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Representation theory for the Križ model

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The natural action of the symmetric group on the configuration spaces $F(X,n)$ induces an action on the Križ model $E(X,n)$. The representation theory for this complex is studied and a big acyclic subcomplex which is $\mathcal{S}_n$–invariant is described.

55R80, 20C30; 55P62, 13A50

1 Introduction

The ordered configuration space of n points $F(X,n)$ of a topological space X is defined as

$$F(X,n) = \{(x_1, x_2, \dots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j\}.$$

For X a smooth complex projective variety, I Križ [16] constructed a rational model $E(X,n)$ for $F(X,n)$, a simplified version of the Fulton–MacPherson model [15].

Let us recall the construction of Križ. We denote by $p_i^*\colon H^*(X) \to H^*(X^n)$ and $p_{ij}^*\colon H^*(X^2) \to H^*(X^n)$ (for $i \neq j$) the pullbacks of the obvious projections and by m the complex dimension of X (for cohomology groups we use rational or complex coefficients). The model $E(X,n)$ is defined as follows: as an algebra $E(X,n)$ is isomorphic to the exterior algebra with generators G_{ij}, $1 \le i, j \le n$ (of degree $2m-1$) and coefficients in $H^*(X)^{\otimes n}$ modulo the relations

$$\begin{aligned} G_{ji} &= G_{ij}, & & \\ p_j^*(x) G_{ij} &= p_i^*(x) G_{ij}, & & i<j,\ x \in H^*(X), \\ G_{ik} G_{jk} &= G_{ij} G_{jk} - G_{ij} G_{ik}, & & i<j<k. \end{aligned}$$

The differential d is given by $d|_{H^*(X)^{\otimes n}} = 0$ and $d(G_{ij}) = p_{ij}^*(\Delta)$, where Δ denotes the class of the diagonal $w \otimes 1 + \cdots + 1 \otimes w \in H^*(X) \otimes H^*(X)$ and $w \in H^{2m}(X)$ is the fundamental class.

This model is a differential bigraded algebra $E(X,n) = \bigoplus_{k,q} E_q^k(X,n)$: the lower degree q (called the exterior degree) is given by the number of exterior generators G_{ij},

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and the upper degree k is given by the total degree; the multiplication is homogeneous and the differential has bidegree $\binom{+1}{-1}$:

$$E_q^k \otimes E_{q'}^{k'} \to E_{q+q'}^{k+k'}, \quad d\colon E_q^k \to E_{q-1}^{k+1}$$

In the next definition $G_{I_* J_*}$ is a product of exterior generators

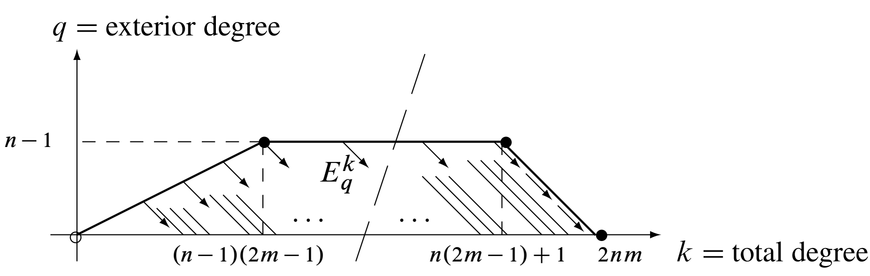
$$G_{I_* J_*} = G_{i_1 j_1} G_{i_2 j_2} \cdots G_{i_q j_q}.$$

Definition 1.1 (Fulton and MacPherson [15], Križ [16]) The symmetric group $\mathcal{S}_n$ acts on $E(X,n)$ by permuting the factors in $H^*(X^n) = H^{*\otimes n}$ and changing the indices of the exterior generators: for an arbitrary permutation $\sigma \in \mathcal{S}_n$,

$$\sigma(p_1^*(x_{h_1}) \cdots p_n^*(x_{h_n}) G_{I_* J_*}) = p_{\sigma(1)}^*(x_{h_1}) \cdots p_{\sigma(n)}^*(x_{h_n}) G_{\sigma(I_*)\sigma(J_*)}.$$

The action of $\mathcal{S}_n$ is well-defined because the set of relations is invariant under this action.

In the next diagram the nonzero bigraded components $E_q^k(X,n)$ lie in the trapezoid with vertices $(0,0)$, $(2mn,0)$, $((n-1)(2m-1), n-1)$ and $(n(2m-1)+1, n-1)$. The arrows show the direction of the differentials $(+1,-1)$:



For each q in the interval $[0, n-1]$ in the horizontal graded module $E_q^* = \bigoplus_k E_q^k$ the components equally distanced from the dotted median line are isomorphic as $\mathcal{S}_n$–modules; see Proposition 2.2 in Section 2. In the same section we introduce the combinatorial "types" of monomials of $E_*^*(X,n)$: these are parameterized by forests in which every tree contains a cohomology class of X. The type decomposition of the bigraded components gives a direct sum of $\mathcal{S}_n$ submodules, each of these being generated by a unique element; see Theorem 2.13. In the next section we describe the $\mathcal{S}_n$ structure of types: explicit decomposition into irreducible representations in many particular cases and, using the results of Lehrer and Solomon [19], we compute the character for the general type. See Propositions 3.1, 3.2, 3.3, 3.4 and Theorem 3.11.

In Section 4 we present some properties of the differential which are consequences of its $\mathcal{S}_n$–equivariance. For all smooth projective varieties, except the projective line, we show that the differential is injective on the "left side" of the trapezoid (the component $E_0^0 \cong \mathbb{Q}$ contributes to the cohomology group $H^0(F(X,n);\mathbb{Q}) \cong \mathbb{Q}$).

Proposition 1.1 *The differentials in the Križ model of a smooth complex projective variety different from $\mathbb{C}P^1$ are injective for any q in the interval $[1, n-1]$:*

$$d\colon E_q^{q(2m-1)}(X,n) \rightarrowtail E_{q-1}^{q(2m-1)+1}(X,n)$$

The top horizontal line has no contribution to the cohomology either, thus we have the following.

Proposition 1.2 *The top differentials in the Križ model are injective for any k in the interval $[(n-1)(2m-1), n(2m-1)+1]$:*

$$d\colon E_{n-1}^{k}(X,n) \rightarrowtail E_{n-2}^{k+1}(X,n)$$

In Section 5 we show that the "right side" of the trapezoid is an acyclic complex.

Proposition 1.3 *All cohomology groups of the subcomplex*

$$0 \to E_{n-1}^{n(2m-1)+1}(X,n) \to E_{n-2}^{n(2m-1)+2}(X,n) \to \cdots \to E_0^{2nm}(X,n) \to 0$$

are zero.

Other (smaller) copies of this subcomplex are contained in the interior of the trapezoid and their sum gives a large acyclic complex which is also $\mathcal{S}_n$–equivariant. This subcomplex $E_*^*(w(X,n))$ and the quotient $E_*^*(X,n) \big/ E_*^*(w(X,n))$ are described in Propositions 5.5 and 5.6; the location of the subcomplex $E_*^*(w(X,n))$ is given in the diagram by the interior lines with slope -1. In [6] a different acyclic subcomplex of $E_*^*(X,n)$ is described by the third author, Markl and Papadima: this is an ideal, but is not an $\mathcal{S}_n$–submodule. The subcomplex $E_*^*(w(X,n))$ is an $\mathcal{S}_n$–subalgebra, but not an ideal. The right side of the trapezoid, denoted by $E_*^{\mathrm{Top}}(X,n)$ in Section 5, is an acyclic ideal which is also an $\mathcal{S}_n$–submodule, but it is quite small.

In the last section the simplest and, from the viewpoint of Proposition 1.1, the exceptional case of $\mathbb{C}P^1$ is analyzed; we recover and we complete the results of Cohen and Taylor [11; 10] and Feichtner and Ziegler [13], computing in this case the Poincaré polynomials in two variables:

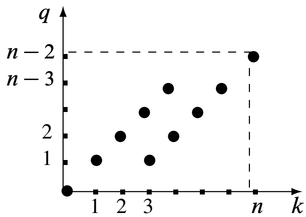
$$P_{F(X,n)}(t,s) = \sum_{k,q \geq 0} (\dim H_q^k) t^k s^q$$

Theorem 1.2 *In the cohomology algebra of the configuration space $F(\mathbb{C}P^1, n)$ $(n \geq 4)$ the nonzero bigraded components are*

$$H_q^q \cong H_{q+1}^{q+3} \quad \text{for } q = 0, 1, \ldots, n-3.$$

Its Poincaré polynomial is

$$P_{F(\mathbb{C}P^1,n)}(t,s) = (1+st^3)(1+2st)(1+3st)\cdots(1+(n-2)st).$$



For the irreducible $\mathcal{S}_n$–modules we will use the standard notation (see Fulton and Harris [14]): $V(\lambda)$ corresponds to the partition of n, $\lambda \vdash n$, $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_t \geq 1)$, and also the stable notation (see Church and Farb [9] or the authors [2]): $V(\mu)_n = V(n - \sum \mu_i, \mu_1, \mu_2, \ldots, \mu_s)$ for $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_s \geq 1)$ satisfying the relation $n - \sum_{i=1}^{s} \mu_i \geq \mu_1$.

Other extensions and applications of the results of this paper could be found by the first and third authors in [3; 4] and the second and third authors [5].

Recently Lambrechts and Stanley [17] constructed a (quasi)model for the configuration space of a topological space with Poincaré duality cohomology; if such a space is formal, the model of Lambrechts and Stanley is reduced to the Križ model and this is the case of Kähler manifolds; see Deligne, Griffiths, Morgan and Sullivan [12]. The results of this paper could be applied to (simply connected) formal closed manifolds (with few changes for the odd-dimensional manifolds).

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2 Cohomology classes in the forest

We will now fix a (monomial) ordered basis for the cohomology algebra $H^*(X; \mathbb{Q})$: $\mathcal{B} = \{x_1 = 1 \prec x_2 \prec \cdots \prec x_B = w\}$, where $w \in H^{2m}(X; \mathbb{Q})$ is the fundamental class of X and $B = \sum \beta_i$ is the sum of Betti numbers; we choose the order $\prec$ such that the sequence $\{\deg x_i\}_{i=1,B}$ is increasing (not necessarily strictly increasing). Using simple computations with Diamond lemma (see Bergman [7]) one can find a monomial basis for the Križ model: we denote by $G_{I_* J_*} = G_{i_1 j_1} G_{i_2 j_2} \cdots G_{i_q j_q}$ the exterior monomial corresponding to the sequences $I_* = (i_1, \dots, i_q)$, $J_* = (j_1, \dots, j_q)$, where $i_a < j_a$ $(a = 1, 2, \dots, q)$ and $j_1 < j_2 < \cdots < j_q$, and by $x_{H_*} = x_{h_1} \otimes x_{h_2} \otimes \cdots \otimes x_{h_n}$ $(x_{h_a} \in \mathcal{B})$ a scalar from $H^{*\otimes n}$. Then

$$\{x_{H_*} G_{I_* J_*} \mid x_{h_a} = 1 \text{ if } a \in J_*,\ \deg x_{H_*} = k - q(2m-1)\}$$

is a basis of $E_q^k(X, n)$ and we call it the *canonical* (Bezrukavnikov) basis (see Bezrukavnikov [8]).

The next result is obvious.

