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The number of strings on essential tangle decompositions of a knot can be unbounded

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We construct an infinite collection of knots with the property that any knot in this family has n–string essential tangle decompositions for arbitrarily high n.

57M25, 57N10

1 Introduction

An *n–string tangle* $(B,\mathcal{T})$ is a ball B together with collection of n disjoint arcs $\mathcal{T}$ properly embedded in B, for $n\in\mathbb{N}$. We say that $(B,\mathcal{T})$ is *essential* if n is 1 and its arc is knotted,[1] or if n is bigger than 1 and there is no properly embedded disk in B disjoint from $\mathcal{T}$ and separating the components of $\mathcal{T}$ in B. Otherwise, we say that the tangle is *inessential*. (See Figure 1 for examples.)

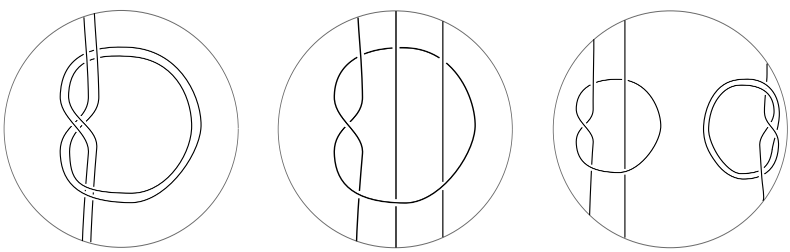

Figure 1: Examples of essential tangles (left and middle), and an inessential tangle (right)

Let K be a knot in S^3 and S a 2–sphere in general position with K. Each ball bounded by S in S^3 intersects K in the same number n of arcs. So these balls together with the arcs of intersection with K are n–string tangles. In this case, we say that S defines a *n–string tangle decomposition* of K, and if both tangles are essential we say that the tangle decomposition of K defined by S is *essential*. A knot is composite if and only if it has a 1–string essential tangle decomposition; otherwise

[1] An arc of $\mathcal{T}$ is *unknotted* if it cobounds a disk embedded in B together with an arc in ∂B; otherwise, it is said to be *knotted*.

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the knot is prime. Note also that S defines an essential tangle decomposition for K if and only if the intersection of S with the exterior of K, $E(K)$,[2] is an essential surface in $E(K)$; see Definition 3.

A tangle decomposition of a knot is natural and has been relevant for knot theory and its applications. The concept of a "tangle" was first used in the work of Conway [3], where he defines and classifies (2–string) rational tangles and uses it as an instrument to list knots. The concept of an essential tangle was first used in [8], where Kirby and Lickorish prove that any knot is concordant to a prime knot. They actually define *prime tangle*, that is an essential tangle with no local knots.[3] Another example is the work of Lickorish in [9], where he proves, for instance, that if a knot has a 2–string prime tangle decomposition, then the knot is prime. Tangles are also used in applied mathematics to study the DNA topology. The paper [2] by Buck surveys the subject concisely and also explains how tangles are useful to the study of the topological properties of DNA, an application pioneered by Ernst and Sumners in [5].

This paper addresses the question of if the number of strings on essential tangle decompositions of a fixed knot is bounded. There are results showing some evidence for this to be true. For instance, knots with no closed essential surfaces (see Culler, Gordon, Luecke and Shalen [4]), tunnel number one knots (see Gordon and Reid [6]) and free genus-one knots (see Matsuda and Ozawa [10]) have no essential tangle decompositions. There also are knots with a unique essential tangle decomposition; see Ozama [12]. Furthermore, in Proposition 2.1 of [11], Mizuma and Tsutsumi proved that, for a given knot, the number of strings in essential tangle decompositions, without parallel strings,[4] is bounded. The proof of this result allows a more general statement. That is, the number of strings that are not parallel to other strings in an essential tangle decomposition of a fixed knot is bounded. So, from this flow of results and intuition on essential tangle decompositions, the following theorem and its corollary are surprising.

Theorem 1 *There is an infinite collection of prime knots such that, for all $n \geq 2$, each knot has a n–string essential tangle decomposition.*

Corollary 2 *There is an infinite collection of knots such that, for all $n \geq 1$, each knot has a n–string essential tangle decomposition.*

[2] We denote by $E(K)$ the *exterior* of a knot K, that is, $S^3 - \operatorname{int} N(K)$, where $N(K)$ is a regular neighborhood of K.

[3] A tangle $(B, \mathcal{T})$ has no local knots if any 2–sphere intersecting $\mathcal{T}$ transversely in two points bounds a ball in B meeting $\mathcal{T}$ in an unknotted arc.

[4] Two strings of a tangle in a ball B are parallel if there is an embedded disk in B cobounded by these strings and two arcs in ∂B.

