Algebraic & Geometric Topology 13 (2013) 3009–3029

Detection of a nontrivial product in the stable homotopy groups of spheres

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In this paper, we prove that there exists a new family of nontrivial homotopy elements in the stable homotopy groups of spheres with dimension $q(p^n+sp+2)-4$. These nontrivial homotopy elements are represented by $\widetilde{\beta}_s h_0 h_n$ in the $E_2^{s+2,t}$–term of the Adams spectral sequence, where $p \geqslant 5$, $n > 4$, $p+1 < s < 2p-1$, $t = q(p^n+sp+2)+s-2$, $q = 2(p-1)$.

55Q45; 55T15, 55S10

1 Introduction

Let S be the sphere spectrum localized at an odd prime p and let A be the mod p Steenrod algebra. To determine the stable homotopy groups of spheres π_*S is one of the central problems in homotopy theory. One of the main tools for investigating this problem is the Adams spectral sequence (ASS) $E_2^{s,t} = \mathrm{Ext}_A^{s,t}(\mathbb{Z}_p, \mathbb{Z}_p) \Rightarrow \pi_{t-s}S$, where the $E_2^{s,t}$–term is the cohomology of A and the Adams differential is $d_r\colon E_r^{s,t} \to E_r^{s+r,t+r-1}$. In detecting nontrivial elements of π_*S with the ASS, three problems arise: calculation of the E_2–terms $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$, computation of the differentials and questions of extensions from E_∞ to π_*S.

The known results on $\mathrm{Ext}_A^{*,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ are as follows. From Liulevicius [8], we know that $\mathrm{Ext}_A^{1,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ has a $\mathbb{Z}_p$–basis consisting of $a_0 \in \mathrm{Ext}_A^{1,1}(\mathbb{Z}_p, \mathbb{Z}_p)$ and $h_i \in \mathrm{Ext}_A^{1,p^iq}(\mathbb{Z}_p, \mathbb{Z}_p)$ for all $i \geqslant 0$. $\mathrm{Ext}_A^{2,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ has $\mathbb{Z}_p$–basis consisting of α_2, a_0^2, a_0h_i $(i > 0)$, g_i $(i \geqslant 0)$, k_i $(i \geqslant 0)$, b_i $(i \geqslant 0)$ and h_ih_j $(j \geqslant i+2, i \geqslant 0)$ whose internal degrees are $2q+1$, 2, p^iq+1, $q(p^{i+1}+2p^i)$, $q(2p^{i+1}+p^i)$, $p^{i+1}q$ and $q(p^i+p^j)$ respectively. $\mathrm{Ext}_A^{3,*}(\mathbb{Z}_p, \mathbb{Z}_p)$ for $p > 2$ has been computed by Aikawa [1].

Let M be the Moore spectrum modulo an odd prime p given by the cofibration

$$S \xrightarrow{p} S \xrightarrow{i_1} M \xrightarrow{j_1} \Sigma S. \tag{1-1}$$

Published: 31 July 2013 DOI: 10.2140/agt.2013.13.3009

Let $\alpha\colon \Sigma^q M \to M$ be the Adams map and $V(1)$ be its cofibre given by the cofibration

$$\Sigma^q M \xrightarrow{\alpha} M \xrightarrow{i_2} V(1) \xrightarrow{j_2} \Sigma^{q+1} M. \tag{1-2}$$

Let $\beta\colon \Sigma^{(p+1)q} V(1) \to V(1)$ be the v_2–mapping. It is well known that, in the ASS, the β–element $\beta_s = j_1 j_2 \beta^s i_2 i_1$ is a nontrivial element of order p in $\pi_{(p+1)sq-q-2} S$, where $p \geqslant 5$; see Miller, Ravenel and Wilson [10].

From Wang and Zheng [13], we know that $\beta_s \in \pi_{spq+(s-1)q-2} S$ is represented by the second Greek letter family element, $\widetilde{\beta}_s \in \mathrm{Ext}_A^{s,spq+(s-1)q+s-2,*}(\mathbb{Z}_p, \mathbb{Z}_p)$, in the ASS and $\widetilde{\beta}_s$ is represented by the element $s(s-1)a_2^{s-2} h_{2,0} h_{1,1}$ in the May spectral sequence (MSS).

