. ☐

In the following lemma we formulate the convergence of the free boundary in terms of the function γ.

Lemma 43. *There exists $C_1 > 0$ and $C_2 > 0$ such that, if u is a solution in D, $0 \in D$ and $0 \in \Gamma$ is a regular free boundary point with blowup limit u_w, then, with $\epsilon > 0$ and γ as in Lemma 42, we have*

$$|\gamma(x_1) - x_1| \le C_1\big(\sigma_0(C_2|x_1|)\big)^{1/2}|x_1| \quad \text{for } 0 < x_1 < \tfrac{1}{4}\epsilon,$$

where σ_0 is as defined in (2-6).

Proof. By Theorem 8 we have

$$d(x, \Gamma_{u_w}) \le C_1\big(\sigma_0(C_2|x|)\big)^{1/2}|x|.$$

For $x_1 > 0$ we estimate

$$d(x, \Gamma_{u_w}) \ge \tfrac{\sqrt{2}}{2}|x_2 - x_1|;$$

thus

$$\begin{aligned} |\gamma(x_1) - x_1| \le C_3\big(\sigma_0(C_2|x|)\big)^{1/2}|x| &\le C_4\big(\sigma_0(C_2|x|)\big)^{1/2}\big(|\gamma(x_1)| + |x_1|\big) \\ &\le C_4\big(\sigma_0(C_2|x|)\big)^{1/2}\big(|\gamma(x_1) - x_1| + 2|x_1|\big). \end{aligned} \tag{9-10}$$

By the continuity of γ at 0 we have that $\gamma(x_1) \to \gamma(0) = 0$ as $x_1 \to 0$. Hence $|x| \le C_5\big(|\gamma(x_1)| + |x_1|\big) \to 0$ as $x_1 \to 0$. From this convergence we obtain $\sigma_0(C_2|x|) \to 0$ as $x_1 \to 0$.

Thus, from (9-10) it follows that

$$|\gamma(x_1) - x_1| \le C_6\big(\sigma_0(C_2|x|)\big)^{1/2}|x_1|. \tag{9-11}$$

In turn, from (9-11) it follows that

$$\begin{aligned} |x| \le C_5\big(|\gamma(x_1)| + |x_1|\big) \le C_5\big(|\gamma(x_1) - x_1| + 2|x_1|\big) &\le C_5\Big(C_6\big(\sigma_0(C_2|x|)\big)^{1/2}|x_1| + 2|x_1|\Big) \\ &= C_5\Big(C_6\big(\sigma_0(C_2|x|)\big)^{1/2} + 2\Big)|x_1| \\ &\le C_7|x_1|. \end{aligned} \tag{9-12}$$

Now, by (9-11) and (9-12) the lemma is proved. □

In the following lemma we formulate the convergence of the normals in terms of the function γ.

Lemma 44. *There exists $C_1 > 0$ and $C_2 > 0$ such that, if u is a solution in D, $0 \in D$ and $0 \in \Gamma$ is a regular free boundary point with blowup limit u_w, and $\epsilon > 0$ and γ are as in Lemma 42, then we have $\gamma \in C^1\big(0, \frac{1}{4}\epsilon\big)$ and*

$$|\gamma'(x_1) - 1| \le C_1\big(\sigma_1(C_2|x_1|)\big)^{1/2},$$

where σ_1 is as defined in (2-6).

Proof. By Theorem 39, for small enough $\epsilon > 0$ all points of $\Gamma \cap \{x_1 > 0\} \cap B_\epsilon$ are usual regular points. Let $0 < x_1 < \frac{1}{4}\epsilon$. Hence (see [Petrosyan et al. 2012]) Γ is a C^1 curve in a neighbourhood of $(x_1, \gamma(x_1))$. From (8-28) it follows that for small enough ϵ and $|x| < \epsilon$ we have $n(x) \notin \{-e_1, e_1\}$. It follows that $\gamma'(x_1)$ exists and

$$n(x) = \frac{(-\gamma'(x_1), 1)}{\sqrt{1 + (\gamma'(x_1))^2}}.$$

From here it follows that there exists $C > 0$ such that for $n(x)$ close enough to ν_w we have

$$|\gamma'(x_1) - 1| \le C|n(x) - \nu_w|. \tag{9-13}$$

Now, by (8-28) and (9-13) we obtain

$$|\gamma'(x_1) - 1| \le C_2\|u_{4|x|} - u_w\|^{1/2}_{C^1(B_1)}. \tag{9-14}$$

By (9-12) together with the definition of σ_1 and (9-14), the lemma is proved. □

Proof of Theorem 9. This follows from Lemmas 42, 43 and 44 and the symmetry of the problem with respect to the line $\{x_1 = 0\}$. □

