Algebraic & Geometric Topology 15 (2015) 3047–3056

The Morava K–theory of $BO(q)$ and $MO(q)$

NITU KITCHLOO
W STEPHEN WILSON

We give an easy proof that the Morava K–theories for $BO(q)$ and $MO(q)$ are in even degrees. Although this is a known result, it had followed from a difficult proof that $BP^*(BO(q))$ was Landweber flat. Landweber flatness follows from the even Morava K–theory. We go further and compute an explicit description of $K(n)_*(BO(q))$ and $K(n)_*(MO(q))$ and reconcile it with the purely algebraic construct from Landweber flatness.

55R45, 55N15; 55N20, 55N22

1 Introduction

We are concerned with the (co)homology theory, Morava K–theory $K(n)^*(-)$, where $K(n)_* = \mathbb{Z}/2[v_n^{\pm 1}]$ with the degree of v_n equal to $2(2^n-1)$ (we are only concerned with $p=2$).

What brought us to the problem of computing the Morava K–theories of the spaces $BO(q)$ was a real need to have $BP^*(BO(q))$ be Landweber flat [5] for Kitchloo and Wilson's [2]. $BP^*(BO(q))$ had been computed by Wilson [9] and was shown to be Landweber flat by Kono and Yagita [3], with some seriously complex computations. They went on to show that $K(n)^*(BO(q))$ was concentrated in even degrees because $BP^*(BO(q))$ was.

The computation in [3] does not give an explicit answer to what $K(n)^*(BO(q))$ is, only that it is even degree. If it is known that $K(n)^*(BO(q))$ is even degree for all n, then the results of Ravenel, Wilson, and Yagita [7] show that $BP^*(BO(q))$ is Landweber flat, without having to compute it.

We present here an easy proof that $K(n)_*(BO(q))$ is even degree and then go further and give a basis. Duality for Morava K–theory is straightforward, so $K(n)^*(BO(q))$ is also even degree.

Theorem 1-1 [3] (i) *$K(n)^*(BO(q))$ and $K(n)^*(MO(q))$ are even degree for all n.*

(ii) *$BP^*(BO(q))$ is Landweber flat.*

Published: 10 December 2015 DOI: 10.2140/agt.2015.15.3047

As mentioned, (ii) follows directly from (i) using [7] but Kono and Yagita prove (ii) first and then (i). We prove (i) in Section 3.

We work with the homology version of the theories and have:

Theorem 1-2 (i) *There are elements* $b_{2i} \in K(n)_{2i}(BO(1))$ *for* $0 < i < 2^n$ *coming from* $K(n)_{2i}(\mathbb{RP}^\infty)$.

(ii) *There are elements* $c_{4i} \in K(n)_{4i}(BO(2))$ *for* $2^n \le i$.

(iii) *Using products from the standard maps* $BO(i) \times BO(j) \to BO(i+j)$, *a basis for the reduced homology* $\widetilde{K(n)}_*(BO(q))$ *is*

$$\{b_{2i_1} b_{2i_2} \cdots b_{2i_k} c_{4j_1} c_{4j_2} \cdots c_{4j_m}\},$$

where $0 < k + 2m \le q$, *and* $0 < i_1 \le i_2 \le \cdots \le i_k < 2^n \le j_1 \le j_2 \le \cdots \le j_m$.

(iv) $\widetilde{K(n)}_*(MO(q))$ *is as above with* $k + 2m = q$.

In [9], it was shown that

$$BP^*(BO(q)) \simeq BP^*[\![c_1, c_2, \ldots, c_q]\!]/(c_1 - c_1^*, c_2 - c_2^*, \ldots, c_q - c_q^*), \tag{1-3}$$

where c_j is the Conner–Floyd Chern class and c_j^* is its complex conjugate. In [3], Kono and Yagita show that $BP^*(BO(q))$ is Landweber flat and that

$$K(n)^*(BO(q)) \simeq K(n)^* \widehat{\otimes}_{BP^*} BP^*(BO(q)). \tag{1-4}$$

This shows that the Morava K–theory is even degree. We have computed Morava K–theory directly to show it is even degree, so the results of [7] also give us Landweber flatness for $BP^*(BO(q))$. Either approach gives us

$$K(n)^*(BO(q)) \simeq K(n)^*[\![c_1, c_2, \ldots, c_q]\!]/(c_1 - c_1^*, c_2 - c_2^*, \ldots, c_q - c_q^*). \tag{1-5}$$

This is a purely algebraic construct that looks nothing like the answer given in this paper. A direct independent proof of this can also be found in Kriz [4]. In Section 5 we reconcile it with our direct computation of $K(n)_*(BO(q))$ by finding a basis for it that is consistent with what we find for $K(n)_*(BO(q))$.