Essential surfaces are very important in the study of 3–manifold topology. And as observed above, to each n–string essential tangle decomposition of a knot corresponds a meridional essential surface in the exterior of the knot, with $2n$ boundary components. Therefore, from the results in this paper, there are knots with meridional planar essential surfaces in their exteriors with all possible numbers of boundary components. Furthermore, from Lemma 1.2 in Bleiler [1], the double cover of S^3 along these knots contains genus-g closed incompressible surfaces, meeting the fixed point set of the covering action in $2(g+1)$ points, and separating the double cover in irreducible and ∂–irreducible components, for all $g \geq 1$.

The reference used for standard definitions and results of knot theory is Rolfsen's book [13], and throughout this paper we work in the piecewise-linear category.

In Section 2, we show the existence of handlebody-knots (see Definition 4) with incompressible planar surfaces in their exteriors with b boundary components for all $b \geq 2$. In Section 3, we use these handlebody-knots to prove Theorem 1 and its corollary. The main techniques used are standard in 3–manifold topology. Throughout the paper, the number of connected components of a topological space X is denoted by $|X|$.

2 Meridional incompressible planar surfaces in handlebody-knots complements

To prove Theorem 1, we use the correspondence between n–string essential tangle decompositions of a knot and meridional planar essential surfaces in the knot exterior. We start by defining these surfaces.

Definition 3 A *planar surface* is a surface obtained from a 2–sphere by removing the interior of a finite number of disks.

Let H be a handlebody embedded in S^3.

A surface P properly embedded in $E(H) = S^3 - \text{int}\, H$ is *meridional* if each boundary component of P bounds a disk in H.

An embedded disk D in $E(H)$ is a *compressing disk* for P if $D \cap P = \partial D$ and ∂D does not bound a disk in P. We say that P is *incompressible* if there is no compressing disk for P in $E(H)$.

An embedded disk D in $E(H)$ is a *boundary compressing disk* for P if $\partial D \cap P = \alpha$, with α a connected arc not cutting a disk from P, and $\partial D - \alpha = \beta$ a connected arc in ∂H. We say that P is *boundary incompressible* if there is no boundary compressing disk for P in $E(H)$.

The surface P is *essential* if it is incompressible and boundary incompressible.

In this section, we present handlebody-knots whose exteriors contain meridional incompressible planar surfaces with n boundary components for any $n \geq 2$. This embedding will later be used in the proof of Theorem 1. We consider next the definition of handlebody-knot.

Definition 4 A *handlebody-knot* of genus g in S^3 is an embedded handlebody of genus g in S^3. A *spine* γ of a handlebody-knot Γ is an embedded graph in S^3 with Γ as a regular neighborhood.

Let Γ be the genus-two handlebody-knot 4_1 from the list of [7], with spine γ, as in Figure 2. Consider also a collection of distinct knots C_i, for $i \in \mathbb{N}$, and C some other nontrivial knot. We work with γ as if defined by two vertices, two loops e_1 and e_2 (one for each vertex), and an edge e between the two vertices.

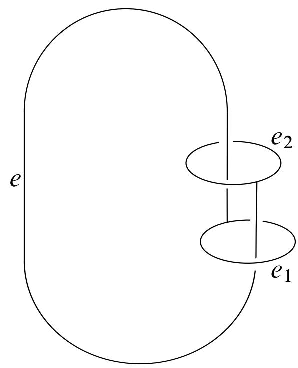

Figure 2: The spine γ of the handlebody-knot Γ, with labels of the two loops e_1 and e_2, and of the edge e

Consider two disjoint closed arcs a_1 and a_2 in e, as in Figure 3 (left). In this figure we also have represented an embedded 2–sphere S_2 in S^3 that intersects γ in e at two points, p_1 and p_2, and separates the arcs a_1 and a_2. Denote the ball bounded by S_2 containing a single component of e by $B_{2,1}$ and the other by $B_{2,2}$. Denote by l_1 and l_2 the components of $B_{2,2} \cap \gamma$ that contain e_1 and e_2, respectively, and note that l_j intersects S_2 at p_j, for $j = 1, 2$.

We perform an unusual connected sum operation between γ and the knots C and C_i along the arcs a_1 and a_2. That is, we take a ball in S^3 intersecting γ in a_1, and a ball in S^3 intersecting C_i at a single unknotted arc. A connected sum operation is obtained by removing both balls and gluing their boundaries through a homeomorphism in a way that the boundary points of a_1 are mapped to the boundary points of the chosen arc in C_i. A similar operation is obtained from the arc a_2 and C. From these operations