In the case when 0 is a regular point but with $u_w(x_1, -x_2)$ as the blowup limit, we consider the even reflection $\tilde{u}(x_1, x_2) = u(x_1, -x_2)$, apply Theorem 9 to $\tilde{u}$ and obtain that the free boundary of u is a graph with properties as in Theorem 9 but reflected with respect to the line $\{x_2 = 0\}$.

By the following two lemmas we prove that if $W(+0, u) = 2W(1, u_w)$ then u might be decomposed into the sum of two functions each having 0 as a regular point.

Lemma 45. *If u is a solution in D, $0 \in D$, $0 \in \Gamma$ and $W(+0, u) = 2W(1, u_w)$, then there exists an $\epsilon > 0$ such that $u(x_1, 0) = 0$ for $|x_1| < \epsilon$.*

Proof. Let $u_0 = u_w + u_w(x_1, -x_2)$. We have

$$d\big(\pm\tfrac{1}{4}e_1, \{u_0 > 0\}\big) = \tfrac{\sqrt{2}}{8}.$$

We compute

$$d^2\big(\pm\tfrac{1}{4}e_1, \{u_0 > 0\}\big)\big(d\big(\pm\tfrac{1}{4}e_1, \{u_0 > 0\}\big) + \tfrac{1}{4}\big) = \big(\tfrac{\sqrt{2}}{8}\big)^2\big(\tfrac{\sqrt{2}}{8} + \tfrac{1}{4}\big).$$

Now, if $|x_1| > 0$ is small enough that

$$\|u_{4|x_1|} - u_0\|_{L^\infty(B_1)} < c\big(\tfrac{\sqrt{2}}{8}\big)^2\big(\tfrac{\sqrt{2}}{8} + \tfrac{1}{4}\big)$$

with c as in Lemma 32 then, by (7-2), we have $u_{4|x_1|}\left(\pm\frac{1}{4}e_1\right)=0$. Thus $u(x_1,0)=0$. □

Lemma 46. *If u is a solution in D, $0\in D$, $0\in\Gamma$ and $W(+0,u)=2W(1,u_w)$, then there exists an $\epsilon>0$ such that $u_+=\chi_{\{x_2>0\}}u$ and $u_-=\chi_{\{x_2<0\}}u$ are solutions in B_ϵ. We have $W(+0,u_\pm)=W(1,u_w)$, the blowup limit of u_+ is u_w and the blowup limit of u_- is $u_w(x_1,-x_2)$.*

Proof. By Lemma 45 there exists an $\epsilon>0$ such that $u(x_1,0)=0$ for $|x_1|<\epsilon$.

Because $u\ge 0$, $u\in C^1_{\text{loc}}(D)$ and $u(x_1,0)=0$ for $|x_1|<\epsilon$, it follows that $\nabla u(x_1,0)=0$ for $|x_1|<\epsilon$.

From this it follows that u_+ and u_- are solutions in B_ϵ. We have $u_r(x)\to u_w+u_w(x_1,-x_2)$ in $C^1(B_1)$ as $r\to 0$. Thus $\chi_{\{x_2>0\}}u_r\to u_w$ in $C^1(B_1)$ and $u_{+,r}(x)=r^{-3}\chi_{\{x_2>0\}}(rx)u(rx)=\chi_{\{x_2>0\}}u_r(x)$; hence $u_{+,r}(x)\to u_w$ in $C^1(B_1)$ and

$$W(+0,u_+)=\lim_{r\to+0}W(r,u_+)=\lim_{r\to+0}W(1,u_{+,r})=W(1,u_w).$$

We argue similarly for u_-. □

In the case $W(+0,u)=2W(1,u_w)$, by Lemma 46 and Theorem 9 it follows that the free boundary near to 0 is the union of two graphs, one graph as in Theorem 9 and the other a graph with properties as in Theorem 9 but reflected with respect to the line $\{x_2=0\}$.