By Kriz [4], there are finite groups G with $K(2)^*(BG)$ not concentrated in even degrees. It seems conceivable that $BP^*(BG)$ may still be concentrated in even degrees for all compact Lie groups G. This is part of Yagita's conjecture [10, Conjecture 12.2] that for a complex algebraic group G (not necessarily connected), the Levine–Morel algebraic cobordism of BG maps isomorphically to $MU^*(BG)$. Yagita's conjecture in turn would imply a conjecture by Totaro [8] that for a complex algebraic group, the Chow ring of BG maps isomorphically to $MU^*(BG) \otimes_{MU^*} \mathbb{Z}$.

This paper is about a good situation where Totaro's conjecture is valid, by the calculation of $\mathrm{CH}^*BO(n)$ [8, Section 15] and Wilson's calculation of $MU^*(BO(n))$ [9].

The authors thank the referee for pointing out the connection with Totaro's work and Totaro for his help in describing the connection.

We review some facts about the standard homology of $BO(q)$ in Section 2 and prove the details of Theorem 1-2 in Section 4.

2 The standard homology of $BO(q)$ and $MO(q)$

We begin with some review of basic facts about the homology of BO and $BO(n)$. All of our (co)homology will be with $\mathbb{Z}/2$ coefficients. We start with elements

$$b_i \in \widetilde{H}_i(\mathbb{RP}^\infty = BO(1)), \quad i > 0.$$

We have

$$\widetilde{H}_*(BO(1)) = \mathbb{Z}/2\{b_i : i > 0\}$$

and maps

$$BO(1) \to \cdots \to BO(q-1) \to BO(n) \to \cdots \to BO.$$

The image of the above b_i in $H_*(BO)$ give us the well-known homology of BO as a polynomial algebra:

$$H_*(BO) = \mathbb{Z}/2[b_1, b_2, \ldots].$$

We also have the usual maps

$$BO(q) \times BO(k) \longrightarrow BO(q+k). \tag{2-1}$$

For homology we only need

$$\prod^q BO(1) \longrightarrow BO(q).$$

Because $b_i b_j = b_j b_i$, we have elements

$$b_{i_1} b_{i_2} \cdots b_{i_k} \in \widetilde{H}_*(BO(q)) \quad \text{for } 0 < k \le q \text{ and } 0 < i_1 \le i_2 \le \cdots \le i_k.$$

These elements form a basis for the reduced homology of $BO(q)$.

As an aside, if that is not commonly understood, we can quickly use the better-known cohomology of $BO(q)$ to see that the size is right. We have

$$H^*(BO(q)) = \mathbb{Z}/2[w_1, w_2, \ldots, w_q]$$

as a polynomial algebra on the Stiefel–Whitney classes. If, by induction, we know $H_*(BO(q-1))$, all we have to do to see the size is right is show that the elements with $k = q$ above are in one-to-one correspondence with the ideal generated by $w_q \in H^*(BO(q))$. That correspondence is easily given by

$$0 < i_1 \le i_2 \le \cdots \le i_q \;\mapsto\; w_q^{i_1} w_{q-1}^{i_2-i_1} w_{q-2}^{i_3-i_2} \cdots w_1^{i_q-i_{q-1}}.$$

The Steenrod algebra operates on the mod 2 homology of BO and $BO(q)$. As an element of the Steenrod algebra operates on an element $b_{i_1} b_{i_2} \cdots b_{i_k}$, it does not alter the number of b's, so we can define

$$M_q = \mathbb{Z}/2\{b_{i_1} b_{i_2} \cdots b_{i_q}\} \quad \text{for } 0 < i_1 \le i_2 \le \cdots \le i_q$$

and we get the reduced homology

$$\widetilde{H}_*(BO(q)) = \bigoplus_{j=1}^{q} M_j, \tag{2-2}$$

$$\widetilde{H}_*(BO) = \bigoplus_{j=1}^{\infty} M_j \tag{2-3}$$

as modules over the Steenrod algebra.

From [6] we know that stably $BO(q) \simeq \bigvee_{1 \le i \le q} MO(i)$, so stably we have

$$BO(q) \simeq BO(q-1) \vee MO(q).$$

From this we see that $M_q = H_*(MO(q))$.