Using the ASS, X Wang [12] proved in 1994 that the product $\widetilde{\beta}_s h_0 b_n$ is a permanent cycle in the ASS and converges to a nontrivial element of order p in $\pi_* S$. In 1998, X Wang and Q Zheng [13] proved the convergence of $\widetilde{\beta}_s h_0 h_n$. Recently, X Liu [4; 5] and Liu and Li [6] proved the convergence of $\widetilde{\beta}_s h_0 h_n b_0$, $\widetilde{\gamma}_{s+3} h_0 h_n h_m$ and $\widetilde{\beta}_s h_0 h_n h_m$. However, all of them are working under the condition $s < p$. If $s > p$, the computation becomes much more complicated.

In this paper, we interest ourselves in the problem of convergence of the product $\widetilde{\beta}_s h_0 h_n$ $(p+1 < s < 2p-1)$, and get the following theorem.

Theorem 1.1 *If $n > 4$, $p \geqslant 5$ and $p+1 < s < 2p-1$, the product $\widetilde{\beta}_s h_0 h_n$ survives to E_∞ in the ASS and it converges to an element in $\pi_* S$.*

Remark 1.2 If $p+1 < s < 2p-2$, we believe that $\widetilde{\beta}_s h_0 b_n$ also survives. However, this must be more complicated.

So far, not so many families of homotopy elements in $\pi_* S$ have been detected. In [2], Cohen detected a family $\zeta_n \in \pi_{p^n q+q-3} S$, for $n \geqslant 1$, which has filtration 3 in the ASS and is represented by $h_0 b_n \in \mathrm{Ext}_A^{3,p^n q+q}(\mathbb{Z}_p, \mathbb{Z}_p)$. Lee [3] proved that $\beta_1^{p-1} \zeta_n$ is nontrivial for all n, ie, $b_0^{p-1} h_0 b_n$ is a permanent cycle in the ASS and converges nontrivially to $\beta_1^{p-1} \zeta_n$. This result gave another infinite family of elements in the stable homotopy of spheres. In [9], M Mahowald detected a family $\eta_j \in \pi_{p^j q+pq-2} S$, for $p = 2$, $j \neq 2$, which has filtration 2 in the ASS and is represented by $h_1 h_j \in \mathrm{Ext}_A^{2,p^j q+pq}(\mathbb{Z}_p, \mathbb{Z}_p)$.

For the convenience of the reader, let us briefly indicate the main idea in the proof of Theorem 1.1.

Note that $\widetilde{\beta}_s$ and $h_0 h_n$ are both permanent cycles, so $\widetilde{\beta}_s h_0 h_n$ is a permanent cycle, that is $d_r(\widetilde{\beta}_s h_0 h_n) = 0$. Thus, to prove the convergence of the product $\widetilde{\beta}_s h_0 h_n$, it is enough to show that the product

$$\widetilde{\beta}_s h_0 h_n \neq 0 \in \mathrm{Ext}_A^{s+2,t}(\mathbb{Z}_p, \mathbb{Z}_p)$$

and that it is not a d_r boundary in the ASS. For the latter, it is enough to show that

$$\mathrm{Ext}_A^{s+2-r,q(p^n+sp+s)+s-r+1}(\mathbb{Z}_p, \mathbb{Z}_p) = 0, \quad s+2 > r \geqslant 2.$$

The MSS is a powerful tool to prove both of the above.

This paper is organized as follows. In Section 2, we introduce a good method used to compute the generators of the MSS E_1–term. In Section 3, we use this method to prove some important results on Ext groups. The proof of Theorem 1.1 will be given in the last section.