10. An irregularity result for the free boundary near degenerate points

Lemma 47. *Let u be a solution in D with $0\in D$. Suppose also that there exists $\delta>0$ such that $B_\delta\subset D$, $\partial_{x_2}u\le 0$ in $B_\delta\cap\{x_1>0,\ x_2>0\}$, $\Gamma\cap B_\delta\cap\{x_1=0,\ x_2>0\}\ne\varnothing$ and $B_\delta\cap\{x_1>0,\ x_2>0\}\subset\Omega$; then $u=u_{hs}$ in $B_\delta\cap\{x_1>0,\ x_2>0\}$.*

Proof. For ease of notation let us write $v=-\partial_{x_2}u$. We have that v is harmonic in Ω and $v\ge 0$ in $B_\delta\cap\{x_1>0,\ x_2>0\}$.

Assume $y\in\Gamma\cap B_\delta\cap\{x_1=0,\ x_2>0\}$, then by the optimal growth (Theorem 11) we have $\partial_{x_1}v(y)=0$. For small enough $r>0$ we have $B_r(re_1+y)\subset\Omega$. Now, because v is nonnegative and harmonic in $B_r(re_1+y)$ and $\partial_{x_1}v(y)=0$, by Hopf's lemma we conclude that $v=0$ in $B_r(re_1+x)$. Because v is harmonic in Ω we obtain that $v=0$ in $B_\delta\cap\{x_1>0,\ x_2>0\}$. Hence $u=u(x_1)$ in $B_\delta\cap\{x_1>0,\ x_2>0\}$. By this and the assumption $\Gamma\cap B_\delta\cap\{x_1=0,\ x_2>0\}\ne\varnothing$ the claim follows. □

Lemma 48. *Let u be a solution in D with $0\in D$. Suppose also that there exists $\delta>0$ such that $B_\delta\subset D$, $\partial_{x_2}u\le 0$ in $B_\delta\cap\{x_1>0,\ x_2>0\}$, and there exists $\rho\in C\left(\left[0,\frac{1}{2}\delta\right)\right)\cap C^1\left(\left[0,\frac{1}{2}\delta\right)\right)$ such that $\rho(0)=\rho'(+0)=0$, $\rho>0$ in $\left(0,\frac{1}{2}\delta\right)$, ρ is convex and*

$$\Omega\cap B_\delta\cap\left\{x_1>0,\ 0<x_2<\tfrac{1}{2}\delta\right\}=B_\delta\cap\left\{0<x_2<\tfrac{1}{2}\delta,\ \rho(x_2)<x_1\right\};\tag{10-1}$$

then for every $q>1$ there exist $c>0$ and $t_0>0$ such that

$$\rho(t)\ge ct^q\quad\text{and}\quad\rho'(t)\ge ct^{q-1}\quad\text{for } 0<t<t_0.\tag{10-2}$$

Proof. Again, for ease of notation let us write $v=-\partial_{x_2}u$. The proof is divided into multiple steps.

Step 1. In this step we show that $v > 0$ in $B_\delta \cap \{0 < x_2 < \frac{1}{2}\delta,\ \rho(x_2) < x_1\}$.

If there is $x \in B_\delta \cap \{0 < x_2 < \frac{1}{2}\delta,\ \rho(x_2) < x_1\}$ such that $v(x) = 0$ then, because v is harmonic and nonnegative in $B_\delta \cap \{0 < x_2 < \frac{1}{2}\delta,\ \rho(x_2) < x_1\}$, it follows that $v = 0$ in $B_\delta \cap \{0 < x_2 < \frac{1}{2}\delta,\ \rho(x_2) < x_1\}$, but then because $u(\rho(t), t) = 0$ for $0 < t < \frac{1}{2}\delta$ we come to contradiction with (10-1).