3 The Morava K–theories of $BO(q)$ and $MO(q)$ are even

The first differential in the Atiyah–Hirzebruch spectral sequence (AHSS) $H_*(X; K(n)_*)$ is just the Milnor primitive Q_n, which is easy to evaluate in $H_*(BO(1))$ as it just takes b_{2k} to $b_{2k+1-2^{n+1}}$ as long as $2k > 2^{n+1} - 1$.

Remark 3-1 After the first differential, the AHSS collapses for $K(n)_*(BO(1))$ because the AHSS is even degree. The reduced homology is $K(n)_*$, which is free on $\{b_2, b_4, \ldots, b_{2^{n+1}-2}\}$.

Remark 3-2 More interesting is that after the first differential for BO we are also done, with the polynomial result, from the AHSS:

$$K(n)_*(BO) \simeq K(n)_*[b_2, b_4, \ldots, b_{2^{n+1}-2}] \otimes K(n)_*[b_{2i}^2 : i \ge 2^n],$$

which was done in [7]. The differential, or as we prefer to say, the Q_n homology, is computed by pairing up what is missing above as

$$P(b_{2i+1}) \otimes E(b_{2i+2^{n+1}}).$$

Each of these has trivial Q_n homology. This collapses after this first differential because it is even degree. Since b_{2i}, $i \geq 2^n$, is not an element, the notation is misleading. Later, we will give this generator the name c_{4i}. The element exists in $k(n)_*(BO)$ and reduces to b_{2i}^2 in $H_*(BO)$.

Proof of Theorem 1-1 Now we know that the first differential of the AHSS is all it takes to get $K(n)_*(BO)$ and see that it is all in even degrees. The first differential is just an operation from the Steenrod algebra Q_n. By Equation (2-3), we must have the Q_n homology of each M_j in even degrees. From this we see that $K(n)_*(BO(q))$ and $K(n)_*(MO(q))$ must be in even degrees, and by standard Morava K–theory duality, $K(n)^*(BO(q))$ is in even degrees. This completes the proof of Theorem 1-1. □

4 The details of the Morava K–theories of $BO(q)$ and $MO(q)$

All of the homology of $BO(q)$ came from products of elements from $BO(1)$. For Morava K–theory we have to use elements from $BO(2)$ as well.

Two kinds of elements in $K(n)_*(BO(2))$ come from $K(n)_*(BO(1))$. First we have the image coming from the map $BO(1) \to BO(2)$, ie $K(n)_*\{b_2, b_4, \dots, b_{2^{n+1}-2}\}$. Our second kind comes from the product $BO(1) \times BO(1) \to BO(2)$, which gives

$$K(n)_*\{b_{2i_1} b_{2i_2}\}, \quad 0 < i_1 \leq i_2 < 2^n.$$

There are more elements that come from M_2 in $K(n)_*(BO(2))$. In particular, from the computation of $K(n)_*(BO)$ we know that all b_{2j}^2 survive. These elements live in M_2, and so actually survive to $K(n)_*(BO(2))$. Consequently, between $K(n)_*(BO(1))$ and $K(n)_*(BO(2))$ we have all the multiplicative generators of $K(n)_*(BO)$. We easily see which M_q these multiple products live in by the number of b's.

We can now pretty much read off the description of a basis for $K(n)_*(BO(q))$. To make the description a little easier to read, we can consider the part that comes from M_q and call it $M_q^K = K(n)_*(MO(q))$. Then we have

$$K(n)_*(BO(q)) \simeq K(n)_*(BO(q-1)) \oplus K(n)_*(MO(q)).$$

We are not using the splitting from [6] to compute $K(n)_*(BO(q))$, only to compute $K(n)_*(MO(q))$.

Let's give new names to the elements in $K(n)_*(BO(2))$ represented by b_{2j}^2 so we won't have the nonexistent product hanging around. Let's set $c_{4j} = b_{2j}^2$ for $j \geq 2^n$.

We can now give an explicit description of $M_q^K = K(n)_*(MO(q))$ as

$$M_q^K \simeq K(n)_*\{b_{2i_1} b_{2i_2} \dots b_{2i_k} c_{4j_1} c_{4j_2} \dots c_{4j_m}\},$$

where $k + 2m = q$ and $0 < i_1 \leq i_2 \leq \dots \leq i_k < 2^n \leq j_1 \leq j_2 \leq \dots \leq j_m$. This completes the proof of Theorem 1-2.