Proposition 2.1 *The bigraded components $E_q^k(X, n)$ are invariant under the action of the symmetric group and the differential d is $\mathcal{S}_n$–equivariant:*

$$d(\sigma(x_{H_*} G_{I_* J_*})) = \sigma(d(x_{H_*} G_{I_* J_*}))$$

This proposition and the Schur lemma give a splitting of the Križ complex into subcomplexes corresponding to the decomposition of E_*^* into isotypic components $E_*^*(V(\lambda))$, for λ an arbitrary partition of n:

$$(E_*^*(X, n), d) = \bigoplus_{\lambda \vdash n} (E_*^*(V(\lambda)), d_\lambda)$$

The cohomology algebra $H^*(X; \mathbb{Q})$ satisfies Poincaré duality; denote by $\mathcal{B}^*$ the Poincaré dual basis

$$\mathcal{B}^* = \{y^1 = w, y^2, \dots, y^B = 1 \mid \text{if } \deg x_i + \deg y^j = 2m, \text{ then } x_i y^j = \delta_{ij} w\}.$$

Proposition 2.2 *For any $q = 0, 1, \dots, n-1$ and any k in the interval of integers $[(2m-1)q, 2mn - q]$, there is an isomorphism of $\mathcal{S}_n$–modules*

$$E_q^k(X, n) \cong E_q^{2mn + 2q(m-1) - k}(X, n).$$

Proof Define the map Φ: $E_q^k \to E_q^{2mn+2q(m-1)-k}$ on the basis by

$$\Phi(x_{H_*} G_{I_* J_*}) = x'_{H_*} G_{I_* J_*},$$

where the factors of $x'_{H_*} = x'_{h_1} \otimes x'_{h_2} \otimes \cdots \otimes x'_{h_n}$ are given by

$$x'_{h_a} = \begin{cases} 1 (= x_{h_a}) & \text{if } h_a \in J_*, \\ y^{h_a} & \text{if } h_a \text{ is not in } J_*. \end{cases}$$

It is easy to see that Φ is $\mathcal{S}_n$–equivariant and the sum of the total degree of $x_{H_*}G_{I_*J_*}$ and the total degree of $\Phi(x_{H_*}G_{I_*J_*})$ is $2mn+2q(m-1)$. □

In [19] Lehrer and Solomon studied the representation theory of the Arnold algebra $\mathcal{A}^*(n)$, the cohomology algebra of $F(\mathbb{C},n)$ (see Arnold [1]).

Definition 2.1 The Arnold algebra $\mathcal{A}^*(n)$ is defined by

$$\mathcal{A}^*(n) = \bigwedge(G_{ij}, 1 \le i < j \le n) \big/ (G_{ij}G_{ik} - G_{ij}G_{jk} + G_{ik}G_{jk}),$$

where the generators G_{ij} have degree 1.

In Section 5 we will define a differential on $\mathcal{A}^*(n)$.

A basis of this algebra is given by monomials $G_{I_*J_*} = G_{i_1j_1}G_{i_2j_2}\cdots G_{i_qj_q}$, where $2 \le j_1 < j_2 < \cdots < j_q \le n$ and $1 \le i_a < j_a$ for any $a = 1,\dots,q$. The symmetric group $\mathcal{S}_n$ acts naturally on $\mathcal{A}^*(n)$: $\sigma.G_{I_*J_*} = G_{\sigma(I_*)\sigma(J_*)}$. To simplify the proofs, Lehrer and Solomon associated graphs to the monomials in $\mathcal{A}^*(n)$: to any monomial $G_{I_*J_*}$ from the Arnold algebra $\mathcal{A}^*(n)$, they associated the graph γ with vertices $\{1,2,\dots,n\}$ and edges $\{i,j\}$ corresponding to the factors G_{ij} of the given monomial.

Conversely, to any simple graph γ (ie no double edges, no loops) on the set $\{1,2,\dots,n\}$ one can associate an element of $\mathcal{A}^*(n)$: to the graph without edges we associate $1 \in \mathcal{A}^*(n)$; otherwise to any edge $\{i,j\}$ we consider the factor G_{ij} and take their product in the lexicographic order.

Remark 2.2 [19] If the graph γ contains a cycle, the associated element in $\mathcal{A}^*(n)$ is zero.

Proof Start an induction on the length of the cycle with length 3:

$$G_{ij}G_{ik}G_{jk} = G_{ij}(G_{ij}G_{jk} - G_{ij}G_{ik}) = 0$$

To a cycle of length $l+1$ corresponds an element containing as a factor the product

$$G_{i_1i_2}G_{i_2i_3}\cdots G_{i_li_{l+1}}G_{i_{l+1}i_1} = G_{i_1i_2}\cdots G_{i_{l-1}i_l}(G_{i_1i_l}G_{i_1i_{l+1}} - G_{i_1i_l}G_{i_li_{l+1}}),$$

both terms having associated graphs with cycles of length l. □

Now we extend Lehrer and Solomon's construction to the Križ model: we add to their construction "marks" which are elements of a (fixed) monomial basis of the cohomology algebra $H^*(X;\mathbb{Q})$.

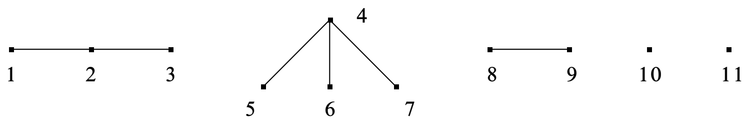
Definition 2.3 We associate to the monomial $\mu = p_1^*(x_{h_1})\cdots p_n^*(x_{h_n})G_{I_*J_*}$ from the canonical basis of $E_*^*(X,n)$:

(a) The Lehrer–Solomon graph γ associated to $G_{I_*J_*}$: vertices $1,\dots,n$ and for each factor G_{ij} we take the edge $\{i,j\}$.

(b) To each connected component of this γ we associate the "mark" x_{h_i}, the cohomology class lying on the position i, where i is the smallest index in the given connected component.

Example 2.4 (a) Consider the monomial $\mu_0 = G_{I_*J_*} = G_{12}G_{23}G_{45}G_{46}G_{47}G_{89} \in \mathcal{A}^6(11)$. Its associated Lehrer–Solomon graph γ is

γ:

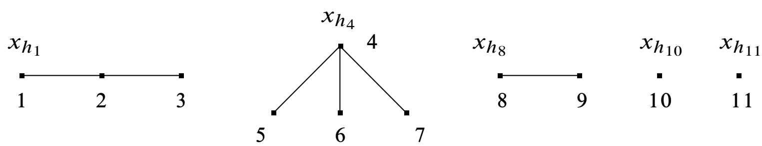


(b) Consider the monomial $\mu = x_{H_*}G_{I_*J_*} \in E_6^*(X,11)$ given by

$$x_{h_1}\otimes 1\otimes 1\otimes x_{h_4}\otimes 1\otimes 1\otimes 1\otimes x_{h_8}\otimes 1\otimes x_{h_{10}}\otimes x_{h_{11}} G_{12}G_{23}G_{45}G_{46}G_{47}G_{89}.$$

Its associated marked graph Γ is

Γ:



Conversely, we associate to a given marked (simple) graph Γ an element in $E_*^*(X,n)$: if there is no edge and the marks of the vertices $1,2,\dots,n$ are $x_{h_1},\dots,x_{h_n}$, the corresponding element is $x_{h_1}\otimes\cdots\otimes x_{h_n}\in E_0^*$; otherwise we take the product, in the lexicographic order, of the exterior factors G_{ij} corresponding to the edges $\{i,j\}$ and the scalar is the product of marks $x_{h_1}\otimes 1\otimes\cdots\otimes x_{h_i}\otimes\cdots$, where the factor x_{h_i} is the mark on the connected component having i as the smallest element (all the other factors are equal to 1).

Due to the acyclicity of Lehrer–Solomon graphs (see Remark 2.2), we will consider only marked forests (all the connected components are trees). If we restrict the correspondence $\{\text{monomials in } E_*^*(X,n)\}\to\{\text{marked graphs}\}$ to the canonical basis (Bezrukavnikov), we obtain only marked monotonic graphs.

Definition 2.5 A tree with vertices $\{1 \leq i_1 < i_2 < \cdots < i_p \leq n\}$ is *monotonic* if, for any vertex i_k, the unique path from i_1 to i_k is strictly increasing:

$$i_1 < i_a < i_b < \cdots < i_k$$

$$\overset{i_1}{\bullet}\!\!-\!\!\!-\!\!\overset{i_a}{\bullet}\!\!-\!\!\!-\!\!\overset{i_b}{\bullet}\ \cdots\ \bullet\!\!-\!\!\!-\!\!\overset{i_k}{\bullet}$$

(choosing the root i_1, the rooted tree is monotonic). A forest with vertices $\{1, 2, \ldots, n\}$ is *monotonic* if all its trees are monotonic.

Example 2.6 The tree

$$\overset{3}{\bullet}\!\!-\!\!\!-\!\!\overset{2}{\bullet}\!\!-\!\!\!-\!\!\overset{5}{\bullet}$$

is monotonic, but

$$\overset{3}{\bullet}\!\!-\!\!\!-\!\!\overset{5}{\bullet}\!\!-\!\!\!-\!\!\overset{2}{\bullet}$$

is not.

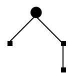
Remark 2.7 There is one to one correspondence

$$\{\text{monomials in the canonical basis of } E^*_*(X,n)\} \leftrightarrow \{\text{marked monotonic forests}\}.$$

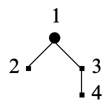
Proof Let us suppose that the graph Γ associated to a canonical monomial $G_{I_*J_*}$ ($j_1 < j_2 < \cdots < j_q, i_a < j_a$ for $a = 1, 2, \ldots, q$) is connected; from its Euler characteristic we find that $\operatorname{card}(I_* \cup J_*) = q + 1$. If Γ is not monotonic, there is a path $i_1 - \cdots - j - k - h$ such that $(i_1 \leq) j < k$, $k > h$, and this corresponds to a forbidden product $G_{jk}G_{hk}$ in $G_{I_*J_*}$. Conversely, to any monotonic tree (or forest) corresponds a product $G_{I_*J_*}$ from the canonical basis: a vertex j, distinct from the minimal vertex i in the same connected component, is joined with a unique vertex h, and this is smaller than j, namely the second last vertex on the path from i to j; therefore j appears only once on the second position, hence in the sequence J_*. □

There is an obvious action of the symmetric group $\mathcal{S}_n$ on the set of marked graphs: the natural action of $\mathcal{S}_n$ on the set of vertices $\{1, 2, \ldots, n\}$ induces an action on the set of edges and an action on the connected components and the corresponding marks. The set of monotonic marked forests is not $\mathcal{S}_n$–stable, like the set of monomials in the canonical basis of the Križ model. But $E^*_*(X,n)$ and the $\mathbb{Q}$ vector space which is generated by marked monotonic forests are $\mathcal{S}_n$–stable and these two vector spaces will be identified.

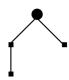
In the next examples two $\mathcal{S}_4$–orbits in the $\mathbb{Q}$–span of monotonic marked forests are described: $\oplus$ and $\ominus$ stand for the sum and difference in this vector space and $\tau_i = (i, i+1)$ $(i = 1, 2, 3)$ are the Coxeter generators of $\mathcal{S}_4$. In order to save space, we have that the bullet $\bullet$ corresponds to the vertex 1, the root of the tree, and the vertices connected to 1 are, from left to right, written in increasing order. Hence,



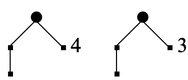
is the short form of:



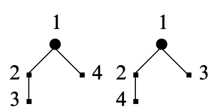
But the rooted tree



is ambiguous, and therefore we will write

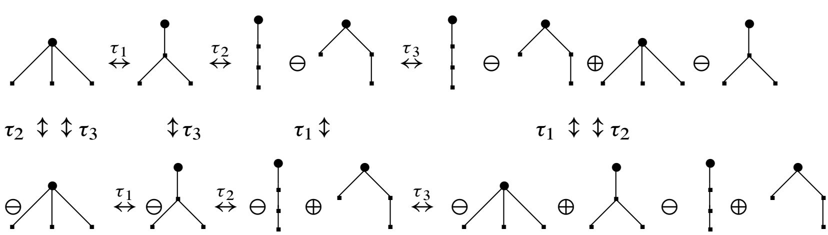


for the short form of

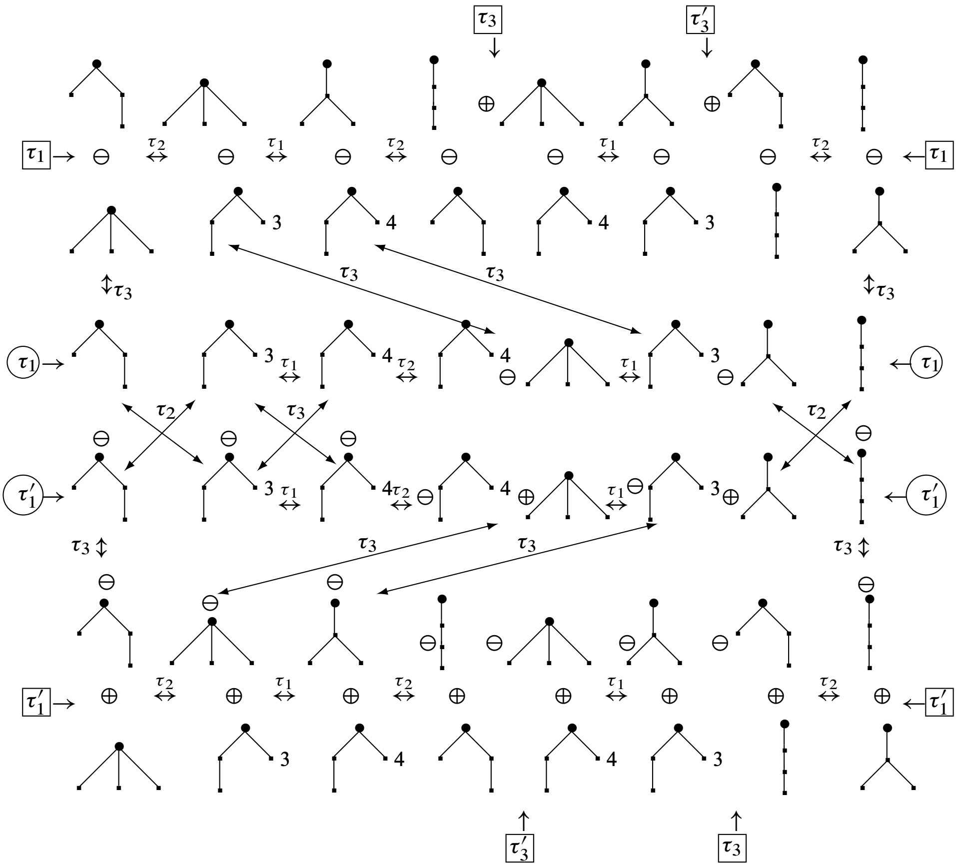


respectively. For the same reason the (unique) mark $x_{h_1} \in \mathcal{B}$ is omitted.

Example 2.8



Example 2.9



In the first example we have a "small" orbit: the dimension of the representation is 3 and it is isomorphic to $V(2,1,1)$. In the second one we see a "complete" orbit: the dimension is 6 and the representation is $V(3,1) \oplus V(2,1,1)$. The second example suggests the next definition.

Definition 2.10 We say that two monotonic marked forests are of *the same type*, $(\Gamma, H_*) \sim (\Gamma', H'_*)$, if there is a permutation $\sigma \in \mathcal{S}_n$ which induces a bijection between the connected components of the two graphs, preserving the number of elements of the corresponding components and their marks.

It is clear that marked monotonic forests in the same $\mathcal{S}_n$–orbit are of the same type, but not conversely. A complete system of representatives for this equivalence relation is given by forests of bamboos:

$$(\Gamma_{L_*},H_*):\quad \overset{x_{h_1}}{\underset{1}{\bullet}-\underset{2}{\bullet}-\cdots-\underset{L_1}{\bullet}}\qquad \overset{x_{h_2}}{\underset{L_1+1}{\bullet}-\cdots-\underset{L_2}{\bullet}}\quad\cdots\quad \overset{x_{h_t}}{\underset{L_{t-1}+1}{\bullet}-\cdots-\underset{L_t}{\bullet}}$$

where the sequence of lengths of the bamboos $L_* = (\lambda_1, \lambda_2, \dots, \lambda_t)$ is decreasing $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_t$, $L_i = \lambda_1 + \lambda_2 + \dots + \lambda_i$, $L_t = n$, and, for equal lengths, the marks are in a decreasing order. We split the $\mathcal{S}_n$–modules $E_q^k(X,n)$ into smaller pieces using the type of the associated monotonic marked forests.

Definition 2.11 For a given marked forest of bamboos (Γ_{L_*}, H_*) we define *the subspace of type* (L_*, H_*) as the linear span of monomials with the associated marked graph of type (Γ_{L_*}, H_*); this will be denoted by $E_*^*(L_*, H_*)$.

Example 2.12 For the type $(\Gamma_{(3,1,1)}, (h_1, h_2, h_3))$, where $x_{h_2} \succ x_{h_3}$,

$$(\Gamma_{L_*},H_*):\quad \overset{x_{h_1}}{\underset{1}{\bullet}-\underset{2}{\bullet}-\underset{3}{\bullet}}\qquad \overset{x_{h_2}}{\underset{4}{\bullet}}\quad \overset{x_{h_3}}{\underset{5}{\bullet}}$$

the associated space $E_*^*(L_*, H_*)$ is of dimension 40 and its canonical basis is given by the monomials

$$\begin{array}{cc}
x_{h_1} \otimes 1 \otimes 1 \otimes x_{h_2} \otimes x_{h_3} G_{12} G_{13}, & x_{h_1} \otimes 1 \otimes 1 \otimes x_{h_3} \otimes x_{h_2} G_{12} G_{13}, \\
x_{h_1} \otimes 1 \otimes 1 \otimes x_{h_2} \otimes x_{h_3} G_{12} G_{23}, & x_{h_1} \otimes 1 \otimes 1 \otimes x_{h_3} \otimes x_{h_2} G_{12} G_{23}, \\
\cdots & \cdots \\
x_{h_2} \otimes x_{h_3} \otimes x_{h_1} \otimes 1 \otimes 1 G_{34} G_{45}, & x_{h_3} \otimes x_{h_2} \otimes x_{h_1} \otimes 1 \otimes 1 G_{34} G_{45}.
\end{array}$$

If in this example $x_{h_2} = x_{h_3}$, the dimension of $E_*^*(L_*, H_*)$ is 20.

To a given type (L_*, H_*), $L_* = (\lambda_1, \dots, \lambda_t)$, $H_* = (h_1, \dots, h_t)$, we will associate two integers,

$$|L_*| = \sum_{i=1}^{t} (\lambda_i - 1), \quad |H_*| = \sum_{i=1}^{t} \deg(x_{h_i}).$$

Theorem 2.13 *The bihomogenous components $E_q^k(X,n)$ can be decomposed into a direct sum of monogenic $\mathcal{S}_n$–submodules*

$$E_q^k(X,n) = \bigoplus_{\substack{|L_*|=q \\ |H_*|=k-q(2m-1)}} E_q^k(L_*, H_*)$$

In particular, we have that the multiplicities of the irreducible $\mathcal{S}_n$–submodules of each term $E_q^k(L_*, H_*) = \bigoplus_{\lambda \vdash n} m_\lambda V(\lambda)$ satisfy the relations $m_\lambda \leq \dim V(\lambda)$.

Proof The $\mathcal{S}_n$–module $E^k_q(X,n)$ is the direct sum $\bigoplus E^k_q(L_*, H_*)$ by the very definition of monomials of type (L_*, H_*). We will show that

(a) for any type (L_*, H_*), the vector space $E(L_*, H_*)$ is $\mathcal{S}_n$–stable;

(b) a generator of the $\mathcal{S}_n$–module $E^*_*(L_*, H_*)$ is the monomial corresponding to the marked monotonic bamboo (L_*, H_*):

$$\mu_{(L_*,H_*)} = p^*_1(x_{h_1})p^*_{L_1+1}(x_{h_2})\cdots p^*_{L_{t-1}+1}(x_{h_t})\overline{G_{1L_1}}\cdot\overline{G_{L_1+1,L_2}}\cdots\overline{G_{L_{t-1}+1,L_t}}$$

(here $\overline{G_{ab}} = G_{a,a+1}G_{a+1,a+2}\cdots G_{b-1,b}$).

The last claim of the theorem is a consequence of the $\mathcal{S}_n$– equivariant surjection

$$\mathbb{Q}[\mathcal{S}_n] \to E(L_*, H_*), \quad \sigma \mapsto \sigma\cdot\mu_{(L_*,H_*)}.$$

To prove (a) it is enough to consider the action of the transpositions $\tau_i = (i, i+1)$ on the tree T containing the vertices $i, i+1$ or on the disjoint union of two trees, T' and T'', containing i and $i+1$ respectively. In the case of one tree, the transform $\tau_i T$ is again a monotonic tree (and $\tau_i T \sim T$) if the path from 1 to $i+1$ does not contain i. Otherwise the monotonic path $1-\cdots-h-i-(i+1)$ is transformed into $1-\cdots-h-(i+1)-i$ and the corresponding factor $G_{h,i+1}G_{i,i+1}$ should be replaced by $G_{h,i}G_{i,i+1}-G_{h,i}G_{h,i+1}$. The resulting monomials have monotonic trees of the same type with T. The case of two trees, i in T' and $i+1$ in T'', is simpler: $\tau_i(T'\sqcup T'')$ is a union of two monotonic trees ($h<i<j$ is equivalent to $h<i+1<j$) and obviously this union is of the same type with $T'\sqcup T''$.

It is enough to prove (b) for a monotonic tree: we will show by induction on n that the bamboo $B_n = \mu_{(L_*=(n),H_*=(x_h))}$ generate the module $\mathrm{Res}^{\mathcal{S}_n}_{\mathcal{S}_{n-1}} E^*_*(X,n)$. If σ is a permutation in $\mathcal{S}_{n-1}$, we denote by $\widetilde{\sigma}$ its extension to $\mathcal{S}_n$: $\widetilde{\sigma}(n) = n$. In the case $n=3$ there is a unique monotonic tree which is not the monotonic bamboo B_3, but this belongs to the $\mathcal{S}_2$–orbit of B_3:

$$2 - 1 - 3 = \tau_1(1 - 2 - 3) = \tau_1(B_3), \qquad \tau_1 \in \mathcal{S}_2.$$

Let us consider a monotonic tree T_n with n vertices and its monotonic subtree $T_{n-1} = T_n\setminus\{n\}$ (by monotonicity, the vertex n is connected to a unique other vertex, h). From the set of permutations $\pi\in\mathcal{S}_{n-1}$ with the property that $\pi(T_{n-1})$ is still monotonic we choose one such that $\pi(h) = j$ is maximal. Now we start a second induction on $n-j$, the number of vertices of $\pi(T_n)$ lying on the branches starting from j. If this number is equal to 1, then $j = n-1$ and, by induction on n there are permutations σ_a in $\mathcal{S}_{n-2}$ and constants $c_a \in \mathbb{Q}$ such that

$$\pi(T_{n-1}) = \sum_a c_a \sigma_a(B_{n-1}).$$

The extension of a permutation $\widetilde{\sigma}$ of a permutation in $\mathcal{S}_{n-2}$ does not change the edge $(n-1)-n$ and we find $\widetilde{\pi}(T_n) = \sum_a c_a \widetilde{\sigma}_a(B_n)$ and therefore

$$T_n = \sum_a c_a \widetilde{\pi^{-1}\sigma}_a(B_n), \quad \text{where } \pi^{-1}\sigma_a \in \mathcal{S}_{n-1}.$$

If j is less than $n-1$, then $j+1$ is connected to j (by the maximality condition); applying the transposition τ_j, we obtain a nonmonotonic tree $\tau_j\pi(T_n)$ (if $j \neq 1$). The sequence $k-(j+1)-j$ should be replaced: we obtain a difference of monotonic trees $T_n' - T_n''$, in each of them n is connected with $j+1$. The second induction will give two expansions

$$T_n' = \sum_p c_p' \widetilde{\sigma}_p'(B_n), T_n'' = \sum_q c_q'' \widetilde{\sigma}_q''(B_n)$$

with $\sigma_p', \sigma_q'' \in \mathcal{S}_{n-1}$. Therefore T_n is in $\mathbb{Q}[\mathcal{S}_{n-1}](B_n)$:

$$T_n = \sum_p c_p' \widetilde{\pi^{-1}\tau_j\sigma_p'}(B_n) - \sum_q c_q'' \widetilde{\pi^{-1}\tau_j\sigma_q''}(B_n).$$

In the case $j = 1$ the transposition τ_1 transforms $\pi(T_n)$ into a monotonic tree where n is connected with 2 and we can apply the second induction. □

Corollary 2.14 *The $\mathcal{S}_{n-1}$ orbit of the monomial $x_h \otimes 1 \otimes \cdots 1 G_{12} G_{23} \cdots G_{n-1,n}$ coincides with $E_{n-1}^{(n-1)(2m-1)+|x_h|}(n,h)$ and*

$$\mathrm{Res}_{\mathcal{S}_{n-1}}^{\mathcal{S}_n} E_{n-1}^{(n-1)(2m-1)+|x_h|}(n,h) \cong \mathbb{Q}[\mathcal{S}_{n-1}].$$

Proof In the proof of the theorem we obtained a surjective $\mathcal{S}_{n-1}$–map

$$\mathbb{Q}[\mathcal{S}_{n-1}] \to \mathrm{Res}_{\mathcal{S}_{n-1}}^{\mathcal{S}_n} E_{n-1}^{(n-1)(2m-1)+|x_h|}(n,h), \quad \sigma \mapsto \sigma \cdot B_n.$$

On the other hand the number of monotonic trees with n vertices is $(n-1)!$. □

3 The $\mathcal{S}_n$–module $E_q^k(L_*, H_*)$

In this section we will study the symmetric structure of the modules $E_q^k(L_*, H_*)$ using two methods. We extend the main result of Lehrer and Solomon [19] and we will give a general formula, but in an implicit form; next, in some particular cases, we will present explicit computations of the irreducible components and their multiplicities, using direct methods.