Step 2. In this step we show that for each $q > 1$ and $\eta > \big(\tan(\pi/(2q))\big)^{-1}$ there exist $c_1 > 0$ (depending on u) and $t_1 > 0$ such that

$$v(x_t) \geq c_1 t^{2q} \quad \text{for } 0 < t < t_1, \tag{10-3}$$

where

$$x_t = (\eta t, t) \in \Omega.$$

Let $q > 1$ and

$$\alpha_q = \frac{\pi}{2q}.$$

Because $\rho'(+0) = 0$ there exists $t_q > 0$ such that $\rho(t) < t/\tan\alpha_q$ for $0 < t < t_q$.

Let us denote

$$r_q = \frac{t_q}{\tan\alpha_q}.$$

It follows that

$$\Omega_q = \{x = re^{i\theta} \mid 0 < r < r_q,\ 0 < \theta < \alpha_q\} \subset \Omega.$$

Let us define the function

$$v_q(x) = r^{2q}\sin(2q\theta) \quad \text{for } x = re^{i\theta} \in \Omega_q.$$

We have

$$\partial\big(\tfrac{1}{2}\Omega_q\big) = S_q \cup A_q,$$

where

$$S_q = \big\{x = re^{i\theta} \,\big|\, 0 \leq r < \tfrac{1}{2}r_q,\ \theta \in \{0, \alpha_q\}\big\}$$

and

$$A_q = \big\{x = re^{i\theta} \,\big|\, r = \tfrac{1}{2}r_q,\ 0 \leq \theta \leq \alpha_q\big\}.$$

Let $a = \frac{1}{2}r_q e_1$ and $b = \frac{1}{2}r_q e^{i\alpha_q}$ be the endpoints of the arc A_q. We have $b \in \Omega$, hence $v(b) > 0$. Either $v(a) > 0$ or $v(a) = 0$ and, by Hopf's lemma, we have $\partial_{x_2} v(a) > 0$. Also we have $v > 0$ on $A_q \backslash \{a, b\}$.

Thus there exists $\epsilon > 0$ such that

$$\epsilon v_q \leq v \quad \text{on } A_q. \tag{10-4}$$

We have $v_q = 0$ and $v \geq 0$ on S_q, thus

$$\epsilon v_q \leq v \quad \text{on } S_q. \tag{10-5}$$

Putting (10-4) and (10-5) together we have

$$\epsilon v_q \leq v \quad \text{on } \partial\big(\tfrac{1}{2}\Omega_q\big).$$

Now, by the maximum principle, we obtain that

$$\epsilon v_q \le v \quad \text{in } \tfrac{1}{2}\Omega_q. \tag{10-6}$$

We compute $|x_t| = \sqrt{1+\eta^2}\,t$, so for

$$0 < t < \frac{1}{2}\frac{r_q}{\sqrt{1+\eta^2}}$$

we have $|x_t| < \frac{1}{2} r_q$; also we compute

$$\frac{x_{t,2}}{x_{t,1}} = \frac{1}{\eta} < \tan\alpha_q,$$

thus we have

$$x_t \in \frac{1}{2}\Omega_q \quad \text{for } 0 < t < \frac{1}{2}\frac{r_q}{\sqrt{1+\eta^2}}. \tag{10-7}$$

Now, by (10-6) and (10-7) we have

$$v(x_t) \ge \epsilon v_q(x_t) = \epsilon |x_t|^{2q} \sin\Bigl(2q \arctan\frac{1}{\eta}\Bigr) = c_1 t^{2q} \quad \text{for } 0 < t < \frac{1}{2}\frac{r_q}{\sqrt{1+\eta^2}},$$

where

$$c_1 = \epsilon(1+\eta^2)^q \sin\Bigl(2q\arctan\frac{1}{\eta}\Bigr) > 0.$$

Step 3. In this step we show that there exists $c_2 > 0$ (independent of u) and $t_2 > 0$ such that if

$$0 < t < t_2 \quad \text{and} \quad \eta < 1$$

then there exists $y_t = (\rho(y_{t,2}), y_{t,2}) \in \Gamma$ with $0 < y_{t,2} < t_q$ such that

$$d_t = |y_t - x_t| = d(\Gamma, x_t)$$

and

$$\partial_{n(y_t)} v(y_t) \ge \frac{c_2}{d_t} v(x_t). \tag{10-8}$$

Here $n(y)$ is the normal to Γ at y, pointing into Ω.