There is still one bit of structure unaccounted for that we should mention. Although $K(n)_*(BO(q))$ is not an algebra, it is a coalgebra. The coalgebra structure for the b's comes from $BO(1)$, so for $p < 2^n$ we get

$$\psi(b_{2p}) = \sum_{i+j=p} b_{2i} \otimes b_{2j}.$$

The c_{4j} are written in terms of the b's in the AHSS, so we also know their coproduct modulo (v_n). It is just

$$\psi(c_{4p}) = \psi(b_{2p}^2) = \sum_{i+j=p} b_{2i}^2 \otimes b_{2j}^2 \quad \mathrm{mod}\ (v_n).$$

If $i \geq 2^n$, replace b_{2i}^2 with c_{4i}. Do the same with j. We can work modulo (v_n) because this single differential also computes $k(n)_*(BO(q))$ where we only have nonnegative powers of v_n.

We know that $K(n)_*(BO) \subset K(n)_*(BU)$. In [1], there are elements of $K(n)_*(BU)$ named z_q that are our $c_{4(2^n+q)}$. In [1, Theorem 3.14], the z_q are computed in terms of $K(n)_*(BU)$ modulo (v_n^2), and their complexity, and consequently the complexity of the coproduct, shows up here already. This is to be expected given the complexity of the dual algebra structure from Equation (1-5).

5 Reconciliation

The map $BO(q) \to BU(q)$ automatically gives a map of the algebraic construct on the right side of Equation (1-3) to $BP^*(BO(q))$. The work of [9] first involves showing the map is surjective, which is done with the Adams spectral sequence. To show injectivity, the algebraic construct is analyzed. We can use that analysis here to show what we want. We have to establish some notation first.

We have $BP^*(\mathbb{CP}^\infty) \simeq BP^*[\![x]\!]$, $x \in BP^2(\mathbb{CP}^\infty)$ and

$$\begin{array}{ccc} BP^*\Big(\prod^q \mathbb{CP}^\infty\Big) & \simeq & BP^*[\![x_1, x_2, \ldots, x_q]\!] \\ \cup & & \cup \\ BP^*(BU(q)) & \simeq & BP^*[\![c_1, c_2, \ldots, c_q]\!]. \end{array}$$

The inclusion is given by all of the symmetric functions, which are generated by the elementary symmetric functions given by the c_k.

For $I = (i_1, \ldots, i_q)$, let $x^I = x_1^{i_1} \cdots x_q^{i_q}$. Two monomials are equivalent if some permutation of the x_i takes one to the other. Define the symmetric function

$$s_I = \sum x^I,$$

where the sum goes over all monomials equivalent to x^I. The elementary symmetric function is $c_k = \sum x_1 \cdots x_k$. [9, Theorem 1.30, page 358] computes c_k^* for BP as

$$c_k^* = (-1)^k c_k + \sum_{i>0} v_i s_{2^i, \underbrace{1,1,\ldots,1}_{q-1}} \mod J^2,$$

where $J = (2, v_1, v_2, \ldots)$. We know that the generators of $BP^*(BO(q))$ all map nontrivially to the cohomology $H^*(BO(q))$. As a result, we can look at this relation using only the coefficients of $k(n)^* = \mathbb{Z}/2[v_n]$ and consider the relation modulo (v_n^2). Inductively, the only relation we need is $k = q$. This reduces to

$$c_q - c_q^* = v_n s_{2^n, \underbrace{1,1,\ldots,1}_{q-1}} \mod (v_n^2).$$

Note that for $BU(q)$, our relation is divisible by $c_q = x_1 \cdots x_q$, ie

$$s_{2^n,1,1,\ldots,1} = c_q s_{2^n-1}.$$

Because $K(n)^*(BU(q)) \simeq K(n)^* \widehat{\otimes} BP^*(BU(q))$, we can be quite sloppy with our powers of v_n because we are going to invert v_n to get our algebraic description in the end. The degree of v_n is negative, so the more powers of v_n, the higher the degree of the symmetric function.

The following theorem will reconcile our two different descriptions of $K(n)^*(BO(q))$.