The symmetric structure of the bottom horizontal line is completely elementary: the types are given by $L_* = (1^{(n)}),\ H_* = (h_1^{(m_1)}, h_2^{(m_2)}, \dots, h_t^{(m_t)})$:

$$\underbrace{\overset{x_{h_1}}{\underset{1}{\blacksquare}} \qquad \overset{x_{h_1}}{\underset{2}{\blacksquare}}}_{m_1} \ \cdots \qquad \cdots \qquad \cdots \ \underbrace{\overset{x_{h_t}}{\underset{n-1}{\blacksquare}} \qquad \overset{x_{h_t}}{\underset{n}{\blacksquare}}}_{m_t}$$

(here $m_1, m_2, \dots, m_t$ are multiplicities of the elements $x_{h_1} \succ x_{h_2} \succ \cdots \succ x_{h_t}$ in $\mathcal{B}$ and $m_1 + \cdots + m_t = n$).

Example 3.1 In the case of distinct marks (all the multiplicities m_i are equal to 1) we obtain the largest possible "type" submodule

$$E_0^{|H_*|}(1^{(n)}, (h_1, h_2, \dots, h_n)) \cong \mathbb{Q}[\mathcal{S}_n].$$

Proof As $\sigma^{-1}(x_1 \otimes \cdots \otimes x_n) = \pm x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}$, the character of this module is given by

$$\chi_{E_0^{|H_*|}}(\sigma) = \begin{cases} n! & \sigma = \mathrm{id}, \\ 0 & \sigma \neq \mathrm{id}. \end{cases}$$

This completes the proof. □

Example 3.2 In the case of a unique mark $(m_1 = n)$

$$E_0^{n|x_h|}(1^{(n)}, h^{(n)}) \cong \begin{cases} V(n) & \text{if } |x_h| \text{ is even or } n = 1, \\ V(1, 1, \dots, 1) & \text{if } |x_h| \text{ is odd and } n \geq 2. \end{cases}$$

Proof This is the consequence of the relation

$$\tau_i(x_h \otimes \cdots \otimes x_h) = (-1)^{|x_h|}(x_h \otimes \cdots \otimes x_h).$$ □

For the general type corresponding to the discrete graph we will use the notation

$$V^{\epsilon(m,h)} = \begin{cases} V(m) & \text{if } m = 1 \text{ or } |x_h| \text{ is even,} \\ V(1,1,\dots,1) & \text{if } m \geq 2 \text{ and } |x_h| \text{ is odd.} \end{cases}$$

Proposition 3.1 *The $\mathcal{S}_n$ structure of the type (L_*, H_*), where $L_* = (1^{(n)})$, $H_* = (h_1^{(m_1)}, h_2^{(m_2)}, \dots, h_t^{(m_t)})$ is given by*

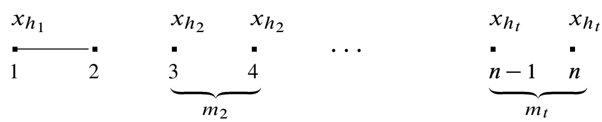
$$E_0^{|H_*|}(1^{(n)}, H_*) \cong \mathrm{Ind}_{\mathcal{S}_{m_1} \times \mathcal{S}_{m_2} \times \cdots \times \mathcal{S}_{m_t}}^{\mathcal{S}_n} V^{\epsilon(m_1,h_1)} \otimes V^{\epsilon(m_2,h_2)} \otimes \cdots \otimes V^{\epsilon(m_t,h_t)}.$$

Proof The symmetric group $\mathcal{S}_n$ acts transitively on the set of 1–dimensional spaces $\mathbb{Q}\langle x_1 \otimes \cdots \otimes x_n \rangle$, where m_1 factors (on different positions) coincide with $x_{h_1}, \dots, m_t$ factors coincide with x_{h_t}. The subgroup leaving invariant the 1–dimensional subspace

$$\underbrace{x_{h_1} \otimes \cdots \otimes x_{h_1}}_{m_1} \otimes \cdots \otimes \underbrace{x_{h_t} \otimes \cdots \otimes x_{h_t}}_{m_t}$$

is the direct product $\mathcal{S}_{m_1} \times \mathcal{S}_{m_2} \times \cdots \times \mathcal{S}_{m_t}$ (with the obvious notation: $\mathcal{S}_{m_1}$ acts on the subset $\{1, 2, \dots, m_1\}$, $\mathcal{S}_{m_2}$ acts on the subset $\{m_1+1, m_1+2, \dots, m_1+m_2\}$, and so on) and the corresponding representation of this subgroup is $V^{\epsilon(m_1,h_1)} \otimes V^{\epsilon(m_2,h_2)} \otimes \cdots \otimes V^{\epsilon(m_t,h_t)}$. General facts from the theory of induced representations (see for instance Serre [21, Chapter 7]) imply the result. □

On the next horizontal line the types are given by a unique graph:



(here $x_{h_2} \succ \cdots \succ x_{h_t}$, but h_1 could be equal to one of $h_2, \dots, h_t$ and we have $2 + m_2 + \cdots + m_t = n$).

Proposition 3.2 *The $\mathcal{S}_n$ structure corresponding to the type $H_* = (h_1, h_2^{(m_2)}, \dots, h_t^{(m_t)})$, $L_* = (2, 1^{(n-2)})$, is given by*

$$E_1^{2m-1+|H_*|}(L_*, H_*) \cong \mathrm{Ind}_{\mathcal{S}_2 \times \mathcal{S}_{m_2} \times \cdots \times \mathcal{S}_{m_t}}^{\mathcal{S}_n} V(2) \otimes V^{\epsilon(m_2,h_2)} \otimes \cdots \otimes V^{\epsilon(m_t,h_t)}.$$

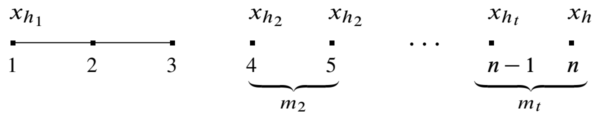
Proof This is similar to the previous proof: the module $E_*^*(L_*, H_*)$ is the direct sum of 1–dimensional subspaces $\mathbb{Q}\langle x_1 \otimes \cdots \otimes x_n G_{ij} \rangle$, where on the i^{th} position lies x_{h_1}, on the j^{th} position is 1, and x_{h_2} is lying on m_2 (arbitrary) positions, $\dots, x_{h_t}$ on m_t positions, and these subspaces are permuted by $\mathcal{S}_n$. The subgroup leaving invariant the line

$$\mathbb{Q}\Big\langle x_{h_1} \otimes 1 \otimes \underbrace{x_{h_2} \otimes x_{h_2} \otimes \cdots}_{m_2} \otimes \cdots \otimes \underbrace{\cdots \otimes x_{h_t}}_{m_t} G_{12} \Big\rangle$$

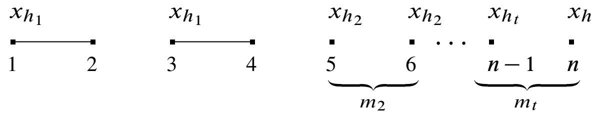
is the product $\mathcal{S}_2 \times \mathcal{S}_{m_2} \times \cdots \times \mathcal{S}_{m_t}$ and the representation on this line is equivalent with $V(2) \otimes V^{\epsilon(m_2,h_2)} \otimes \cdots \otimes V^{\epsilon(m_t,h_t)}$, hence the result. $\square$

The following types appear on the third horizontal line:

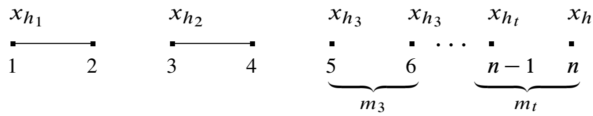
(a) $L_* = (3, 1^{(n-3)}),\ H_* = (h_1, h_2^{(m_2)}, \ldots, h_t^{(m_t)})$



(b) $L_* = (2^{(2)}, 1^{(n-4)}),\ H_* = (h_1^{(2)}, h_2^{(m_2)}, \ldots, h_t^{(m_t)})$



(c) $L_* = (2^{(2)}, 1^{(n-4)}),\ H_* = (h_1, h_2, h_3^{(m_3)}, \ldots, h_t^{(m_t)})$



(In the first two cases $h_2, \ldots, h_t$ are distinct and h_1 could be one of them, in the third case $x_{h_1} \succ x_{h_2}$, $x_{h_3} \succ \cdots \succ x_{h_t}$ and h_1 or h_2 are not necessarily different from $h_3, \ldots, h_t$).

Example 3.3 For $n \geq 6$, the module $E_2^{2(2m-1)}(L_* = (3, 1^{(n-3)}), H_* = (1, 1^{(n-3)}))$ has the stable decomposition

$$V(1)_n \oplus V(2)_n \oplus V(1,1)_n \oplus V(2,1)_n.$$

Proof Computing directly the corresponding character, we find for an arbitrary permutation $\sigma \in \mathcal{S}_n$ of type $(i_1; i_2; \ldots; i_n)$ (here i_q is the number of cycles of length q): for any triple of 1–cycles $(i)(j)(k)$, $1 \leq i < j < k \leq n$, σ fixes the monomials $G_{ij}G_{ik}$ and $G_{ij}G_{jk}$ and this gives $2\binom{i_1}{3}$ such monomials. The 3–cycle (i, j, k) changes the sign of $G_{ij}G_{jk}$ and there is no other combination of cycles leaving $\mathbb{Q}\langle G_{ij}G_{jk}\rangle$ or $\mathbb{Q}\langle G_{ij}G_{ik}\rangle$ invariant. Therefore the character is

$$\chi_{E_2^{2(2m-1)}(\bullet\!-\!\bullet\!-\!\bullet\,\bullet\cdots\bullet)}(i_1; \ldots; i_n) = 2\binom{i_1}{3} - i_3.$$

Using Frobenius' formula we obtain the next results:

	$\chi_V(i_1;\dots;i_n)$
$V(1)_n$	i_1-1
$V(1,1)_n$	$\binom{i_1-1}{2}-i_2$
$V(2)_n$	$\frac{1}{2}i_1(i_1-3)+i_2$
$V(3)_n$	$\frac{1}{6}i_1(i_1-1)(i_1-5)+i_2(i_1-1)+i_3$
$V(2,1)_n$	$\frac{1}{3}i_1(i_1-2)(i_1-4)-i_3$
$V(3,1)_n$	$\frac{1}{8}i_1(i_1-1)(i_1-3)(i_1-6)+i_2\binom{i_1-1}{2}-\binom{i_2}{2}-i_4$

As a consequence, we obtain the above decomposition. □

In the unstable cases, $n=3,4,5$, this module decomposes as

$$\begin{aligned} n=3: &\quad V(2,1),\\ n=4: &\quad V(3,1)\oplus V(2,2)\oplus V(2,1,1),\\ n=5: &\quad V(4,1)\oplus V(3,2)\oplus V(3,1,1). \end{aligned}$$

The same decompositions appear if H_* is replaced by $H_*=(h_1,h_2^{(n-3)})$ with $|x_{h_1}|$, $|x_{h_2}|$ even.