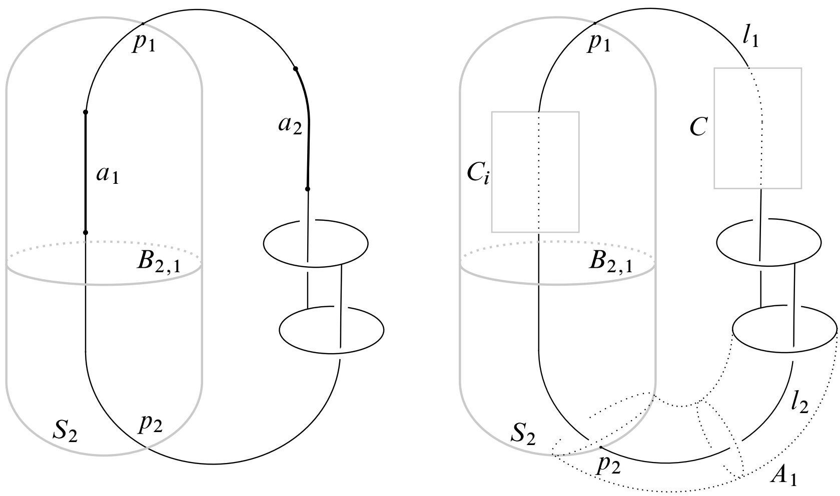

Figure 3: The arcs a_1 and a_2 in γ and the sphere S_2 (left); the spines γ_i of the handlebody-knots Γ_i and the annulus A_1 (right). Note that C_i and C label the pattern of the respective knots.

we get the handlebody-knots as represented schematically in Figure 3 (right), which we denote by Γ_i with a respective spine γ_i. For each handlebody-knot Γ_i we consider the swallow-follow torus X_i defined by the connected sum of C with C_i. A minimal JSJ–decomposition for the complement of Γ_i is defined by the torus X_i, cutting from $E(\Gamma_i)$ the exterior of $C_i \# C$, and a JSJ–decomposition of $E(C_i \# C)$. Also, the torus X_i cuts from $E(\Gamma_i)$ the only component obtained from the JSJ–decomposition containing the boundary of $E(\Gamma_i)$. Hence, from the unicity of minimal JSJ–decomposition of compact 3–manifolds, for any other minimal JSJ–decomposition of $E(\Gamma_i)$, the torus cutting the component with the boundary of $E(\Gamma_i)$ is isotopic to X_i. Consequently, if Γ_i is ambient isotopic to Γ_j for $i \neq j$, the torus X_i is isotopic to X_j, which means that $E(C_i \# C)$ is ambient isotopic to $E(C_j \# C)$. This is a contradiction with the torus $C_i \# C$ and $C_j \# C$ being distinct. Then, the handlebody-knots Γ_i are not ambient isotopic.

Both loops e_1 and e_2 cobound an embedded annulus in $B_{2,2}$, parallel to the component of e in $B_{2,2}$ each encircles, with interior disjoint from γ_i and intersecting S_2 in the other boundary component. Consider such an annulus with a boundary component in e_1, denoted A_1, as it is illustrated in Figure 3 (right). We proceed with an isotopy of γ_i along A_1, taking l_1 passing through S_2, and we obtain γ_i as in Figure 4 (left). We refer to this isotopy as an *annulus isotopy* of γ_i. After this isotopy we denote S_2 by S_3, considering its relative position with Γ_i, and the respective balls it bounds by $B_{3,1}$ and $B_{3,2}$. We assume that l_1 intersects S_3 at p_1. Note that all intersections

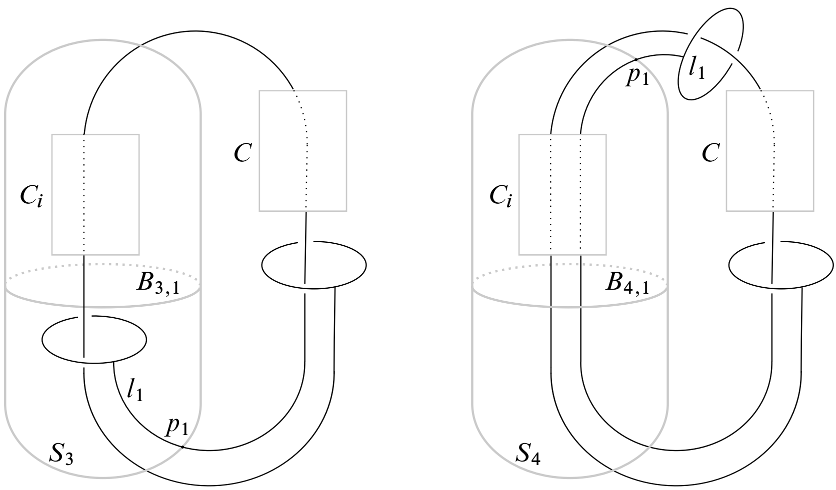

Figure 4: The spine γ_i after one (left), and two (right), annulus isotopies, and the spheres S_3 and S_4

of γ_i and S_3 are in the arc of e between p_1 and p_2. Again, we consider an embedded annulus A_2 in $B_{3,1}$, cobounded by e_1 and its intersection with S_3, parallel to the component of $e \cap B_{3,1}$ disjoint from e_1 and in the direction of the local knot C_i, following its pattern. By an annulus isotopy of γ_i along A_2 taking l_1 passing through S_3, we obtain γ_i as in Figure 4 (right). After this isotopy, we denote S_3 by S_4, considering its relative position with Γ_i, and the respective balls it bounds by $B_{4,1}$ and $B_{4,2}$. The ball $B_{4,1}$ intersects γ_i in two parallel arcs, and we still assume that $l_1 \cap S_4$ is p_1. Note again that all intersections of γ_i and S_4 are in the arc of e between p_1 and p_2.