2 Detecting generators in the May E_1–term

From Ravenel [11], there is an MSS $\{E_r^{s,t,*}, d_r\}$ which converges to $\mathrm{Ext}_A^{s,t}(\mathbb{Z}_p.\mathbb{Z}_p)$ with E_1–term

$$E_1^{*,*,*} = E(h_{i,j} \mid i > 0, j \geqslant 0) \otimes P(b_{i,j} \mid i > 0, j \geqslant 0) \otimes P(a_i \mid i \geqslant 0),$$

where $E(\cdot)$ denotes the exterior algebra, $P(\cdot)$ denotes the polynomial algebra, and

$$h_{i,j} \in E_1^{1,2(p^i-1)p^j,2i-1}, \quad b_{i,j} \in E_1^{2,2(p^i-1)p^{j+1},p(2i-1)}, \quad a_i \in E_1^{1,2p^i-1,2i+1}.$$

One has $d_r\colon E_r^{s,t,M} \to E_r^{s+1,t,M-r}$ for $r \geqslant 1$, and if $x \in E_r^{s,t,M}$ and $y \in E_r^{s',t',M}$, then

$$d_r(x \cdot y) = d_r(x) \cdot y + (-1)^s x \cdot d_r(y).$$

From Liu and Wang [7, Proposition 2.5], there exists a graded commutativity in the May E_1–term as follows:

$$\text{(2-1)} \qquad \begin{cases} a_m h_{n,j} = h_{n,j} a_m, & h_{m,k} h_{n,j} = -h_{n,j} h_{m,k}, \\ a_m b_{n,j} = b_{n,j} a_m, & h_{m,k} b_{n,j} = b_{n,j} h_{m,k}, \\ a_m a_n = a_n a_m, & b_{m,n} b_{i,j} = b_{i,j} b_{m,n}. \end{cases}$$

The first May differential d_1 is given by

$$\text{(2-2)} \qquad \begin{cases} d_1(h_{i,j}) = -\sum_{0<k<i} h_{i-k,k+j} h_{k,j}, \\ d_1(a_i) = -\sum_{0 \leqslant k<i} h_{i-k,k} a_k, \\ d_1(b_{i,j}) = 0. \end{cases}$$

For each element $x \in E_1^{s,t,M}$, we define $\dim x = s$, $\deg x = t$. Then we have

$$\text{(2-3)} \qquad \begin{cases} \dim h_{i,j} = \dim a_i = 1, \quad \dim b_{i,j} = 2, \\ \deg h_{i,j} = 2(p^i - 1)p^j = q(p^{i+j-1} + \cdots + p^j), \\ \deg b_{i,j} = 2(p^i - 1)p^{j+1} = q(p^{i+j} + \cdots + p^{j+1}), \\ \deg a_i = 2p^i - 1 = q(p^{i-1} + \cdots + 1) + 1, \\ \deg a_0 = 1, \end{cases}$$

where $i \geqslant 1,\ j \geqslant 0$.

We denote $a_i, h_{i,j}$ and $b_{i,j}$ by x, y and z respectively. By the graded commutativity of $E_1^{*,*,*}$, we can consider a generator

$$g = (x_1, \ldots, x_b)(y_1, \ldots, y_m)(z_1, \ldots, z_l) \in E_1^{b+m+2l,t+b,*},$$

where $t = (\overline{c}_0 + \overline{c}_1 p + \cdots + \overline{c}_n p^n)q$ with $0 \leqslant \overline{c}_i < p$ $(\overline{c}_n > 0)$, $0 < b < q$.

Note that the degrees of x_i, y_i and z_i can be uniquely expressed as

$$\begin{aligned} \deg x_i &= q(x_{i,0} + x_{i,1} p + \cdots + x_{i,n} p^n) + 1, \\ \deg y_i &= q(y_{i,0} + y_{i,1} p + \cdots + y_{i,n} p^n), \\ \deg z_i &= q(0 + z_{i,1} p + \cdots + z_{i,n} p^n). \end{aligned}$$