Let

$$\Pi_q = \{0 < x_1 < r_q,\ 0 < x_2 < t_q\};$$

then we have

$$\Gamma_q = \Gamma \cap \Pi_q = \{(\rho(t), t) \mid 0 < t < t_q\}.$$

One may see that

$$d(x_t, \partial\Pi_q) = \min\bigl\{\eta t,\ r_q - \eta t,\ t,\ t_q - t\bigr\} = \eta t \tag{10-9}$$

if

$$t < \min\Bigl(\frac{r_q}{2\eta}, \frac{t_q}{1+\eta}\Bigr) \quad \text{and} \quad \eta < 1.$$

Because $\eta > (\tan\alpha_q)^{-1}$ and $0 < t < t_q$, we have that $\rho(t) < t/\tan\alpha_q < \eta t$. Also we have $\rho(t) > 0$, thus

$$d(x_t, (\rho(t), t)) = \eta t - \rho(t) < \eta t.$$

Now, because $(\rho(t), t) \in \Gamma_q$ we have

$$d(x_t, \Gamma) < \eta t. \tag{10-10}$$

By (10-9) and (10-10) there exists $y_t \in \Gamma_q$ such that

$$d_t = |y_t - x_t| = d(\Gamma, x_t). \tag{10-11}$$

Because

$$d(x_t, \partial\Pi_q) = \eta t > d(\Gamma, x_t) = d_t,$$

we have

$$B_{d_t}(x_t) \subset \Pi_q \subset \Omega.$$

Because $y_t \in \partial B_{d_t}(x_t)$, by the quantitative Hopf lemma (see [Han and Lin 2011]) there exists $c_2 > 0$ (independent of u and t) such that (10-8) holds.

Step 4. In this step we show that

$$\partial_{n(y)} v(y) = -n_2(y) y_1 \quad \text{for } y \in \Gamma_q. \tag{10-12}$$

By the equation $\triangle u = |x_1| \chi_{\{u>0\}}$ and the smoothness of the free boundary Γ_q, i.e., smoothness of ρ, it follows that in a neighbourhood of $y \in \Gamma_q$ we have

$$\triangle v = -n_2 |x_1| \mathcal{H}^1 \llcorner \Gamma. \tag{10-13}$$

From (10-1) and (10-13), the equation (10-12) follows.

Step 5. In this step we show that for $0 < t < t_2$ we have

$$y_{t,2} < (1+\eta)t. \tag{10-14}$$

We have

$$n(y) = \frac{(1, -\rho'(y_2))}{\sqrt{1 + (\rho'(y_2))^2}} \quad \text{for } y \in \Gamma_q \tag{10-15}$$

and

$$y_t = x_t - d_t n(y_t).$$

Thus

$$y_{t,2} = t + d_t \frac{\rho'(y_{t,2})}{\sqrt{1 + (\rho'(y_{t,2}))^2}}$$

and

$$y_{t,2} \le t + d_t < t + \eta t = (1+\eta)t.$$

Step 6. In this step we show that there exists $c_3 > 0$ and $t_3 > 0$ such that

$$\rho(y_{t,2}) \rho'(y_{t,2}) \ge c_3 t^{2q-1} \quad \text{for } 0 < t < t_3. \tag{10-16}$$

Set $t_3 = \min(t_1, t_2)$. From (10-3), (10-8) and (10-12) it follows that

$$-n_2(y_t)y_{t,1} = \partial_{n(y_t)}v(y_t) \geq \frac{c_2}{d_t}v(x_t) \geq \frac{c_2}{d_t}c_1 t^{2q} \quad \text{for } 0 < t < t_3. \tag{10-17}$$

From (10-17), (10-15), (10-10) and (10-11) we get

$$\begin{aligned}\rho(y_{t,2})\rho'(y_{t,2}) = \rho'(y_{t,2})y_{t,1} &\geq \frac{\rho'(y_{t,2})}{\sqrt{1+(\rho'(y_{t,2}))^2}}y_{t,1} \\ &= -n_2(y_t)y_{t,1} \geq \frac{c_2}{d_t}c_1 t^{2q} \geq \frac{1}{\eta}c_1c_2t^{2q-1} = c_3t^{2q-1}.\end{aligned}$$

Step 7. In this step, using the convexity of ρ we finish the proof of the lemma.