For a canonical position, we isotope e_1 along the component of $e \cap B_{4,2}$, disjoint from e_1 and e_2, encircling l_2; see Figure 5 (left). We can now continue the previous process. Consider again an annulus A_3 in $B_{4,2}$, cobounded by e_1 and its intersection with S_4, parallel to the components of $e \cap B_{4,2}$ other than l_1, and in the opposite direction of the local knot C. By an annulus isotopy of γ_i along A_3, taking l_1 passing through S_4, we obtain γ_i as in Figure 5 (right). After this isotopy, we denote S_4 by S_5, considering its relative position with Γ_i, and we denote the balls it bounds by $B_{5,1}$ and $B_{5,2}$. Again, l_1 intersects S_5 at p_1, and all intersections of S_5 with γ_i are in the arc of e between p_1 and p_2. For the next step, proceed with an annulus isotopy along an annulus A_4 in $B_{5,1}$ cobounded by e_1, parallel to the components of $e \cap B_{5,1}$ disjoint from e_1, in the direction of the local knot C_i, following its pattern.

After $2(k-1)$ (for $k = 1, 2, \ldots$) annulus isotopies as the ones explained above, we get γ_i as in Figure 6 (left). From S_2, we obtain S_{2k} and the balls it bounds, $B_{2k,1}$ and $B_{2k,2}$. The ball $B_{2k,1}$ intersects γ_i in k parallel arcs with the pattern of C_i, and

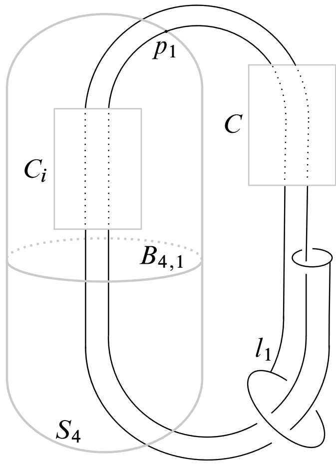

Figure 5: The spine γ_i of Figure 4 (left) in a canonical position (left), and γ_i after another annulus isotopy (right)

the ball $B_{2k,2}$ intersects γ_i in $k-2$ parallel arcs with the pattern of C, another arc with the pattern of C encircled by l_2, and l_1 that encircles all these other components.

After $2k-1$ (for $k=1,2,\dots$) annulus isotopies, we obtain γ_i as in Figure 6 (right). From S_2, we obtain S_{2k+1} and the balls it bounds, $B_{2k+1,1}$ and $B_{2k+1,2}$. The ball $B_{2k+1,1}$ intersects γ_i in k parallel arcs with the pattern of C_i and l_1 encircling these arcs, and the ball $B_{2k+1,2}$ intersects γ_i in $k-1$ parallel arcs with the pattern of C, together with another arc with the pattern of C and l_2 which encircles this arc.

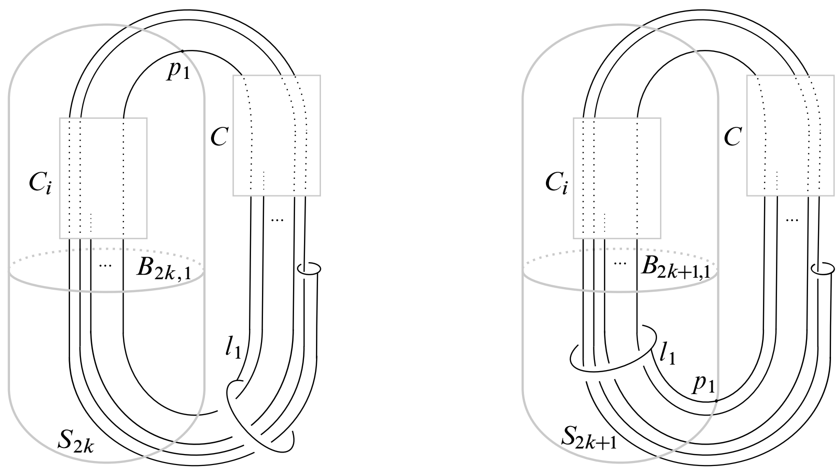

Figure 6: The spine γ_i after an even number (left), and an odd number (right), of annulus isotopies, and the corresponding spheres S_{2k} and S_{2k+1}, $k \in \mathbb{N}$

Note after each isotopy we assume that l_j intersects S_n, for $n = 2, 3, \dots,$ in p_j and that all points of $S_n \cap \gamma_i$ are in the arc between p_1 and p_2 in e.

We denote $S^3 - \operatorname{int} \Gamma_i$ by $E(\Gamma_i)$, and $S^3 - \gamma_i$ by $E(\gamma_i)$. Let Q_n, for $n = 2, 3, \dots,$ be the intersection of S_n with $E(\Gamma_i)$ in S^3.