Example 3.4 The module $E_2^{2(2m-1)}(L_*=(2^{(2)},1^{(n-4)}),H_*=(1^{(2)};1^{(n-4)}))$, for $n\geq 7$, has the stable decomposition

$$V(1)_n\oplus V(2)_n\oplus V(1,1)_n\oplus V(3)_n\oplus V(2,1)_n\oplus V(3,1)_n.$$

Proof Direct computation of the character gives nonzero contributions only for factors of the form $(i)(j)(k)(l)$, $(i)(j)(k,l)$, $(i,j)(k,l)$ and (i,j,k,l); the result is

$$\chi_{E_*^*(\bullet-\bullet\bullet-\bullet\bullet\bullet\cdots\bullet)}(i_1;\dots;i_n)=3\binom{i_1}{4}+i_2\binom{i_1}{2}-\binom{i_2}{2}-i_4$$

and from the same table we obtain the above decomposition. □

In the unstable cases we obtain the decompositions

$$\begin{aligned} n=4: &\quad V(3,1),\\ n=5: &\quad V(4,1)\oplus V(3,2)\oplus V(3,1,1)\oplus V(2,2,1),\\ n=6: &\quad V(5,1)\oplus V(4,2)\oplus V(4,1,1)\oplus V(3,3)\oplus V(3,2,1). \end{aligned}$$

Example 3.5 In the case $n=4$ we have the decompositions

$$E_2^{2(2m-1)+2|x_h|}(\bullet_1^{x_h}-\bullet_2\,\bullet_3^{x_h}-\bullet_4)\cong\begin{cases}V(3,1) & \text{if } |x_h|=\text{even},\\ V(4)\oplus V(2,2) & \text{if } |x_h|=\text{odd}.\end{cases}$$

Proof The even case is the same as in the previous example; for the odd case direct computation of the character gives:

σ	id	(12)	(123)	(1234)	(12)(34)
$\chi(\sigma)$	3	1	0	1	3

For instance,

$$\begin{aligned}(1234)\,p_1^*(x_h)\,p_2^*(x_h)G_{13}G_{24} &= p_2^*(x_h)\,p_3^*(x_h)G_{24}G_{13}\\ &= -p_2^*(x_h)\,p_3^*(x_h)G_{13}G_{24}\\ &= -p_2^*(x_h)\,p_1^*(x_h)G_{13}G_{24}\\ &= (-1)^{|x_h|+1}\,p_1^*(x_h)\,p_2^*(x_h)G_{13}G_{24}.\end{aligned}$$

This completes the proof. □

Using some of these particular cases and the same proof as in Propositions 3.1 and 3.2, we obtain the following.

Proposition 3.3 *The $\mathcal{S}_n$ structure of the modules corresponding to the types on the third horizontal line is given by:*

(a) $$E_2^*\big(L_* = (3, 1^{(n-3)}), H_* = (h_1, h_2^{(m_2)}, \dots, h_t^{(m_t)})\big) \cong \mathrm{Ind}_{\mathcal{S}_3\times\mathcal{S}_{m_2}\times\cdots\times\mathcal{S}_{m_t}}^{\mathcal{S}_n}\big(V(2,1)\otimes V^{\epsilon(m_2,h_2)}\otimes\cdots\otimes V^{\epsilon(m_t,h_t)}\big)$$

(b) $$E_2^*\big(L_* = (2^{(2)}, 1^{(n-4)}), H_* = (h_1^{(2)}, h_2^{(m_2)}, \dots, h_t^{(m_t)})\big) \cong \begin{cases}\mathrm{Ind}_{\mathcal{S}_4\times\mathcal{S}_{m_2}\times\cdots\times\mathcal{S}_{m_t}}^{\mathcal{S}_n}\big(V(3,1)\otimes V^{\epsilon(m_2,h_2)}\otimes\cdots\otimes V^{\epsilon(m_t,h_t)}\big) & |x_{h_1}| = \text{even}\\ \mathrm{Ind}_{\mathcal{S}_4\times\mathcal{S}_{m_2}\times\cdots\times\mathcal{S}_{m_t}}^{\mathcal{S}_n}\big((V(4)\oplus V(2,2))\otimes V^{\epsilon(m_2,h_2)}\otimes\cdots\otimes V^{\epsilon(m_t,h_t)}\big) & |x_{h_1}| = \text{odd}\end{cases}$$

(c) $$\begin{aligned}E_2^*(L_* = (2^{(2)}, 1^{(n-4)}), H_* = (h_1, h_2, h_3^{(m_3)}, \dots, h_t^{(m_t)}))\\ \cong \mathrm{Ind}_{\mathcal{S}_2\times\mathcal{S}_2\times\mathcal{S}_{m_3}\times\cdots\times\mathcal{S}_{m_t}}^{\mathcal{S}_n}\big(V(2)\otimes V(2)\otimes V^{\epsilon(m_3,h_3)}\otimes\cdots\otimes V^{\epsilon(m_t,h_t)}\big)\\ \cong \mathrm{Ind}_{\mathcal{S}_4\times\mathcal{S}_{m_3}\times\cdots\times\mathcal{S}_{m_t}}^{\mathcal{S}_n}\big((V(4)\oplus V(3,1)\oplus V(2,2))\otimes V^{\epsilon(m_3,h_3)}\otimes\cdots\otimes V^{\epsilon(m_t,h_t)}\big)\end{aligned}$$

Now we will analyze the top horizontal line; we start with the upper-left vertex of the trapezoid, $E_{n-1}^{(n-1)(2m-1)}(X,n)$.

Example 3.6 For small values of n the structure of this module is a consequence of the absence (for $n \geq 2$) of the one-dimensional submodules of $E^*_{n-1}(X,n)$ and of the result of Section 2, $\mathrm{Res}^{\mathcal{S}_n}_{\mathcal{S}_{n-1}} E^{(n-1)(2m-1)}_{n-1}(X,n) \cong \mathbb{Q}[\mathcal{S}_{n-1}]$:

$$\begin{aligned}
n=2: &\quad E_1^{2m-1} \cong V(2),\\
n=3: &\quad E_2^{2(2m-1)} \cong V(2,1),\\
n=4: &\quad E_3^{3(2m-1)} \cong V(3,1) \oplus V(2,1,1),\\
n=5: &\quad E_4^{4(2m-1)} \cong V(4,1) \oplus V(3,2) \oplus V(3,1,1) \oplus V(2,2,1) \oplus V(2,1,1,1).
\end{aligned}$$

By Poincaré duality, the right side of the trapezoid has the same structure.

Example 3.7 For $n=6$ there are seven $\mathcal{S}_6$–modules without one-dimensional submodules satisfying $\mathrm{Res}^{\mathcal{S}_6}_{\mathcal{S}_5} E_5^{5(2m-1)} \cong \mathbb{Q}[\mathcal{S}_5]$. A direct combinatorial computation of the character of $E_5^{5(2m-1)}$ will give its nonzero values:

σ	id	(123456)	(123)(456)	(12)(34)(56)
$\chi(\sigma)$	120	-1	-3	8

Therefore we get the next (asymmetric) decomposition:

$$\begin{aligned}
E_5^{5(2m-1)} \cong &\quad V(5,1) \oplus 2V(4,2) \oplus V(4,1,1) \oplus 3V(3,2,1)\\
&\oplus V(2,2,2) \oplus 2V(3,1,1,1) \oplus V(2,2,1,1) \oplus V(2,1,1,1,1)
\end{aligned}$$

One can find a character table of $\mathcal{S}_6$ in Ledermann [18].

Alternatively, we can use the next general result of Stanley [22] and Lehrer and Solomon [19] and the Frobenius reciprocity formula. Let us recall some notation necessary to present the Lehrer–Solomon theorem: consider the subgroup generated by the n–cycle $c_n = (1,2,\dots,n)$, $\langle c_n \rangle$, the elementary character φ_n of this cyclic group, $\varphi_n(c_n) = e^{(2\pi i)/n}$ and ε_n, the sign character of $\mathcal{S}_n$. More generally, for a partition $L_* \vdash n$, $L_* = (\lambda_1 \geq \lambda_2 \geq \cdots \lambda_t \geq 1)$, let us denote by c_{L_*} the product of cycles $c_{\lambda_1} c_{\lambda_2} \cdots c_{\lambda_t}$, $c_{\lambda_1} = (1,2,\dots,L_1)$, $c_{\lambda_2} = (L_1+1, L_1+2, \dots, L_2), \dots,$ $c_{\lambda_t} = (L_{t-1}+1, L_{t-1}+2, \dots, L_t)$. Associated to L_* we have the next diagram of groups and characters of one-dimensional representations:

$$\begin{tikzcd}
& (\mathcal{S}_n, \varepsilon_n) \arrow[dl, dash] \arrow[dr, dash] & & & & \\
\mathcal{S}_{L_*} \arrow[dr, dash] & & (Z_{L_*}, \xi_{L_*}) \arrow[dl, dash] \arrow[ddrr, dash] & & & \\
& (C_{L_*}, \varphi_{L_*}) \arrow[dr, dash] & & & & \mathcal{S}_t \arrow[dl, dash] \\
& & \langle c_\lambda \rangle \arrow[dr, dash] & & (N_{L_*}, \alpha_{L_*}) \arrow[dl, dash] & \\
& & & 1 & &
\end{tikzcd}$$

where $\langle c_{L_*} \rangle$ is the subgroup generated by c_{L_*}, $\mathcal{S}_{L_*}$ is the direct product

$$\mathcal{S}_{L_*} \cong \mathcal{S}_{\lambda_1} \times \mathcal{S}_{\lambda_2} \times \cdots \times \mathcal{S}_{\lambda_t},$$

and C_{L_*} is the centralizer of c_{L_*} in $\mathcal{S}_{L_*}$:

$$C_{L_*} = \langle c_{\lambda_1} \rangle \times \langle c_{\lambda_2} \rangle \times \cdots \times \langle c_{\lambda_t} \rangle$$

The group N_{L_*} is generated by the elements v_i, "block transpositions" corresponding to equal parts $\lambda_i = \lambda_{i+1}$ in the partition L_*:

$$v_i = (L_{i-1}+1, L_i+1)(L_{i-1}+2, L_i+2)\cdots(L_i = L_{i-1}+\lambda_i, L_{i+1} = L_i + \lambda_{i+1})$$

The last group, Z_{L_*}, is the centralizer of c_{L_*} in $\mathcal{S}_n$ and it is a semidirect product

$$Z_{L_*} = C_{L_*} \rtimes N_{L_*}.$$

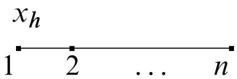
Lehrer and Solomon's main formula involves the next characters:

$$\begin{aligned}
\varphi_{L_*}\colon C_{L_*} \to \mathbb{C}^*, &\quad \varphi_{L_*} = (\varphi_{\lambda_1} \otimes \cdots \otimes \varphi_{\lambda_t}) \cdot \varepsilon_n \,|_{C_{L_*}} \\
\alpha_{L_*}\colon N_{L_*} \to \mathbb{C}^*, &\quad \alpha_{L_*}(v_i) = (-1)^{\lambda_i+1} \\
\xi_{L_*}\colon Z_{L_*} \to \mathbb{C}^*, &\quad \xi_{L_*} = \alpha_{L_*} \cdot \varphi_{L_*}
\end{aligned}$$

Theorem 3.8 (Stanley [22], Lehrer and Solomon [19]) *The representation of $\mathcal{S}_n$ on the top component $\mathcal{A}^{n-1}(n)$ of the Arnold algebra has the character*

$$\chi_{\mathcal{A}^{n-1}(n)} = \varepsilon_n \operatorname{Ind}_{\langle c_n \rangle}^{\mathcal{S}_n}(\varphi_n).$$

On the upper horizontal line the types are parameterized by the n–monotonic bamboo and the monomials from the fixed basis $\mathcal{B}$:



Proposition 3.4 *The $\mathcal{S}_n$ structure of the top component is given by*

$$\chi_{E_{n-1}^{(n-1)(2m-1)+|x_h|}(n,h)} \cong \varepsilon_n \operatorname{Ind}_{\langle c_n\rangle}^{\mathcal{S}_n}(\varphi_n),$$
$$\chi_{E_{n-1}^{(n-1)(2m-1)+i}(X,n)} \cong \beta_i \cdot \varepsilon_n \operatorname{Ind}_{\langle c_n\rangle}^{\mathcal{S}_n}(\varphi_n),$$

where β_i is the i^{th} Betti number.