Theorem 5-1 *A basis for $K(n)^*[\![c_1, \ldots, c_q]\!]/(c_1 - c_1^*, \ldots, c_q - c_q^*)$ in terms of symmetric functions is given by*

$$s_{IJ} = \sum x_1^{i_1} \cdots x_m^{i_m} x_{m+1}^{j_1} \cdots x_{m+p}^{j_p},$$

where $0 < i_1 < \cdots < i_m < 2^n$ and $0 \le j_1 \le \cdots \le j_p$ with $j_{2i-1} = j_{2i}$ and $m + p \le q$.

Remark 5-2 The definition forces p to be even. If we drop the $i_m < 2^n$ condition, any s_K can be written in this form. First, just find all the pairs of equal exponents and create J. Finding I is easy after that.

Remark 5-3 All we do in our proof is reduce arbitrary elements to those in our theorem. Because we know $K(n)_*(BO(q))$, we know that there can be no further reduction, so this is a basis. This does reconcile the two descriptions though.

Proof The proof is by double induction. First, it is by induction on q. This is easy to start with $q = 1$ where the result is well known and straightforward, but worth talking about anyway as it illustrates things to come in the proof.

The relation in $k(n)^*(BU(1))$ that gives $k(n)^*(BO(1))$ and then $K(n)^*(BO(1))$ is just $0 = c_1 - c_1^* = v_n s_{2^n} = v_n x^{2^n}$ modulo (v_n^2). The induction is on the degree of the symmetric function, which in this case is just powers of x. Inverting v_n, we see that x^{2^n} is zero modulo higher powers of x.

For any $s_{2^n+k} = x^{2^n+k}$, we have

$$0 = s_{2^n} s_k = s_{2^n+k} \quad \text{mod higher powers of } x.$$

That is, each s_{2^n+k} is zero modulo higher degree symmetric products. By induction on the degree of the symmetric product (ie induction on k) we push the relation to higher and higher degrees. In the topology on $K(n)^*(BU(1)) \simeq K(n)^*[\![x]\!]$, this converges to zero, and so each s_{2^n+k}, $k \geq 0$, is really zero. We remind the reader that our relation isn't really $s_{2^n,1,\dots,1} = 0$ modulo higher degree symmetric functions. The relation has a v_n in front. Since our relation really is in $k(n)^*(-)$ because it comes from $BP^*(-)$, all powers of v_n are positive. Since we are going to invert v_n at the end to get $K(n)^*(-)$, we can be quite loose with our v_n's.

The same thing will happen in the general, arbitrary q case. However, for $q > 1$, there are nontrivial basis elements in high degrees, so this process doesn't have to go to zero in the limit, but could settle on a basis element. Either way, it works for our proof.

From our induction on q, we assume the result for $q-1$. From [6], we know that stably

$$BO(q) \simeq BO(q-1) \vee MO(q),$$
$$BU(q) \simeq BU(q-1) \vee MU(q).$$

From [9], we know that $BP^*(MO(q))$ is the ideal in $BP^*(BO(q))$ generated by c_q, and so the same is true for $K(n)^*(BO(q))$. Of course, the same is true for $BP^*(MU(q))$, $BP^*(BU(q))$ and $K(n)^*(BU(q))$. Consequently, we can focus our attention on the symmetric functions divisible by c_q when there are only q variables.

We know that $H^*(BU(q))$ is free on the symmetric functions s_I with $I=(i_1,\ldots,i_q)$. If all $i_k>0$, this is a basis for $H^*(MU(q))$ and if some are not greater than 0, they are part of the basis for $H^*(BU(q-1))$. This splitting is only additive, not multiplicative. Because there is no torsion, this is all also true for $BP^*(-)$, $k(n)^*(-)$ and $K(n)^*(-)$.

Our next induction is on the degree of the symmetric functions. We will show that elements not of the form in our theorem are zero modulo higher degree elements. We know that $K(n)^*(BO(q))$ is $K(n)^*(BU(q))$ modulo the relations already described and that $K(n)^*(BU(q))$ is just given by the usual symmetric functions. To prove our result, we will not mod out our relations, but work with $BU(q)$ and just describe how the relations accomplish what we want. This will suffice for our purposes. We begin our induction by noticing that all elements in degrees less than the degree of $s_{2^n,1,\ldots,1}=c_q s_{2^n-1}$ are in our desired basis. The only element in the degree of $s_{2^n,1,\ldots,1}$ not in the basis is our relation element, which is zero modulo higher degree symmetric functions (ignoring the v_n as discussed above).