Lemma 5 *The surface Q_n is incompressible in $E(\Gamma_i)$.*

Proof As Γ_i is a regular neighborhood of γ_i, if Q_n is compressible in $E(\Gamma_i)$, then S_n is compressible in $E(\gamma_i)$. Hence it suffices to prove that S_n is incompressible in $E(\gamma_i)$.

Case 1 Suppose n is even. Then S_n is as in Figure 6 (left).

(i) In this case, the ball $B_{n,1}$ intersects γ_i in a collection of $k = n/2$ parallel knotted arcs. Then $(B_{n,1}, B_{n,1} \cap \gamma_i)$ is an essential tangle. In fact, suppose there is a compressing disk D for S_n in $B_{n,1} - (B_{n,1} \cap \gamma_i)$. Then D separates the arcs $B_{n,1} \cap \gamma_i$ into two collections. Let s_1 and s_2 be two arcs in $B_{n,1}$ which are separated by D. As s_1 and s_2 are parallel, there is a disk E with boundary $s_1 \cup s_2$ and two arcs, α_1 and α_2, in S_n, each with one end in s_1 and the other in s_2. Consider D and E in general position and suppose that $|D \cap E|$ is minimal. If D intersects E in simple closed curves or in arcs with both ends in α_1 or both in α_2, we can reduce $|D \cap E|$ by an innermost arc type of argument, which is a contradiction. Therefore, all arcs of $D \cap E$ have one end in α_1 and the other end in α_2. Hence both s_1 and s_2 are parallel to outermost arcs of $D \cap E$ in D, which implies that s_1 and s_2 are parallel to S_n. This is a contradiction because the arcs s_1 and s_2 are knotted by construction.

(ii) If $n \leq 4$, then the ball $B_{n,2}$ intersects γ_i in l_1 and l_2, and when $n = 4$, also in an arc encircled by both l_1 and l_2. In this case, if there is a compressing disk for S_n in $B_{n,2} - (B_{n,2} \cap \gamma_i)$ it separates a component l_1 or l_2 from the other components. This implies that e_1 or e_2 bound a disk in the complement of γ_i, which is a contradiction with Γ_i being a knotted handlebody-knot. Otherwise, suppose that $n > 4$. Thus $B_{n,2}$ intersects γ_i in $(n/2) - 2$ parallel arcs with the pattern of C, another arc with the pattern of C encircled by l_2, and the component l_1 that encircles the arc encircled by l_1 and the $(n/2) - 2$ parallel arcs. With exception to l_1 and l_2, all other arcs are parallel as properly embedded arcs in $B_{n,2}$. Thus if a compressing disk for S_n in $B_{n,2} - (B_{n,2} \cap \gamma_i)$ separates these arcs, following an argument as in Case 1(i) we have a contradiction with these arcs being knotted. Therefore, a compressing disk for S_n in $B_{n,2} - (B_{n,2} \cap \gamma_i)$ separates a single component l_1 or l_2 from all the other components, or it separates both components l_1 and l_2 from the other parallel arcs. As e_1 bounds a disk disjoint from l_2, in both cases e_1 bounds a disk in the complement of γ_i, which is a contradiction with Γ_i being a knotted handlebody-knot.

Case 2 Suppose now that n is odd. Then S_n is as in Figure 6 (right).

(i) The ball $B_{n,1}$ intersects γ_i in a collection of $(n-1)/2$ parallel arcs and l_1 which encircles these arcs. If there is a compressing disk D of S_n in $B_{n,1}-(B_{n,1}\cap\gamma_i)$ separating the parallel arcs, following an argument as in Case 1(i) we have a contradiction with these arcs being knotted. If D separates the component l_1 from the other components, following an argument as in Case 1(ii) we have a contradiction with Γ_i being a knotted handlebody-knot.

(ii) If $n=3$, the ball $B_{n,2}$ intersects γ_i in an arc with pattern C and l_2 which encircles the arc. If there is a compressing disk for S_n in $B_{n,2}-(B_{n,2}\cap\gamma_i)$ in this case, then it separates the component l_2 from the arc with pattern C. From the same argument used in Case 1(ii), we have a contradiction with Γ_i being a knotted handlebody-knot. If $n>3$, then the ball $B_{n,2}$ intersects γ_i in $(n-1)/2$ parallel arcs and l_2 which encircles one of the previous arcs. Without considering l_2, if a compressing disk for S_n in $B_{n,2}-(B_{n,2}\cap\gamma_i)$ separates the parallel arcs, then following an argument as in Case 1(i) we have a contradiction with the arcs being knotted. If S_n has a compressing disk in $B_{n,2}-(B_{n,2}\cap\gamma_i)$, then this disk isolates the component l_2 from the other components, and following the argument as in Case 1(ii) we have a contradiction with Γ_i being a knotted handlebody-knot. □

