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Correction to the article Frobenius and valuation rings

Rankeya Datta and Karen E. Smith

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Correction to the article Frobenius and valuation rings

Rankeya Datta and Karen E. Smith

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Theorem 5.1 is corrected in the paper *Frobenius and valuation rings*.

The characterization of F-finite valuation rings of function fields (Theorem 5.1) is incorrect. The corrected characterization is as follows:

Theorem 0.1. *Let K be a finitely generated extension of an F-finite ground field k. Let V be any nontrivial valuation ring of K/k. Then V is F-finite if and only if V is divisorial.*

A proof of this corrected theorem follows a complete accounting of affected statements in the paper. Notation and theorem and page numbers are as in [Datta and Smith 2016].

1. List of affected statements

The source of the error is Proposition 2.2.1 (page 1061), which is misquoted from the original source [Bourbaki 1998]. Specifically, the last sentence in the statement of Proposition 2.2.1 should read: *Furthermore, equality holds **if** the integral closure of R_ν in L is a finitely generated R_ν-module,* not ***if and only if***. The statement *is* correct, however, under the additional assumption that the valuation ring R_ν is *Noetherian* [Bourbaki 1998, VI, §8.5, Remark (1)].

This error has consequences in the following statements from the paper.

THEOREM 4.3.1 should read: *Let V be a valuation ring of an F-finite field K of prime characteristic p. If V is F-finite, then*

$$[\Gamma : p\Gamma][\kappa : \kappa^p] = [K : K^p],$$

where Γ is the value group and κ is the residue field of V.

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The paper had incorrectly stated the converse, which *does* hold if the valuation ring is Noetherian [Bourbaki 1998, VI, §8.5, Remark (1)].

COROLLARY 4.3.2 should read: *With hypotheses as in Theorem 4.3.1, if* $[K : K^p] = [\kappa : \kappa^p]$, *then* V *is F-finite.* See Section 2 below for the proof. The original statement holds as stated if V is Noetherian.

EXAMPLES 4.4.1 and 4.4.2 (page 1069): The computations here are correct but the conclusions are not. Rather, Theorem 0.1 ensures these valuation rings are *not* F-finite.

EXAMPLE 4.5.1 (page 1070) is correct as stated, as it follows from the correct implication of Theorem 4.3.1.

PROPOSITION 4.6.1 (page 1071): This statement is correct. However, the proof of (iii) is not, because it cited the incorrect implication of Proposition 2.2.1. The corrected proof is in Section 3 below.

THEOREM 5.1 (page 1072): The equivalence of (ii) and (iii) is correct, and both imply (i). Under the additional assumption that the valuation is discrete, also (i) implies (ii) and (iii). The proof of Theorem 5.1 in the paper proves:

Theorem 1.1. *Let* K *be a finitely generated field extension of an F-finite ground field* k *of characteristic* p. *The following are equivalent for a valuation* v *on* K/k:

(i) *The valuation* v *is Abhyankar.*

(ii) $[\Gamma : p\Gamma][\kappa : \kappa^p] = [K : K^p]$, *where* Γ *is the value group, and* κ *is the residue field of* v.

COROLLARY 5.2 (page 1072) is correct as stated; it is a consequence of Theorem 0.1.

COROLLARY 6.6.3 (page 1086) is correct as stated; the proof should invoke Theorem 0.1 instead of Theorem 5.1.

The remaining statements in Section 5 (Remark 5.3, Proposition 5.4, Lemma 5.5, Proposition 5.6, Lemma 5.7, Lemma 5.8) and Section 6 are correct, because they do not rely on Proposition 2.2.1 or its consequences.

2. Proof of Theorem 0.1

Lemma 2.1. *Let* $(V, \mathfrak{m})$ *be a valuation ring of prime characteristic* p. *If* $\mathfrak{m}$ *is not principal, then*

$$\mathfrak{m} = \mathfrak{m}^{[p]}.$$

Proof. Take any $x \in \mathfrak{m}$. It suffices to show that there exists $y \in \mathfrak{m}$ such that $y^p \mid x$. Note that since $\mathfrak{m}$ is not principal, the value group Γ of the corresponding valuation v *does not have a smallest element* > 0. This means we can choose $\beta_0 \in \Gamma$ such that $0 < \beta_0 < v(x)$, and then also choose $\beta_1 \in \Gamma$ such that