Furthermore, the sequence $(x_{i,0}, x_{i,1}, \ldots, x_{i,n})$ is of the form $(1, \ldots, 1, 0, \ldots, 0)$, while the sequences $(y_{i,0}, y_{i,1}, \ldots, y_{i,n})$ and $(0, z_{i,1}, \ldots, z_{i,n})$ are of the form $(0, \ldots, 0, 1, \ldots, 1, 0, \ldots, 0)$. Denote the sequences by columns, then the generator g determines a matrix:

$$\text{(2-4)} \qquad \begin{array}{c} \qquad\qquad A \qquad\qquad\qquad\qquad\qquad B \qquad\qquad \\ \left(\begin{array}{ccc|cccccc} x_{1,0} & \cdots & x_{b,0} & y_{1,0} & \cdots & y_{m,0} & 0 & \cdots & 0 \\ x_{1,1} & \cdots & x_{b,1} & y_{1,1} & \cdots & y_{m,1} & z_{1,1} & \cdots & z_{l,1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{1,n} & \cdots & x_{b,n} & y_{1,n} & \cdots & y_{m,n} & z_{1,n} & \cdots & z_{l,n} \end{array}\right) \end{array}$$

The entries of the matrix (2-4) are 0 or 1. Because of the graded commutativity of $E_1^{*,*,*}$, by interchanging columns in part A and B respectively, the matrix (2-4) can always be transformed into a new one whose entries $x_{i,j}, y_{i,j}, z_{i,j}$ satisfy the following conditions:

$$
\begin{array}{ll}
(1) & x_{1,j} \geqslant x_{2,j} \geqslant \cdots \geqslant x_{b,j},\, x_{i,0} \geqslant x_{i,1} \geqslant x_{i,n} \text{ for } i \leqslant b \text{ and } j \leqslant n. \\
(2) & \text{If } y_{i,j-1} = 0 \text{ and } y_{i,j} = 1, \text{ then for all } k < j,\, y_{i,k} = 0. \\
(3) & \text{If } y_{i,j} = 1 \text{ and } y_{i,j+1} = 0, \text{ then for all } k > j,\, y_{i,k} = 0. \\
(4) & y_{1,0} \geqslant y_{2,0} \geqslant \cdots \geqslant y_{m,0}. \\
(5) & \text{If } y_{i,0} = y_{i+1,0},\, y_{i,1} = y_{i+1,1}, \dots, y_{i,j} = y_{i+1,j}, \text{ then} \\
 & y_{i,j+1} \geqslant y_{i+1,j+1}. \\
(6) & \text{Conditions (2)–(5) apply also to } z_{i,j}.
\end{array}
\tag{2-5}
$$

For example, part A of the matrix (2-4) may be transformed into the following form:

$$
\begin{pmatrix}
1 & \cdots & 1 & \cdots & \cdots & 1 & \cdots & 1 \\
1 & \cdots & 1 & \cdots & \cdots & 1 & & \\
\vdots & \vdots & \vdots & \vdots & & & & \\
1 & \cdots & 1 & & & & & \\
1 & & & & & & &
\end{pmatrix}
\tag{2-6}
$$

The entries not displayed are all 0 and a column $(\overbrace{1, \dots, 1}^{i}, 0, \dots, 0)^T$ denotes a_i. Part B of the matrix (2-4) may be transformed into the following form:

$$
\begin{pmatrix}
I_1 & & & \\
 & I_2 & & \\
 & & \ddots & \\
 & & & I_n
\end{pmatrix}
$$

The entries of each I_i are all 1, the others are all 0. Unfortunately, we can not determine which columns $(0, \dots, 0, 1, \dots, 1, 0, \dots, 0)^T$ (j 0s, i 1s, $0, \dots, 0$) in the above matrix denote $h_{i,j}$ or $b_{i,j-1}$. For this reason, we give the following definition.

Definition 2.1 Define the polynomial algebra

$$\widetilde{E}_1^{s,t,*} = P[h_{i,j} \mid i > 0,\, j \geqslant 0] \otimes P[b_{i,j} \mid i > 0,\, j \geqslant 0] \otimes P[a_i \mid i \geqslant 0]$$

and note the obvious identification $E_1^{s,t,*} = \widetilde{E}_1^{s,t,*}/(h_{i,j}^2)$. If, in the above, $b_{i,j}$ is replaced by $h_{i,j+1}$, then we get

$$F_1^{s,t,*} := P[a_i \mid i \geqslant 0] \otimes P[h_{i,j} \mid i > 0,\ j \geqslant 0].$$