The surface Q_n is boundary compressible in $E(\Gamma_i)$ as there are boundary compressing disks over the regular neighborhoods of l_1 and l_2. However, our construction of the handlebody-knots Γ_i could have been made in such a way that the surfaces Q_n are incompressible and boundary incompressible in their complements. For that purpose, we could do a connected sum of γ_i with two knots along two arcs in e_1 and e_2. After this operation, there won't be boundary compressing disks of Q_n over the regular neighborhoods of l_1 and l_2 in $E(\Gamma_i)$. And as these are the only possible boundary compressing disks, because all other components $\gamma_i-\gamma_i\cap S_n$ correspond to knotted arcs in their respective balls, after these connected sums the surfaces Q_n would also be boundary incompressible in the complement of the handlebody-knots. But for the purpose of this paper, we will use the handlebody-knots Γ_i.

3 Knots with essential tangle decompositions with an arbitrarily high number of strings

In this section, we use the handlebody-knots Γ_i to construct infinitely many examples of knots with essential tangle decompositions for all numbers of strings.

Let N_1 and N_2 be torus knots in the boundary of the solid tori T_1 and T_2 (that we assume to be in different copies of S^3). Consider a regular neighborhood B_i of an arc of N_i intersecting T_i at a ball, for $i=1,2$. We isotope B_i and $B_i \cap N_i$ away from the interior of T_i such that B_i intersects T_i at a disk, for $i=1,2$. We proceed with a connected sum of N_1 and N_2 by removing the interior of B_1 and attaching the exterior of B_2 in such a way that the disks $B_1 \cap T_1$ and $B_2 \cap T_2$ are identified. Hence the knot $N_1 \# N_2$, denoted by K, is in the boundary of a genus-two handlebody H, obtained by gluing T_1 and T_2 along a disk in their boundaries. We denote the identification disk of T_1 and T_2 in H by D. In Figure 7, we have the example of this connected sum with two trefoils, that we will use as reference for the remainder of the paper.

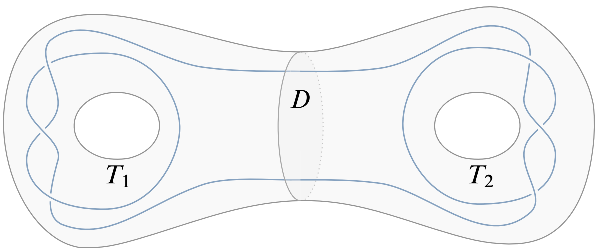

Figure 7: The handlebody H with the connected sum of two trefoil knots

Consider disks D_1 and D_2 parallel to D in H, such that the cylinder $C_{1,2}$ cut by $D_1 \cup D_2$ from H intersects K in two parallel arcs, each with one end in D_1 and the other in D_2. We also keep denoting by T_1 and T_2 the solid tori cut from H by D_1 and D_2, respectively; see Figure 8. Let s be a spine of H that intersects $C_{1,2}$ in a single arc. We denote by d_i the point $D_i \cap s$, and by t_i the intersection of s with T_i, for $i=1,2$.

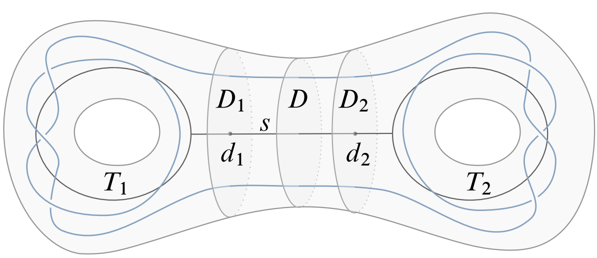

Figure 8: The handlebody H and the spine s with the connected sum of two trefoil knots

We now embed the knot K in Γ_i as follows. Consider an embedding h_i of H in S^3 taking H homeomorphically to Γ_i, such that $h_i(s)=\gamma_i$, $h_i(d_j)=p_j$, $h_i(t_j)=l_j$ and also that $h_i(T_j)=L_j$, for $j=1,2$.

Proof of Theorem 1 Denote by K_i the knots $h_i(K)$, $i \in \mathbb{N}$, for a fixed knot K. To prove that the handlebody-knots Γ_i are distinct, let X_i be the torus cutting from

$E(K_i)$ the exterior of $C_i \# C$. The component cut by X_i from $E(K_i)$ containing the boundary torus is the same for every knot K_i. Hence, from the unicity of minimal JSJ–decomposition of compact 3–manifolds, if two knots K_i and K_j are ambient isotopic, the tori X_i and X_j are also ambient isotopic, contradicting $C_i \# C$ and $C_j \# C$ being distinct. Thus the knots K_i define a collection of distinct knots.

To prove the statement of the theorem, we will show that the spheres S_n, for $n \geq 2$, define n–string essential tangle decomposition for the knots K_i, and that these knots are prime.