Proof As the action of the symmetric group does not change the coefficients

$$\sigma(x_h \otimes 1 \otimes \cdots \otimes 1 G_{I_* J_*}) = p^*_{\sigma(1)}(x_h) \cdot \sigma(G_{I_* J_*}) = p^*_1(x_h) \cdot \sigma(G_{I_* J_*}),$$

we have the $\mathcal{S}_n$–decomposition

$$E_{n-1}^{(n-1)(2m-1)+i}(X,n) \cong \bigoplus_{x_h \in \mathcal{B} \cap H^i(X)} p^*_1(x_h) \cdot \mathcal{A}^{n-1}(n). \qquad \square$$

Example 3.9 If the number of points is an odd prime number p, then the multiplicities m_λ of the irreducible $\mathcal{S}_p$–modules are given by

$$E_{p-1}^{(p-1)(2m-1)+|x_h|}(p,h) \cong \bigoplus_{\lambda \vdash p} m_\lambda V(\lambda),$$
$$m_\lambda = \tfrac{1}{p}(\dim V(\lambda) - \chi_\lambda(c_p)).$$

Proof From the Frobenius reciprocity formula we obtain the expansion (here $V(\lambda^\varepsilon) = V(1,1,\dots,1) \otimes V(\lambda)$ with character $\chi_{\lambda^\varepsilon}$)

$$\begin{aligned} m_\lambda = \langle \chi_V, \varepsilon_p \operatorname{Ind}_{\langle c_p\rangle}^{\mathcal{S}_p}(\varphi_p)\rangle_{\mathcal{S}_p} &= \langle \chi_{\lambda^\varepsilon}, \operatorname{Ind}_{\langle c_p\rangle}^{\mathcal{S}_p}(\varphi_p)\rangle_{\mathcal{S}_p} \\ &= \langle \operatorname{Res}_{\langle c_p\rangle}^{\mathcal{S}_p} \chi_{\lambda^\varepsilon}, (\varphi_p)\rangle_{\mathbb{Z}_p} = \frac{1}{p}\sum_{k=0}^{p-1} \chi_{\lambda^\varepsilon}(c_p^k) e^{(2k\pi i)/p}, \end{aligned}$$

in which all the values $\chi_{\lambda^\varepsilon}(c_p^k)$ are equal but not the first one:

$$\chi_{\lambda^\varepsilon}(c_p^0) = \dim V(\lambda^\varepsilon) = \dim V(\lambda) \qquad \square$$

Now we consider the type $L_* = (\lambda_1, \lambda_2, \dots, \lambda_t)$, $H_* = (1^{(t)})$ $(|L_*| = \sum(\lambda_i - 1))$. Translated into the language of types, the main formula of Lehrer and Solomon is the following.

Theorem 3.10 (Lehrer and Solomon [19]) *The representation of $\mathcal{S}_n$ on the component $E_{|L_*|}^{|L_*|(2m-1)}(L_*, 1^{(t)})$ has the character*

$$\chi_{E_{|L_*|}^{|L_*|(2m-1)}(L_*,1^{(t)})} = \mathrm{Ind}_{Z_{L_*}}^{\mathcal{S}_n}(\xi_{L_*}).$$

To extend this to a general type $L_* = (\lambda_1, \dots, \lambda_t)$, $H_* = (h_1, h_2, \dots, h_t)$, we modify the notation of Lehrer and Solomon as follows: the group $N_{(L_*,H_*)}$ is generated by the elements v_i corresponding to the transposition of equal marked bamboos: $\lambda_i = \lambda_{i+1}$, $h_i = h_{i+1}$ (remember that for equal lengths $\lambda_c = \lambda_{c+1} = \dots = \lambda_d$, the corresponding marks are decreasing, not necessarily strictly: $x_{h_c} \succeq x_{h_{c+1}} \succeq \dots \succeq x_{h_d}$); of course, v_i are given by the same product of disjoint transpositions.

The subgroup $Z_{(L_*,H_*)}$ is defined by the same formula,

$$Z_{(L_*,H_*)} = C_{L_*} \rtimes N_{(L_*,H_*)},$$

but now is, in general, smaller than the centralizer of c_{L_*} in $\mathcal{S}_n$. The character $\varphi_{L_*}\colon C_{L_*} \to \mathbb{C}^*$ is given by the same formula: for a permutation $\sigma \in \mathcal{S}_{L_*} = \mathcal{S}_{l_1} \times \mathcal{S}_{l_2} \times \dots \times \mathcal{S}_{l_t}$, the sign of $\sigma(x_{h_1} \otimes 1 \otimes \dots \otimes x_{h_2} \otimes 1 \cdots G_{I_* J_*})$ is given only by the permutation of the exterior factors G_{ij}, like in Lehrer and Solomon definition of φ_{L_*}. The coefficients $\dots \otimes x_{h_i} \otimes 1 \dots \otimes x_{h_i} \otimes 1 \otimes \cdots$ do have a contribution to the sign after the action of a permutation $\rho \in N_{(L_*,H_*)}$ if the degree $|x_{h_i}|$ is odd; therefore the character $\alpha_{(L_*,H_*)}$ should be modified as follows:

$$\alpha_{(L_*,H_*)} = \begin{cases} (-1)^{\lambda_i+1} & \text{if } |x_{h_i}| \text{ is even,} \\ (-1)^{\lambda_i} & \text{if } |x_{h_i}| \text{ is odd,} \end{cases}$$

and accordingly the character ξ is modified:

$$\xi_{(L_*,H_*)} = \varphi_{L_*} \cdot \alpha_{(L_*,H_*)}.$$

Finally we obtain the character of the Križ algebra $E_*^*(X,n)$.

Theorem 3.11 (a) *The $\mathcal{S}_n$ representation of the submodule $E_*^*(L_*, H_*)$ has the character*

$$\chi_{E_*^*(L_*,H_*)} = \mathrm{Ind}_{Z_{(L_*,H_*)}}^{\mathcal{S}_n}(\xi_{(L_*,H_*)}).$$

(b) *The $\mathcal{S}_n$ representation of the component $E_q^k(X,n)$ has the character*

$$\chi_{E_q^k(X,n)} = \sum_{\substack{|L_*|=q \\ |H_*|+q(2m-1)=k}} \mathrm{Ind}_{Z_{(L_*,H_*)}}^{\mathcal{S}_n}(\xi_{(L_*,H_*)}).$$

(c) *The $\mathcal{S}_n$ representation of the Križ algebra $E_*^*(X,n)$ has the character*

$$\chi_{E_*^*(X,n)} = \sum_{(L_*,H_*)} \operatorname{Ind}_{Z_{(L_*,H_*)}}^{\mathcal{S}_n}(\xi_{(L_*,H_*)}).$$

Proof For a given partition $\Lambda_* = (\Lambda_1, \Lambda_2, \dots, \Lambda_t)$ of the set $\{1,2,\dots,n\}$, where $|\Lambda_i| = \lambda_i$, let us denote by $E_*^*(\Lambda_*, H_*)$ the span of monomials associated to the marked graphs having Λ_* the set of connected components and H_* the corresponding marks. The symmetric group $\mathcal{S}_n$ acts transitively on the components of the direct sum

$$E_*^*(L_*, H_*) = \bigoplus_{\Lambda_*} E_*^*(\Lambda_*, H_*).$$

Now we fix the term $E_*^*(\Lambda_*, H_*)$ with

$$\Lambda_* = (\{1,\dots,L_1\}, \{L_1+1,\dots,L_2\}, \dots, \{L_{t-1}+1,\dots,L_t\})$$

and we follow the proof of Lehrer and Solomon [19, Section 4]: the subgroup of $\mathcal{S}_n$ leaving $E_*^*(\Lambda_*, H_*)$ invariant is $Z_{(L_*,H_*)} = C_{L_*} \rtimes N_{(L_*,H_*)}$ (from the set v_i of permutations of connected components of equal size we have to consider only the permutations of components with the same mark). The action of this subgroup on $E_*^*(\Lambda_*, H_*)$ is a composition of

(C) the action of C_{L_*} on each component separately, and this is given componentwise by Proposition 3.4; its character is φ_{L_*};

(N) the action of $N_{(L_*,H_*)}$ which permutes identical marked trees; the rules of changing the sign were already explained.

The last two formulae of the theorem are direct consequences of the first one. □

4 Proofs of Propositions 1.1 and 1.2

In this section we will give a proof of Propositions 1.1 and 1.2. From the last section in [2] we will use the $\mathcal{S}_n$–decomposition of $E_1^{2m-1}(X,n) \cong \mathcal{A}^1(n)$ and also the bases for the irreducible $\mathcal{S}_n$–submodules described in this paper.

Proposition 4.1 [2] *The structure of the $\mathcal{S}_n$–module $E_1^{2m-1}(X,n)$ is given by*

$$E_1^{2m-1}(X,2) \cong V(2),$$
$$E_1^{2m-1}(X,3) \cong V(3) \oplus V(2,1),$$
$$E_1^{2m-1}(X,n) \cong V(n) \oplus V(n-1,1) \oplus V(n-2,2) \text{ for } n \geq 4.$$

We choose one nonzero element from each $\mathcal{S}_n$–submodule,

$$\begin{aligned}
G^n &= \sum_{i<j} G_{ij} \quad \text{in } V(n),\\
G^n_{12} &= \sum_{k\geq 3}(G_{1k}-G_{2k}) \quad \text{in } V(n-1,1),\\
G_{1234} &= G_{14}-G_{13}+G_{23}-G_{24} \quad \text{in } V(n-2,2),
\end{aligned}$$

and by direct computation we find nonzero differentials.

Lemma 4.1 *The images of the elements G^n and G^n_{12} under the composition*

$$E_1^{2m-1} \xrightarrow{d} E_0^{2m} \xrightarrow{pr} \bigoplus_{i=1}^{n} p_i^*(H^{2m})$$

are given by

$$\begin{aligned}
G^n &\mapsto (n-1)\sum_{i=1}^{n} p_i^*(w),\\
G^n_{12} &\mapsto (n-2)p_1^*(w)-(n-2)p_2^*(w).
\end{aligned}$$

Lemma 4.2 *Let x and y be cohomology classes of positive degree (x in $\mathcal{B}$ and y in the dual bases $\mathcal{B}^*$) such that $xy=w$. The image of the element G_{1234} under the composition*

$$E_1^{2m-1} \xrightarrow{d} E_0^{2m} \xrightarrow{pr} \bigoplus_{i<j} p_i^*(\mathbb{Q}(x))p_j^*(\mathbb{Q}(y))$$

is given by

$$G_{1234} \mapsto p_1^*(x)p_4^*(y)-p_1^*(x)p_3^*(y)+p_2^*(x)p_3^*(y)-p_2^*(x)p_4^*(y).$$

Proposition 4.2 *If X is a smooth complex projective variety ($X \neq \mathbb{C}P^1$), then the "first" differential is injective:*

$$d\colon E_1^{2m-1}(X,n) \rightarrowtail E_0^{2m}(X,n)$$

Proof Using twice the Schur lemma for the $\mathcal{S}_n$–morphisms

$$V(n), V(n-1,1) \hookrightarrow E_1^{2m-1}(X,n) \xrightarrow{d} E_0^{2m}(X,n),$$

these two submodules have isomorphic images through d in E_0^{2m} and trivial kernels because these morphisms are nonzero: $d(G^n)\neq 0$, $d(G^n_{12})\neq 0$. If X is of complex dimension m greater than two, we can take in Lemma 4.2 x to be the Kähler class and $y = x^{m-1}$; if X is a smooth projective curve, but not the projective line, the equation $xy = w$ has also nontrivial solutions. $\square$

Remark 4.3 In the remaining case, the differential

$$d\colon E_1^1(\mathbb{C}P^1, n) \to E_0^2(\mathbb{C}P^1, n)$$

is injective only for $n = 2, 3$; for $n \geq 4$ we obtain

$$H_1^1(F(\mathbb{C}P^1, n)) \cong V(2)_n.$$

We will see in the last section that in the case of $\mathbb{C}P^1$, $n \geq 4$, the differential $d\colon E_q^q(\mathbb{C}P^1) \to E_{q-1}^{q+1}(\mathbb{C}P^1)$ has a nontrivial kernel for $q = 1, 2, \ldots, n-3$ and it is injective for $q = n-2, n-1$.