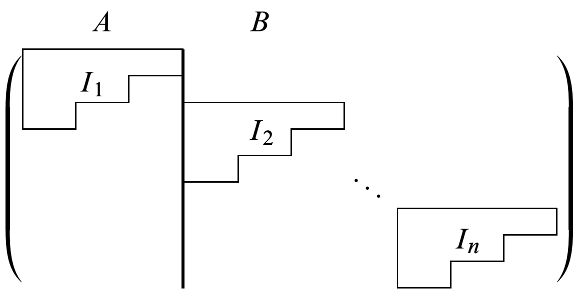
By the graded commutativity of $F_1^{s,t,*}$, we can consider a generator

$$g = (x_1, \dots, x_b)(y_1, \dots, y_m) \in F_1^{b+m,t+b,*},$$

where $t = (\overline{c}_0 + \overline{c}_1 p + \cdots + \overline{c}_n p^n)q$ with $0 \leqslant \overline{c}_i < p (\overline{c}_n > 0), 0 < b < q$. Similarly, the generator g determines a matrix:

$$\text{(2-7)} \qquad \begin{array}{cc} \begin{array}{cc} \quad A \quad & \quad B \quad \end{array} & \\ \left(\begin{array}{ccc|ccc} x_{1,0} & \cdots & x_{b,0} & y_{1,0} & \cdots & y_{m,0} \\ x_{1,1} & \cdots & x_{b,1} & y_{1,1} & \cdots & y_{m,1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{1,n} & \cdots & x_{b,n} & y_{1,n} & \cdots & y_{m,n} \end{array}\right) & \begin{array}{c} c_0 \\ c_1 \\ \vdots \\ c_n \end{array} \end{array}$$

By interchanging columns in parts A and B respectively, the matrix (2-7) can be transformed into a new matrix of the form



where a column $(1, \dots, 1, 0, \dots, 0)^T$ (with i 1s) in part A denotes a_i and a column $(0, \dots, 0, 1, \dots, 1, 0, \dots, 0)^T$ (j 0s, i 1s, $0, \dots, 0$)) in part B denotes $h_{i,j}$. By the properties of the p–adic numbers, we have the following system of equations:

$$\text{(2-8)} \qquad \begin{cases} x_{1,0} + \cdots + x_{b,0} + y_{1,0} + \cdots + y_{m,0} = \overline{c}_0 + k_1 p = c_0, \\ x_{1,1} + \cdots + x_{b,1} + y_{1,1} + \cdots + y_{m,1} = \overline{c}_1 - k_1 + k_2 p = c_1, \\ \quad \vdots \\ x_{1,n-1} + \cdots + x_{b,n-1} + y_{1,n-1} + \cdots + y_{m,n-1} \\ \qquad\qquad = \overline{c}_{n-1} - k_{n-1} + k_n p = c_{n-1}, \\ x_{1,n} + \cdots + x_{b,n} + y_{1,n} + \cdots + y_{m,n} = \overline{c}_n - k_n = c_n, \end{cases}$$

where $k_i \geqslant 0$.

Definition 2.2 In (2-8), the integer sequence $k=(k_1,k_2,\dots,k_n)$ is called the carry sequence.

Definition 2.3 In (2-8), the integer sequence $c=(c_0,c_1,\dots,c_n)$ which is determined by $(\overline{c}_0,\overline{c}_1,\dots,\overline{c}_n)$ and the carry sequence $k=(k_1,k_2,\dots,k_n)$ is called the sum of the row sequence.

Definition 2.4 For the sum of the row sequence c, we denote $m_0=\max\{c_0-b,0\}$, $m_i=\max\{c_i-c_{i-1},0\}$ for $i>0$ and $\widetilde{m}=m_0+m_1+\cdots+m_n$.

We have the following simple method for constructing matrix solutions of (2-8) which satisfy the conditions (2-5)(1–5).