An arbitrary element not of the form in the theorem simply has $i_m\geq 2^n$ instead of $i_m<2^n$. Having fixed a degree, we first consider the cases where $i_m=2^n+k$, with $k>0$. Since we are working with elements divisible by c_q, we can divide by c_q to get a new symmetric function, $s_{I'J'}$, with each i_s replaced by i_s-1 and the same for the j_s. This symmetric function has $i'_m=2^n+k-1$. Since $k>0$, this is known to be zero modulo higher degrees by our induction on degree. Multiplying by c_k to get our original symmetric function, we see it must be zero modulo higher degrees. Note that we are using our induction on q here. If i_1 or j_1 (or both) are equal to 1, then $s_{I'J'}$ is in $K(n)^*(BU(q-1))$ because it is not divisible by c_q. By our induction, we know the behavior of the relations here.

In our fixed degree, we have eliminated all of the bad elements except those with $i_m=2^n$. From such a symmetric function s_{IJ}, we create a new symmetric function $s_{I'J'}$ by eliminating the $x_m^{i_m}=x_m^{2^n}$ term and subtracting 1 from all of the other i_s and j_s. We want to analyze

$$s_{2^n,1,1,\ldots,1}s_{I'J'}.$$

Since $s_{2^n,1,1,\ldots,1}$ is zero modulo higher degrees, this product is too. Multiplying symmetric functions can be tricky because the result can be a sum of symmetric functions. The easy one to deal with is when i_1 and j_1 are greater than one (recall that $m+p=q$). In this case, if your x^{2^n} term is multiplied by any power of x, we are in the situation where our product has x^{2^n+k}, with $k>0$, and we have dealt with those terms already. The only thing left is to multiply the x^{2^n} back into the place it was removed from and then all of the other exponents are raised by 1, giving us back our original s_{IJ}.

Things are slightly more complicated if i_1 or j_1 is 1. (They must be at least 1 because everything is divisible by c_q.) Again, if our x^{2^n} is multiplied by a nonzero power of x, we get x^{2^n+k} and these terms have been handled already. Our x^{2^n} must hit an x^0 term, but by the definition of symmetric functions, these are all equivalent, so the other x_i all just have their exponent raised by 1 in our product and we get our s_{IJ} back, showing it is zero modulo higher degrees. □

References

[1] **N Kitchloo**, **G Laures**, **W S Wilson**, *The Morava K–theory of spaces related to BO*, Adv. Math. 189 (2004) 192–236 MR2093483

[2] **N Kitchloo**, **W S Wilson**, *The $ER(n)$–cohomology of $BO(q)$ and real Johnson–Wilson orientations for vector bundles*, Bull. London Soc. Math. 47 (2015) 835–847

[3] **A Kono**, **N Yagita**, *Brown–Peterson and ordinary cohomology theories of classifying spaces for compact Lie groups*, Trans. Amer. Math. Soc. 339 (1993) 781–798 MR1139493

[4] **I Kriz**, *Morava K–theory of classifying spaces: Some calculations*, Topology 36 (1997) 1247–1273 MR1452850

[5] **P S Landweber**, *Homological properties of comodules over $MU_*(MU)$ and $BP_*(BP)$*, Amer. J. Math. 98 (1976) 591–610 MR0423332

[6] **S A Mitchell**, **S B Priddy**, *A double coset formula for Levi subgroups and splitting BGL_n*, from: "Algebraic topology", (G Carlsson, R L Cohen, H R Miller, D C Ravenel, editors), Lecture Notes in Math. 1370, Springer, Berlin (1989) 325–334 MR1000386

[7] **D C Ravenel**, **W S Wilson**, **N Yagita**, *Brown–Peterson cohomology from Morava K–theory*, K–Theory 15 (1998) 147–199 MR1648284

[8] **B Totaro**, *The Chow ring of a classifying space*, from: "Algebraic K–theory", (W Raskind, C Weibel, editors), Proc. Sympos. Pure Math. 67, Amer. Math. Soc. (1999) 249–281 MR1743244

[9] **W S Wilson**, *The complex cobordism of BO_n*, J. London Math. Soc. 29 (1984) 352–366 MR744106

[10] **N Yagita**, *Coniveau filtration of cohomology of groups*, Proc. Lond. Math. Soc. 101 (2010) 179–206 MR2661245

Department of Mathematics, Johns Hopkins University
Baltimore, MD 21218, USA

nitu@math.jhu.edu, wsw@math.jhu.edu

Received: 12 November 2014 Revised: 2 February 2015