Remark 4.4 As a consequence of [17], Propositions 4.2 and 1.1 are true for any formal space X whose rational cohomology satisfies Poincaré duality but not for cohomology spheres (for $n \geq 4$).

Now we can give a proof of Proposition 1.1 by induction on q.

Proof of Proposition 1.1 Let us suppose that the differential

$$d\colon E_{q-1}^{(q-1)(2m-1)} \to E_{q-2}^{(q-1)(2m-1)+1}$$

is injective and let $u \in E_q^{q(2m-1)}$ be a nonzero cocycle. Let G_{ij} be the smallest exterior generator (in the reverse lexicographic order $G_{12} < G_{13} < G_{23} < G_{14} < \cdots < G_{n-1,n}$) which appears in a nonzero monomial (of the canonical basis) in u: $u = G_{ij}y + z$, $y \in E_{q-1}(G_{\alpha\beta} > G_{ij}) \setminus \{0\}, z \in E_q(G_{\alpha\beta} > G_{ij})$. In the right hand side of the equation

$$0 = d(u) = -G_{ij}dy + p_{ij}^*(\Delta)y + dz$$

the last two terms, $p_{ij}^*(\Delta)y$ and dz, the monomials in the canonical basis contain only factors $G_{\alpha\beta} > G_{ij}$; therefore $dy = 0$ by induction on q, and $y = 0$, and this gives a contradiction. $\square$

Proof of Proposition 1.2 The monomials from the canonical basis lying on the top horizontal line are $p_1^*(x_h)G_{12}G_{i_2 3} \cdots G_{i_{n-1} n}$, where $|x_h| = i \in [0, 2m]$ and $i_a \leq a$.

The next composition is an isomorphism,

$$E_{n-1}^{(n-1)(2m-1)+i} \xrightarrow{d} E_{n-2}^{(n-1)(2m-1)+i+1} \\ \xrightarrow{pr} \bigoplus_{\substack{I_*=(i_2,\ldots,i_{n-1})\\ i_a\leq a}} p_1^*(H^i)p_2^*(H^{2m})G_{i_2 3}\cdots G_{i_{n-1}n},$$

$$p_1^*(x_h)G_{12}G_{i_2 3}\cdots G_{i_{n-1}n} \mapsto p_1^*(x_h)p_2^*(w)G_{i_2 3}\cdots G_{i_{n-1}n},$$

therefore the differential is injective on the top horizontal line. □

5 An acyclic subalgebra of the Križ model

In [6] is introduced a quotient of the Križ model, denoted by J_n, which is quasi-isomorphic to $E_*^*(X,n)$, but the corresponding kernel is not $\mathcal{S}_n$–stable. We will identify an acyclic subcomplex of $E_*^*(X,n)$, denoted by $E_*^*(w(X,n))$ (or simply by $E_*^*(w)$), which is also an $\mathcal{S}_n$–submodule and a subalgebra, giving another smaller complex quasi-isomorphic to the Križ model:

$$SE_*^*(X,n) = E_*^*\big/E_*^*(w), \quad H^*(E_*^*(X,n)) \cong H^*(SE_*^*(X,n))$$

The last isomorphism is now $\mathcal{S}_n$–equivariant.

We start with a well known result in the theory of hyperplane arrangements, see for example Orlik and Terao [20]; a simple proof is included.

Definition 5.1 We define the differential ∂ of degree -1 on the Arnold algebra $\mathcal{A}^*(n)$ by $\partial G_{ij} = 1$ and we call $(\mathcal{A}^*(n),\partial)$ the Arnold differential algebra.

Proposition 5.1 *The Arnold differential algebra $(\mathcal{A}^*(n),\partial)$ is acyclic.*

Proof Define the homotopy $h\colon \mathcal{A}^* \to \mathcal{A}^{*+1}$ by $h(\gamma) = G_{12}\gamma$ and verify that $\partial h + h\partial = \mathrm{id}_{\mathcal{A}^*}$. □

We denote by $E_*^{\mathrm{Top}}(X,n)$ the submodule of the Križ model $(E_*^*(X,n),d)$ given by the sum of the submodules of maximal total degree in each q–exterior degree

$$E_*^{\mathrm{Top}}(X,n) = \bigoplus_{q=0}^{n-1} E_q^{\mathrm{Top}}(X,n) = \bigoplus_{q=0}^{n-1} E_q^{2mn-q}(X,n).$$

It is obvious that $E_*^{\mathrm{Top}}(X,n)$, the right side of the trapezoid, is a subcomplex and an ideal of $E_*^*(X,n)$.

Proposition 5.2 *There is an isomorphism of chain complexes of $\mathcal{S}_n$–modules*

$$(\mathcal{A}^*(n), \partial) \cong (E_*^{\mathrm{Top}}(X,n), d).$$

In particular

$$H^*(E_*^{\mathrm{Top}}(X,n), d) = 0.$$

Proof Using the standard basis $\{G_{I_*J_*} = G_{i_1j_1}G_{i_2j_2}\cdots G_{i_qj_q}\}$ in $\mathcal{A}^q(n)$ and the basis $\{\prod_{h\notin J_*} p_h^*(w)G_{I_*J_*}\}$ in $E_q^{\mathrm{Top}}(X,n)$ (here $2\le j_1<j_2<\cdots<j_q\le n, 1\le i_a<j_a$) we define the isomorphism

$$f\colon \mathcal{A}^q \to E_q^{\mathrm{Top}}, \quad f(G_{I_*J_*}) = \prod_{h\notin J_*} p_h^*(w)G_{I_*J_*}.$$

Obviously f is $\mathcal{S}_n$–equivariant (the degree of w is even) and f preserves the differentials:

$$\begin{aligned}
df(G_{I_*J_*}) &= d\Big(\prod_{h\notin J_*} p_h^*(w)G_{I_*J_*}\Big)\\
&= \sum_{a=1}^{q}(-1)^{a+1}\prod_{h\notin J_*} p_h^*(w)\cdot p_{i_aj_a}^*(\Delta)G_{i_1j_1}\cdots\widehat{G_{i_aj_a}}\cdots G_{i_qj_q}\\
&= \sum_{a=1}^{q}(-1)^{a+1}\prod_{h\notin J_*} p_h^*(w)\cdot p_{j_a}^*(w)G_{i_1j_1}\cdots\widehat{G_{i_aj_a}}\cdots G_{i_qj_q}\\
&= \sum_{a=1}^{q}(-1)^{a+1}\prod_{h\notin J_*\setminus\{j_a\}} p_h^*(w)G_{i_1j_1}\cdots\widehat{G_{i_aj_a}}\cdots G_{i_qj_q}\\
&= f\Big(\sum_{a=1}^{q}(-1)^{a+1}G_{i_1j_1}\cdots\widehat{G_{i_aj_a}}\cdots G_{i_qj_q}\Big)\\
&= f\partial(G_{I_*J_*})
\end{aligned}$$

For the third equality we used $i_a\notin J_*$ and the equality

$$p_{i_a}^*(w)p_{i_aj_a}^*(\Delta) = p_{i_a}^*(w)p_{j_a}^*(w). \qquad \square$$

Proof of Proposition 1.3 Now this is obvious. $\square$

Now we will define three acyclic subcomplexes which generalize the previous subcomplex $(E_*^{\mathrm{Top}}(X,n), d)$. For a fixed nonempty subset $A\subset\{1,2,\dots,n\}$ of cardinality $|A| = a\ge 2$ and a fixed sequence β of length $b=n-a$, $\beta=(x_1,x_2,\dots,x_b)$, where all the elements x_j belong to the fixed basis $\mathcal{B}$ and are different from w, we denote the increasing sequence of elements in $\{1,2,\dots,n\}\setminus A$ by $b_1<b_2<\dots<b_b$, the product $\prod_{j=1}^{b} p_{b_j}^*(x_j)$ by $p^*(\beta)$, and its degree $\sum_{j=1}^{b}\deg(x_j)$ by $|\beta|$. Now we define the subspace

$$E_*^{\mathrm{Top}}(A,\beta)=\sum_{q=0}^{a-1} E_q^{2ma-q+|\beta|}(A,\beta)$$

by

$$E_q^{2ma-q+|\beta|}(A,\beta)=\mathbb{Q}\Big\langle \prod_{i\in A\setminus J_*} p_i^*(w)p^*(\beta)G_{I_*J_*} \;\Big|\; I_*\cup J_*\subset A, |J_*|=q\Big\rangle$$

(in words: the scalars in the "complementary positions," given by β, should be different from w, on the "forbidden positions," corresponding to J_*, there is only 1, and all the other "possible positions" should be filled with the top class w).

Proposition 5.3 *For any A and β as before, the space $E_*^{\mathrm{Top}}(A,\beta)$ is an acyclic subcomplex of the Križ model.*

Proof By definition $E_*^{\mathrm{Top}}(A,\beta)$ is the direct sum of its subspaces $E_q^{2ma-q+|\beta|}(A,\beta)$ and it is stable under the differential:

$$\begin{aligned} d\Big(\prod_{i\in A\setminus J_*} p_i^*(w)p^*(\beta)G_{I_*J_*}\Big) &= \sum_{j_h\in J_*} \pm \prod_{i\in A\setminus J_*} p_i^*(w)p_{j_h}^*(w)p^*(\beta)G_{I_*\setminus\{i_h\}J_*\setminus\{j_h\}} \\ &= \sum_{j_h\in J_*} \pm \prod_{i\in A\setminus (J_*\setminus\{j_h\})} p_i^*(w)p^*(\beta)G_{I_*\setminus\{i_h\}J_*\setminus\{j_h\}} \end{aligned}$$

This subcomplex is acyclic because of the isomorphism

$$(E_*^{\mathrm{Top}}(A,\beta),d)\cong(\mathcal{A}^*(a),\partial)$$

given by

$$\prod_{i\in A\setminus J_*} p_i^*(w)p^*(\beta)G_{I_*J_*}\longleftrightarrow(-1)^{\beta}G_{I_*J_*},$$

the differential of β is zero and the compatibility of the differentials d,∂ was checked in the proof of 5.2. □

Example 5.2 If $A=\{1,2,\dots,n\}$, then β is the empty sequence, $p^*(\beta)=1$ and $E_*^{\mathrm{Top}}(\{1,2,\dots,n\},\phi)=E_*^{\mathrm{Top}}(X,n)$.