Simple method 2.5 *Without loss generality, we suppose the first i rows are as follows:*

$$\left(\begin{array}{l|l}
1 \cdots 1\ 1 \cdots 1 \cdots 1 \cdots 1 \cdots 1 & 0 \cdots 0\ 0 \cdots 0 \cdots 0 \cdots 0\\
1 \cdots 1\ 1 \cdots 1 \cdots 1 \cdots 1 & 1 \cdots 1\ 1 \cdots 1 \cdots 1 \cdots 1\\
\vdots \quad \vdots\ \vdots \quad \vdots \quad \vdots & \vdots \quad \vdots\ \vdots \quad \vdots \quad \vdots\\
1 \cdots 1\ 1 \cdots 1 & 1 \cdots 1\ 1 \cdots 1
\end{array}\right)
\begin{array}{l} c_0\\ c_1\\ \vdots\\ c_{i-1}\end{array}$$

Then the $(i+1)^{th}$ *row is constructed as follows:*

(1) *If* $c_i \geqslant c_{i-1}$, *put* 1s *in the next neighboring* c_i-c_{i-1} *columns, like so:*

$$\left(\begin{array}{l|l|l}
1 \cdots 1\ 1 \cdots 1 \cdots 1 \cdots 1 \cdots 1 & 0 \cdots 0\ 0 \cdots 0 \cdots 0 \cdots 0 & 0 \cdots 0\\
1 \cdots 1\ 1 \cdots 1 \cdots 1 \cdots 1 & 1 \cdots 1\ 1 \cdots 1 \cdots 1 \cdots 1 & 0 \cdots 0\\
\vdots \quad \vdots\ \vdots \quad \vdots \quad \vdots & \vdots \quad \vdots\ \vdots \quad \vdots \quad \vdots & \vdots \quad \vdots\\
1 \cdots 1\ 1 \cdots 1 & 1 \cdots 1\ 1 \cdots 1 & 0 \cdots 0\\
1 \cdots 1\ 1 \cdots 1 & 1 \cdots 1\ 1 \cdots 1 & \underbrace{1 \cdots 1}_{c_i-c_{i-1}}
\end{array}\right)
\begin{array}{l} c_0\\ c_1\\ \vdots\\ c_{i-1}\\ \hline c_i\end{array}$$

(2) *If* $c_i < c_{i-1}$, *delete* 1s *from some former* $c_{i-1}-c_i$ *columns:*

$$\left(\begin{array}{l|l}
1 \cdots 1\ 1 \cdots 1 \cdots 1 \cdots 1 \cdots 1 & 0 \cdots 0\ 0 \cdots 0 \cdots 0 \cdots 0\\
1 \cdots 1\ 1 \cdots 1 \cdots 1 \cdots 1 & 1 \cdots 1\ 1 \cdots 1 \cdots 1 \cdots 1\\
\vdots \quad \vdots\ \vdots \quad \vdots \quad \vdots & \vdots \quad \vdots\ \vdots \quad \vdots \quad \vdots\\
1 \cdots 1\ 1 \cdots 1 & 1 \cdots 1\ 1 \cdots 1\\
1 \cdots 1\ \underbrace{0 \cdots 0}_{r} & 1 \cdots 1\ \underbrace{0 \cdots 0}_{h}
\end{array}\right)
\begin{array}{l} c_0\\ c_1\\ \vdots\\ c_{i-1}\\ \hline c_i\end{array}$$

where $r+h=c_{i-1}-c_i,\ r,h \geqslant 0$.

Sometimes, by Simple method 2.5, we can not detect the generators of $F_1^{b+m,t+b,*}$. For example, assume that the first two rows are as follows:

$$\left(\begin{array}{cccc|c} 1 & \cdots & 1 & 1 & 1 \\ 1 & \cdots & 1 & 0 & 1 \end{array}\right) \begin{array}{l} s \\ s-1 \end{array}$$

By Simple method 2.5(1), we have the following matrix:

$$\left(\begin{array}{cccc|c} 1 & \cdots & 1 & 1 & 1 \\ 1 & \cdots & 1 & 0 & 1 \\ \hline 1 & \cdots & 1 & 0 & 1 \end{array}\right) \begin{array}{l} s \\ s-1 \\ \hline s-1 \end{array}$$

It detects the generator $a_3^{s-2}a_1h_{3,0} \in F_1^{b+m,t+b,*}$, but we can not get the following matrix by Simple method 2.5(1):

$$\left(\begin{array}{cccc|cc} 1 & \cdots & 1 & 1 & 1 & 0 \\ 1 & \cdots & 1 & 0 & 1 & 0 \\ \hline 1 & \cdots & 1 & 0 & 0 & 1 \end{array}\right) \begin{array}{l} s \\ s-1 \\ \hline s-1 \end{array}$$

However, the above matrix actually exists, and it will in fact detect the generator $a_3^{s-2}a_1h_{2,0}h_{1,2} \in F_1^{b+m+1,t+b,*}$. The following discussion gives us a good idea to solve this problem.