$$0 < \beta_1 < \min\{\beta_0, v(x) - \beta_0\}.$$

Let $y_1 \in \mathfrak{m}$ be such that $v(y_1) = \beta_1$. Then

$$v(y_1^2) = 2\beta_1 < \beta_0 + (v(x) - \beta_0) = v(x).$$

Thus, $y_1^2 \mid x$. Repeating this argument inductively, we can find $y_n \in \mathfrak{m}$ such that $y_n^2 \mid y_{n-1}$ for all $n \in \mathbb{N}$. For $n > \log_2 p$, we have that $y_n^{2^n}$, and hence y_n^p, divides x. □

Lemma 2.2. *Let $(V, \mathfrak{m}, \kappa)$ be a valuation ring of characteristic p. Then the dimension of $V/\mathfrak{m}^{[p]}$ over κ^p is*

(a) $[\kappa : \kappa^p]$ *if $\mathfrak{m}$ is not finitely generated.*

(b) $p[\kappa : \kappa^p]$ *if $\mathfrak{m}$ is finitely generated.*

Proof. Consider the short exact sequence of κ^p-vector spaces

$$0 \to \mathfrak{m}/\mathfrak{m}^{[p]} \to V/\mathfrak{m}^{[p]} \to \kappa \to 0. \tag{1}$$

If $\mathfrak{m}$ is not finitely generated, then Lemma 2.1 implies that $\mathfrak{m}/\mathfrak{m}^{[p]} = 0$, and (a) follows. Otherwise, $\mathfrak{m}$ is principal, so $\mathfrak{m}^{[p]} = \mathfrak{m}^p$ and we have a filtration

$$\mathfrak{m} \supsetneq \mathfrak{m}^2 \supsetneq \cdots \supsetneq \mathfrak{m}^{p-1} \supsetneq \mathfrak{m}^{[p]} = \mathfrak{m}^p.$$

Since $\mathfrak{m}^i/\mathfrak{m}^{i+1} \cong \kappa$, we see that $\dim_{\kappa^p}(\mathfrak{m}/\mathfrak{m}^{[p]}) = (p-1)[\kappa : \kappa^p]$. From the short exact sequence (1), $\dim_{\kappa^p}(V/\mathfrak{m}^{[p]}) = p[\kappa : \kappa^p]$, proving (b). □

Proof of Theorem 0.1. If V is divisorial, it is a localization of a finitely generated algebra over the F-finite ground field k, hence it is F-finite.

For the converse, let $\mathfrak{m}$ be the maximal ideal and κ the residue field of V. Since V is F-finite, by (the corrected) Theorem 4.3.1,

$$[\Gamma : p\Gamma][\kappa : \kappa^p] = [K : K^p], \tag{2}$$

where Γ is the value group of V. Thus, the valuation is Abhyankar by Theorem 1.1 above. To show it is divisorial, we need to show it is discrete.

Because the value group of an Abhyankar valuation is finitely generated, $\Gamma \cong \mathbb{Z}^{\oplus s}$ for some integer $s \geq 1$. Hence it suffices to show that $s = 1$.

Since V is F-finite, we know V is free over V^p of rank $[K : K^p]$. Tensoring with the residue field κ^p of V^p, we have that $V/\mathfrak{m}^{[p]}$ is also a free κ^p-module of rank $[K : K^p]$. Thus, from (2) we have

$$\begin{aligned}\dim_{\kappa^p} V/\mathfrak{m}^{[p]} &= [K : K^p] = [\Gamma : p\Gamma][\kappa : \kappa^p] \\ &= \left|\mathbb{Z}^{\oplus s}/p\mathbb{Z}^{\oplus s}\right|[\kappa : \kappa^p] = p^s[\kappa : \kappa^p].\end{aligned} \tag{3}$$

But now, since $p^s \neq 1$, Lemma 2.2 forces $s = 1$, and the proof is complete. □

Remark 2.3. Theorem 0.1 does not hold without *some* assumption on the field K. For instance, if K is perfect, Frobenius is an isomorphism (hence a finite map) for *any* valuation ring of K. The proof of Theorem 0.1 does show that an F-finite valuation ring with finitely generated value group must be Noetherian (hence discrete), without any restriction on its fraction field. Furthermore, the proof also shows that a valuation ring cannot be F-finite if its value group Γ satisfies $[\Gamma : p\Gamma] > p$.