By the convexity of ρ, the function $\rho\rho'$ is nondecreasing; hence, by (10-14) and (10-16), we have

$$\rho((1+\eta)t)\rho'((1+\eta)t) \geq \rho(y_{t,2})\rho'(y_{t,2}) \geq c_3t^{2q-1} \quad \text{for } 0 < t < t_3.$$

Letting $\tau = (1+\eta)t$ we have that

$$\rho(\tau)\rho'(\tau) \geq c_3\Big(\frac{\tau}{1+\eta}\Big)^{2q-1} = c_4\tau^{2q-1} \quad \text{for } 0 < \tau < (1+\eta)t_3 = \tau_0.$$

It follows that

$$(\rho^2)'(\tau) \geq 2c_4\tau^{2q-1} \quad \text{for } 0 < \tau < \tau_0$$

and by integration we obtain

$$\rho(\tau) \geq c_5\tau^q \quad \text{for } 0 < \tau < \tau_0.$$

From the convexity of ρ it follows that $\tau\rho'(\tau) \geq \rho(\tau)$; hence

$$\rho'(\tau) \geq c_5\tau^{q-1} \quad \text{for } 0 < \tau < \tau_0$$

and this completes the proof of the lemma. □

Proof of Theorem 10. By Lemmas 47 and 48 we have that either $\rho = 0$ in $\big(0, \frac{1}{2}\delta\big)$ and $u = u_{hs}$ in $\Omega \cap B_\delta \cap \{x_1 > 0,\ x_2 > 0\}$ or, for all $q > 1$, there exist $c > 0$ and $t_0 > 0$ such that (10-2) holds.

In the latter case, if Γ is $C^{1,\alpha}$ regular for some $0 < \alpha < 1$ at the origin, then there exists $C > 0$ and $\delta_1 > 0$ such that

$$|\rho'(x_2) - \rho'(+0)| \leq C|x_2|^\alpha \quad \text{for } 0 < x_2 < \delta_1.$$

But, because $\rho'(+0) = 0$ and $\rho'(x_2) \geq 0$, we should have

$$\rho'(x_2) \leq Cx_2^\alpha \quad \text{for } 0 < x_2 < \delta_1.$$

This contradicts with (10-2) if we take $1 < q < 1+\alpha$. □

11. Further directions

The problem considered in this paper might be thought of as a prototype of free boundary problems, especially the obstacle problem, with a degenerate force term. There are many open questions in these problems and we are working to complete some works on these questions.

Some further directions are as follows:

(1) Higher dimension. It is interesting to consider the same problem in higher dimensions with possibly different dimensions for the set where the force term vanishes. In [Yeressian 2015] the key nondegeneracy result is proved for such higher-dimensional problems when the force term vanishes on a linear subspace.

(2) More general force terms. Partial results show that, when the force term is of the form $|x_1|^\alpha$ for $\alpha > 0$, the number of homogeneous global solutions — and together with it the possible Weiss balanced energy levels — grows linearly with $\alpha > 0$. Again in [Yeressian 2015] the key nondegeneracy result is proved for such general force terms. Many results in this paper could be written for such more general forces, but to have a reasonable bound on the size of the paper we have opted to consider the case $\alpha = 1$ only.

(3) Degenerate free boundary points and points where $W(+0, x, u) = 2W(1, u_{hs})$. We know that at these points the free boundary converges tangentially to the line $\{x_1 = 0\}$ and we know some topological structure of the set of these points based on the upper semicontinuity of the Weiss balanced energy. Also, in a particular case we have proved an irregularity result for the free boundary at such points. It is interesting to study the structure of the free boundary near to such points in more detail.

(4) Uniform results. For the nondegenerate obstacle problems there are many results which hold uniformly for a class of problems; see [Petrosyan et al. 2012]. But in this paper we have only considered a single solution alone.

(5) Parabolic problem. The problem considered in this paper has a parabolic analogue. It is interesting to know the exact influence of the degeneracy of the force term in the parabolic problems.

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