We start by proving that S_n defines an n–string essential tangle decomposition of K_i. Let $E(K_i)$ be the exterior of K_i in S^3; that is, $E(K_i) = S^3 - \operatorname{int} N(K_i)$. Let P_n be the intersection of S_n with $E(K_i)$ for a fixed n. To prove that S_n defines an essential tangle decomposition for K_i, we need to prove that P_n is essential in $E(K_i)$, ie that P_n is incompressible and boundary incompressible.

First, we observe that P_n is boundary incompressible. In fact, as the strings of $K \cap B_{n,i}$ in $B_{n,i}$ are knotted for $i = 1, 2$, there is no boundary compressing disk for P_n in $E(K_i)$.

Now we prove that P_n is incompressible in $E(K_i)$. Let Δ_j, for $j = 1, \dots, n$, be the disks of intersection between Γ_i and S_n with $\Delta_1 = L_1 \cap S_n$ and $\Delta_n = L_2 \cap S_n$. Denote by $C_{j,j+1}$ the cylinder cut by $\Delta_j \cup \Delta_{j+1}$ from Γ_i. Denote also by $\partial^* C_{j,j+1}$ the annulus $C_{j,j+1} \cap \partial\Gamma_i$; that is, $\partial^* C_{j,j+1} = \partial C_{j,j+1} - (\Delta_j \cup \Delta_{j+1})$. Note that $C_{j,j+1} \cap K$ is a collection of two arcs parallel to $\partial^* C_{j,j+1}$, each with one end in Δ_j and the other in Δ_{j+1}. We also let $\partial^* L_1$ and $\partial^* L_2$ denote $\partial L_1 - \Delta_1$ and $\partial L_2 - \Delta_n$. Furthermore, we denote by s_j the string component of the tangle decomposition of K_i defined by S_n, in L_j, for $j = 1, 2$. Note that s_j is parallel to $\partial^* L_j$. We isotope s_j into $\partial^* L_j$ and denote the annulus $\partial^* L_j \cap E(K_i)$ by Λ_j.

Suppose that P_n is compressible in $E(K_i)$ with D a compressing disk, properly embedded in $B_{n,1}$ or $B_{n,2}$, in general position with Γ_i. If D is disjoint from Γ_i, we have a contradiction with Lemma 5. In this way, we assume that D intersects Γ_i and that $|D \cap \partial\Gamma_i|$ is minimal over all isotopy classes of compressing disks of P_n in $E(K_i)$.

In particular, assume that D intersects an annulus $\partial^* C_{j,j+1}$. If $D \cap \bigcup_{j=1}^{n-1} \partial^* C_{j,j+1}$ contains a simple closed curve or an arc with both ends in the same disk of $\Gamma_i \cap S_n$, by considering an outermost one between such curves and arcs in $\partial^* C_{j,j+1}$, and by cutting and pasting along the disk it bounds or cobounds, we get a contradiction with the minimality of $|D \cap \partial\Gamma_i|$. Thus $D \cap \bigcup_{j=1}^{n-1} \partial^* C_{j,j+1}$ is a collection of arcs with ends in distinct disks of $\Gamma_i \cap S_n$. Consider an outermost arc of $D \cap \bigcup_{j=1}^{n-1} \partial^* C_{j,j+1}$ in D, say a, and without loss of generality, suppose it belongs to $\partial^* C_{j,j+1}$. The arc a

is parallel to a string of the tangle defined by S_n that is in $C_{j,j+1}$, which contradicts the fact that all strings of the tangle decomposition of K_i defined by S_n are knotted. Consequently, we can assume that $D \cap \bigcup_{j=1}^{n-1} \partial^* C_{j,j+1}$ is empty.

Then we are assuming that D intersects $\partial\Gamma_i$ at $\partial^* L_1$ or $\partial^* L_2$, or more precisely, at Λ_1 or Λ_2. We denote by a_j and a'_j the arcs of $\partial\Lambda_j$ parallel to s_j in $\partial^* L_j$, and by b_j and b'_j the arcs cut by ∂a_j and $\partial a'_j$, respectively, in the boundary of $\partial^* L_j$. The boundary components of Λ_j are $a_j \cup b_j$ and $a'_j \cup b'_j$. Note that, as $D \cap s_j$ is empty, the disk D is disjoint from a_j and a'_j. Note also that $a_j \cup b_j$ is a torus knot in the torus $\partial^* L_j \cup (S_n - L_j \cap S_n)$, denoted T'_j. If D intersects Λ_j in inessential simple closed curves or arcs with both ends in b_j or both ends in b'_j, then by cutting and pasting along a disk cut by such curve or arc, we have a contradiction with the minimality of $|D \cap \partial\Gamma_i|$. If D intersects Λ_j in an essential simple closed curve, then $a_j \cup b_j$ is parallel to a simple closed curve in D, which contradicts $a_j \cup b_j$ being knotted. Consequently, D intersects Λ_j in a collection of arcs, each with one end in b_j and the other in b'_j. Let O be an outermost disk in D cut by the arcs of $D \cap \Lambda_j$. Then O is a disk in a solid torus bounded by T'_j and intersects the torus knot $a_j \cup b_j$ in T'_j at a single point. As we are working in S^3, either O is parallel to T'_j or it is a meridian to a solid torus bounded by T_j. In either case, O intersects any torus knot in T'_j at least in two points, which contradicts O intersecting $a_j \cup b_j$ once.