Proof of revised COROLLARY 4.3.2. In general, $[\Gamma : p\Gamma][\kappa : \kappa^p] \leq [K : K^p]$ by [Bourbaki 1998, VI, §8.1, Lemma 2]. Hence our hypothesis forces $[\Gamma : p\Gamma] = 1$. Thus $\Gamma = p\Gamma$, so that Γ cannot have a smallest positive element, which means that the maximal ideal $\mathfrak{m}$ of V is not finitely generated. Lemma 2.2(a) now ensures $\dim_{\kappa^p}(V/\mathfrak{m}^{[p]}) = [\kappa : \kappa^p] = [K : K^p]$. Then V is F-finite by [Bourbaki 1998, VI, §8.5, Theorem 2(c)]. ☐

3. Proof of Proposition 4.6.1(iii)

We recall PROPOSITION 4.6.1(iii): *Let $K \hookrightarrow L$ be a finite extension of F-finite fields of characteristic p. Let w be a valuation on L and v its restriction to K. Then the valuation ring of v is F-finite if and only if the valuation ring of w is F-finite.*

Lemma 3.1. *With notation as in Proposition 4.6.1(iii), the maximal ideal of the valuation ring of v is finitely generated if and only if the maximal ideal of the valuation ring of w is finitely generated.*

Proof. For ideals in a valuation ring, finite generation is the same as being principal. Principality of the maximal ideal is equivalent to the value group having a smallest element > 0. Thus, it suffices to show that the value group Γ_v of v has this property if and only if Γ_w does.

Assume Γ_w has a smallest element $g > 0$. We claim that for each $t \in \mathbb{N}$, the only positive elements of Γ_w less than tg are $g, 2g, \dots, (t-1)g$. Indeed, suppose $0 < h < tg$. Since g is smallest, $g \leq h < tg$, whence $0 \leq h - g < (t-1)g$. So by induction, $h - g = ig$ for some $i \in \{0, 1, \dots, t-2\}$, and hence h is among $g, 2g, \dots, (t-1)g$.

Now, because $[\Gamma_w : \Gamma_v] \leq [L : K] < \infty$ by [Bourbaki 1998, VI, §8.1, Lemma 2], every element of Γ_w/Γ_v is torsion. Let n be the smallest positive integer such that $ng \in \Gamma_v$. We claim that ng is the smallest positive element of Γ_v. Indeed, the only positive elements smaller than ng in Γ_w are $g, 2g, \dots, (n-1)g$, and none of these are in Γ_v by our choice of n.

Conversely, if Γ_v has a smallest element $h > 0$, then the set

$$S := \{g \in \Gamma_w : 0 < g < h\}$$

is finite because for distinct g_1, g_2 in this set, their classes in Γ_w/Γ_v are also distinct, while Γ_w/Γ_v is a finite group. Then the smallest positive element of Γ_w is the smallest element of S, or h if S is empty. □

Proof of Proposition 4.6.1(iii). A *necessary and sufficient* condition for the F-finiteness of a valuation ring $(V, \mathfrak{m}, \kappa)$ with F-finite fraction field K is, by [Bourbaki 1998, VI, §8.5, Theorem 2(c)], that

$$\dim_{\kappa^p}(V/\mathfrak{m}^{[p]}) = [K : K^p]. \tag{4}$$

Lemma 2.2 gives a formula for $\dim_{\kappa^p}(V/\mathfrak{m}^{[p]})$ in terms of $[\kappa : \kappa^p]$ that depends on whether the maximal ideal is finitely generated, which is the same for v and w by Lemma 3.1. Proposition 4.6.1(i) and (ii) tell us that both $[K : K^p] = [L : L^p]$ and $[\kappa_v : \kappa_v^p] = [\kappa_w : \kappa_w^p]$. Thus Lemma 2.2 and (4) guarantee that the valuation ring of v is F-finite if and only if the valuation ring of w is F-finite. □

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rankeya@umich.edu *Department of Mathematics, University of Michigan, 2074 East Hall, 530 Church Street, Ann Arbor, MI 48109-1043, United States*

kesmith@umich.edu *Department of Mathematics, University of Michigan, 2074 East Hall, 530 Church Street, Ann Arbor, MI 48109-1043, United States*

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