Now we fix a number a from 2 to n and a sequence $\beta = (x_1, \dots, x_b)$ of length $b = n - a$ as before. We say that the sequence $\beta' = (y_1, \dots, y_b)$ is similar to β, $\beta \sim \beta'$, if there is a permutation $\sigma \in \mathcal{S}_b$ such that $y_i = x_{\sigma(i)}$ for $i = 1, 2, \dots, b$. We define a new subspace

$$E_*^{\mathrm{Top}}(a, \beta) = \sum_{|A|=a,\, \beta' \sim \beta} E_*^{\mathrm{Top}}(A, \beta').$$

Proposition 5.4 *For any number a and a sequence β as before, the space $E_*^{\mathrm{Top}}(a, \beta)$ is an acyclic subcomplex and $\mathcal{S}_n$–invariant.*

Proof The space is $\mathcal{S}_n$–invariant by construction,

$$\sigma(E_*^{\mathrm{Top}}(A, \beta)) = E_*^{\mathrm{Top}}(\sigma A, \sigma\beta) = E_*^{\mathrm{Top}}(\sigma A, \beta'),$$

and the acyclicity is a consequence of the direct sum decomposition

$$(E_*^{\mathrm{Top}}(a, \beta), d) = \bigoplus_{|A|=a,\, \beta' \sim \beta} (E_*^{\mathrm{Top}}(A, \beta'), d). \qquad \square$$

Finally, we take the whole collection of these subcomplexes,

$$E_*^*(w(X, n)) = \sum_{a=2}^{n} \sum_{\substack{\beta \text{ of length} \\ n-a}} E_*^{\mathrm{Top}}(a, \beta).$$

Proposition 5.5 *The space $E_*^*(w(X, n))$ is an acyclic, $\mathcal{S}_n$–invariant subcomplex.*

Proof It suffices to show the double sum is a direct sum: a monomial $x_1 \otimes \cdots \otimes x_n G_{I_* J_*}$ from the canonical basis in $E_*^*(w(X, n))$ defines in a unique way the subset A and the factor β:

$$A = \{i \in \{1, 2, \dots, n\} \mid x_i = w\} \cup J_*,$$
$$\beta = (x_{h_1}, x_{h_2}, \dots, x_{h_b}),$$

where $h_1 < h_2 < \cdots < h_b$ are the elements of $\{1, 2, \dots, n\} \setminus A$. $\square$

Proposition 5.6 *The projection map*

$$E_*^*(X, n) \longrightarrow SE_*^*(X, n) = E_*^*(X, n)/E_*^*(w(X, n))$$

is a quasi-isomorphism.

Proof This is obvious from the long exact sequence associated to

$$0 \longrightarrow E_*^*(w(X, n)) \longrightarrow E_*^*(X, n) \longrightarrow SE_*^*(X, n) \longrightarrow 0. \qquad \square$$

6 An example: $F(\mathbb{C}P^1, n)$

We analyze the cohomology algebra of the configuration space of the complex projective line using the symmetric structure of the Križ model. We encode the symmetric structure of a bigraded $\mathcal{S}_n$–module H_*^* into the $\mathcal{S}_n$–*Poincaré polynomial*:

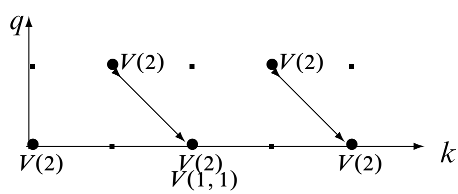
$$SP_{H_*^*}(t,s) = \sum_{\lambda \vdash n} \Big(\sum_{k,q} m_{q,\lambda}^k t^k s^q \Big) V(\lambda),$$

where we have that $m_{q,\lambda}^k$ is the multiplicity of the irreducible representation $V(\lambda)$ in the component H_q^k; the double Poincaré polynomial of H_*^* is a consequence of $SP_{H_*^*}$: $P_{H_*^*}(t,s) = \sum_{k,q} (\sum_{\lambda \vdash n} m_{q,\lambda}^k \dim V(\lambda)) t^k s^q$.

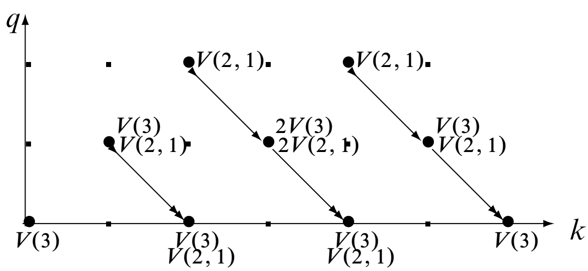
For $n = 2$ and $n = 3$ we have the next tables of the symmetric group structure of the Križ model; using the injectivity properties of the differential, we obtain the first table and for the second table we have to use the vanishing of the cohomology on the left, top and the right side and also the acyclicity of the "interior part" $\bigoplus_{|A|=2} E_*^{\mathrm{Top}}(A,1)$:

$$\begin{aligned} V(3) \oplus V(2,1) &\cong \langle w \otimes 1 \otimes 1 G_{12}, w \otimes 1 \otimes 1 G_{13}, 1 \otimes w \otimes 1 G_{23} \rangle \\ &\overset{d}{\underset{\cong}{\to}} \langle w \otimes w \otimes 1, w \otimes 1 \otimes w, 1 \otimes w \otimes w \rangle \end{aligned}$$

$E_*^*(\mathbb{C}P^1, 2)$:



$E_*^*(\mathbb{C}P^1, 3)$:



As a consequence we obtain the following.

Lemma 6.1 *The nonzero components of the cohomology algebra of* $F(\mathbb{C}P^1, 2)$ *and* $F(\mathbb{C}P^1, 3)$ *are*

$$H_0^0(F(\mathbb{C}P^1, 2)) \cong V(2), \quad H_0^2(F(\mathbb{C}P^1, 2)) \cong V(1,1),$$
$$H_0^0(F(\mathbb{C}P^1, 3)) \cong V(3), \quad H_1^3(F(\mathbb{C}P^1, 3)) \cong V(3).$$

In particular their symmetric Poincaré polynomials are

$$P_{F(\mathbb{C}P^1,2)}(t,s) = V(2) + t^2 V(1,1),$$
$$P_{F(\mathbb{C}P^1,3)}(t,s) = (1 + st^3)V(3).$$

Corollary 6.2 *The Poincaré polynomials of the unordered configuration spaces of the projective line are*

$$P_{C(\mathbb{C}P^1,2)}(t) = 1,$$
$$P_{C(\mathbb{C}P^1,3)}(t) = 1 + t^3.$$

Another consequence of the last computation is the fact that the Serre spectral sequences of the fibrations

$$\mathcal{F}_n : F(\mathbb{C}, n-1) \hookrightarrow F(\mathbb{C}P^1, n) \to \mathbb{C}P^1$$

do not degenerate at $\boldsymbol{E}_2^{*,*}$ (for $n \geq 3$): using the vanishing of the first and second cohomology of $F(\mathbb{C}P^1, 3)$ and the projection $p\colon F(\mathbb{C}P^1, n) \to F(\mathbb{C}P^1, 3)$, we obtain the diagram (we use bold $\boldsymbol{E}_*^{*,*}$ for the components in the spectral sequences):

$$\begin{tikzcd}
\boldsymbol{E}_2^{0,1}(\mathcal{F}_3) = \mathbb{Q}\langle G_{12}\rangle \arrow[r, "p^*"] \arrow[dr, "d_2", "\cong"'] & \boldsymbol{E}_2^{0,1}(\mathcal{F}_n) \cong \mathcal{A}^1(n-1) \arrow[dr, "d_2"] & \\
& \boldsymbol{E}_2^{2,0}(\mathcal{F}_3) \cong \mathbb{Q}\langle w\rangle \arrow[r, "p^*", "\cong"'] & \boldsymbol{E}_2^{2,0}(\mathcal{F}_n) \cong \mathbb{Q}\langle w\rangle
\end{tikzcd}$$

and we find that the differential d_2 is surjective for $n \geq 3$. These spectral sequences degenerate at $\boldsymbol{E}_3$: the two nonzero columns are given by

$$\boldsymbol{E}_\infty^{0,*} = \boldsymbol{E}_3^{0,*} = \ker d_2 \cong \mathcal{A}^*(n-1)\big/(G_{12}),$$
$$\boldsymbol{E}_\infty^{2,*} = \boldsymbol{E}_3^{2,*} \cong \boldsymbol{E}_2^{2,*}\big/\operatorname{Im} d_2 \cong \mathbb{Q}\langle wG_{12}\rangle \otimes \mathcal{A}^*(n-1)\big/(G_{12}).$$

Proposition 6.1 [13] *The cohomology algebra of the configuration space* $F(\mathbb{C}P^1, n)$ *(*$n \geq 3$*) is given by*

$$H^*(F(\mathbb{C}P^1, n)) \cong H^*(F(\mathbb{C}P^1, 3)) \otimes \mathcal{A}^*(n-1)\big/(G_{12}).$$

In particular, its Poincaré polynomial is

$$P_{F(\mathbb{C}P^1,n)}(t) = (1+t^3)(1+2t)(1+3t)\cdots(1+(n-2)t).$$

Using the results from Section 5, we detect the nonzero bigraded components of the cohomology algebra and (partially) its $\mathcal{S}_n$–structure.

Proof of Theorem 1.2 The first cohomology group is

$$H^1(F(\mathbb{C}P^1,n)) = H_1^1 \cong V(n-2,2),$$

and the subalgebra generated by degree 1 elements is contained in $\bigoplus_{q=0}^{n-3} H_q^q$. The element

$$\gamma = 2(n-2)\sum_{i<j} p_i^*(w)G_{ij} - \sum_{i<j}\sum_{k\neq i,j} p_k^*(w)G_{ij} \in E_1^3(\mathbb{C}P^1,n)$$

is a cocycle in the $V(n)$–isotypic component. It can not be a coboundary because $V(n)$ is missing from E_2^2:

$$\begin{aligned} E_2^2(\mathbb{C}P^1,n) &\cong \mathcal{A}^2(n) \\ &\cong 2V(1)_n \oplus 2V(2)_n \oplus 2V(1,1)_n \oplus V(3)_n \oplus 2V(2,1)_n \oplus V(3,1)_n \end{aligned}$$

(this is correct in the stable case $n \geq 7$ (see [9] or [2]); the trivial module $V(n)$ does not appear in the unstable cases either).

As $\beta_3 = 1 + \sum_{2\leq i<j<k\leq n-2} ijk$ and the component H_3^3 contains a submodule of dimension $\beta_3 - 1$, we obtain

$$H_1^3(F(\mathbb{C}P^1,n)) \cong V(n).$$

The ideal generated by γ is contained in $\bigoplus_{q=1}^{n-2} H_q^{q+2}$ and algebra structure shows that all the other bigraded components are zero. □

The module H_2^2 is a quotient of $E_2^2 \cong \mathcal{A}^2(n)$ (its decomposition into irreducible modules was given in the last proof) and also a quotient of

$$\begin{aligned} H_1^1 \wedge H_1^1 &\cong {\textstyle\bigwedge}^2 V(2)_n \\ &\cong V(1,1)_n \oplus V(2,1)_n \oplus V(1,1,1)_n \oplus V(3,1)_n \end{aligned}$$

(see [2]); the intersection of these decompositions gives (for $n \geq 7$) the inclusion

$$H_2^2 < V(1,1)_n \oplus V(2,1)_n \oplus V(3,1)_n$$

and computing their dimensions this inclusion becomes an equality:

$$\begin{aligned}\beta_2 &= \sum_{2\le i<j\le n-2} ij = \frac{(n-4)(n-3)(3n^2-n+2)}{24} \\ &= \frac{(n-1)(n-2)}{2} + \frac{n(n-1)(n-4)}{3} + \frac{n(n-1)(n-3)(n-6)}{8} \\ &= \dim V(1,1)_n + \dim V(2,1)_n + \dim V(3,1)_n\end{aligned}$$

Similar computations give the unstable cases of the next proposition.

Proposition 6.2 *The decomposition of the second cohomology group becomes stable for $n \ge 7$ and it is given by*

$$H^2(F(\mathbb{C}P^1, n)) = H^2_2 \cong V(1,1)_n \oplus V(2,1)_n \oplus V(3,1)_n.$$

In the unstable cases we have

$$\begin{aligned}&H^2(F(\mathbb{C}P^1, n)) = 0, \quad \text{for } n = 2,3,4, \\ &H^2(F(\mathbb{C}P^1, 5)) \cong V(3,1,1), \\ &H^2(F(\mathbb{C}P^1, 6)) \cong V(4,1,1) \oplus V(3,2,1).\end{aligned}$$

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