Therefore, we have that P_n is essential in the complement of K_i, which ends the proof that S_n defines an n–string essential tangle decomposition of K_i.

Now we prove that the knots K_i are prime. From Theorem 1 of [1], if a knot has a 2–string prime tangle decomposition, that is, if the tangles are essential and with no local knots, then the knot is prime. We have that the knot K_i has a 2–string essential tangle decomposition defined by S_2. So to prove that it is prime, we just need to show that the tangle decomposition defined by S_2 has no local knots. The ball $B_{2,1}$ intersects K_i in two parallel arcs. Hence if there is a 2–sphere intersecting only one of the arcs at a single component, this component has to be unknotted. The ball $B_{2,2}$ intersects γ_i in l_1 and l_2; thus it intersects K_i at two strings each with the pattern of a torus knot. Note that even though the pattern of the knot C is in l_2, it does not affect the topological type of the string in L_2. Suppose the tangle in $B_{2,2}$ contains a local knot. That is, there is a ball Q intersecting only one of the strings, and at a knotted arc. As the torus knots are prime, this knotted arc contains the whole pattern of the string; that is, the intersections of Q and $B_{2,2}$ with this string are topologically the same. Therefore, as the strings in $B_{2,2}$ are parallel to the boundary of L_1 and L_2, and Q intersects only one of them, we have that Q contains either e_1 or e_2, or we can isotope e_1 and e_2 in such a way that Q contains either e_1 or e_2. But then, either e_1 or e_2 bound a disk in the complement of γ_i and, as in Case 1(ii) from the proof of Lemma 5, we have a contradiction with Γ_i being a knotted handlebody-knot. Consequently, the tangle decomposition defined by S_2 contains no local knots, and the knots K_i are prime. □

Corollary 2 is now an immediate consequence.

Proof of Corollary 2 In Theorem 1, we proved that the spheres S_n, for $n \geq 2$, define an n–string essential tangle decomposition for the knots K_i. Hence, considering the knots K_i connected sum with some other knot, we have infinitely many knots with n–string essential tangle decompositions for all $n \in \mathbb{N}$, as in the statement of this corollary. □

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References

[1] **S A Bleiler**, *Knots prime on many strings*, Trans. Amer. Math. Soc. 282 (1984) 385–401 MR

[2] **D Buck**, *DNA topology*, from "Applications of knot theory" (D Buck, E Flapan, editors), Proc. Sympos. Appl. Math. 66, Amer. Math. Soc., Providence, RI (2009) 47–79 MR

[3] **J H Conway**, *An enumeration of knots and links, and some of their algebraic properties*, from "Computational Problems in Abstract Algebra" (J Leech, editor), Pergamon, Oxford (1970) 329–358 MR

[4] **M Culler**, **C M Gordon**, **J Luecke**, **P B Shalen**, *Dehn surgery on knots*, Ann. of Math. 125 (1987) 237–300 MR

[5] **C Ernst**, **D W Sumners**, *A calculus for rational tangles: applications to DNA recombination*, Math. Proc. Cambridge Philos. Soc. 108 (1990) 489–515 MR

[6] **C M Gordon**, **A W Reid**, *Tangle decompositions of tunnel number one knots and links*, J. Knot Theory Ramifications 4 (1995) 389–409 MR

[7] **A Ishii**, **K Kishimoto**, **H Moriuchi**, **M Suzuki**, *A table of genus two handlebody-knots up to six crossings*, J. Knot Theory Ramifications 21 (2012) 1250035–1–9 MR

[8] **R C Kirby**, **W B R Lickorish**, *Prime knots and concordance*, Math. Proc. Cambridge Philos. Soc. 86 (1979) 437–441 MR

[9] **W B R Lickorish**, *Prime knots and tangles*, Trans. Amer. Math. Soc. 267 (1981) 321–332 MR

[10] **H Matsuda**, **M Ozawa**, *Free genus one knots do not admit essential tangle decompositions*, J. Knot Theory Ramifications 7 (1998) 945–953 MR

[11] **Y Mizuma**, **Y Tsutsumi**, *Crosscap number, ribbon number and essential tangle decompositions of knots*, Osaka J. Math. 45 (2008) 391–401 MR

[12] **M Ozawa**, *On uniqueness of essential tangle decompositions of knots with free tangle decompositions*, from "Proceedings of applied mathematics workshop, 8" (G T Jin, K H Ko, editors), Korea Adv. Inst. Sci. Tech., Taejŏn (1998) 227–232

[13] **D Rolfsen**, *Knots and links*, Mathematics Lecture Series 7, Publish or Perish, Berkeley, CA (1976) MR

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