For an element $g = x_1x_2\cdots x_b \cdot y_1y_2\cdots y_m \in F_1^{b+m,t+b,*}$, we denote the set of terms in $d_1\{g\}$ by $D_1\{g\}$. Then $D_1\{g\}$ generates a submodule of $F_1^{b+m+1,t+b,*}$ and $D_1^k\{g\} = D_1\{\cdots D_1\{g\}\cdots\}$ generates a submodule of $F_1^{b+m+k,t+b,*}$. Furthermore, we have

$$\begin{aligned} D_1\{a_i\} &= \{a_0h_{i,0}, a_1h_{i-1,1}, \dots, a_{i-1}h_{1,i-1}\}, \\ D_1\{h_{i,j}\} &= \{h_{1,j}h_{i-1,j+1}, h_{2,j}h_{i-2,j+2}, \dots, h_{i-1,j}h_{1,i+j-1}\}. \end{aligned}$$

From [7, Lemmas 5.5 and 5.7], we get the following diagram:

$$\begin{tikzcd}
& & & F_1^{b+\tilde{m}} \arrow[dlll, "*_1"'] \arrow[dll, "*_1"'] \arrow[d, "*_1"] \arrow[drr, "*_1"] & & \\
F_1^{b+\tilde{m}} \arrow[r] \arrow[drrr, "*_2"'] & F_1^{b+\tilde{m}+1} \arrow[r] \arrow[drr, "*_2"'] & \cdots \arrow[r] & F_1^{b+\tilde{m}+k} \arrow[r] \arrow[d, "*_2"] & \cdots \arrow[r] & F_1^{s} \arrow[dll, "*_2"] \\
& & & E_1^{s} & &
\end{tikzcd}$$

where $*_1$ denotes the resolution $h_{i,j} \to h_{i-k,j+k}h_{k,j}$ and $a_i \to a_{i-j}h_{j,i-j}$, $k \geqslant 0$, and $*_2$ denotes the replacement $h_{i,j+1} \to b_{i,j}$.

From the discussion above, the determination of $E_1^{s,t+b,*}$ is reduced to the following steps:

Step 1 Express t/q as a p–adic number so that $t = (\overline{c}_0 + \overline{c}_1 p + \cdots + \overline{c}_n p^n)q$.

Step 2 List all possible carry sequences k such that in the corresponding sum of row sequence c, $\widetilde{m} \leqslant s-b$ (Definition 2.4).

Step 3 For each sum of a row sequence c, we can solve the associated system of equations by Simple method 2.5. Thus, we get all generators of $F_1^{b+\widetilde{m},t+b,*}$.

Step 4 Through the replacement and resolution

(2-9) $$h_{i,j+1} \to b_{i,j}, \quad h_{i,j} \to h_{i-k,j+k}h_{k,j}, \quad a_i \to a_{i-j}h_{j,i-j},$$

we get all generators of $E_1^{s,t+b,*}$.

3 Some results on Ext groups

In this section, we will prove some results on Ext groups which will be used in the proof of the main theorem.

Theorem 3.1 *If $p \geqslant 5$, $n > 4$ and $p+1 < s < 2p-1$, then we have that the product $\widetilde{\beta}_s h_0 h_n \neq 0 \in \mathrm{Ext}_A^{s+2,t,*}(\mathbb{Z}_p, \mathbb{Z}_p)$, where $t = q(p^n + sp + s) + s - 2$.*

Proof Let $s = s' + p$. Then

$$t = q(p^n + p^2 + (s'+1)p + s') + s' + p - 2,$$

where $0 < s' < p-1$.

In the MSS, the product $\widetilde{\beta}_s h_0 h_n$ is represented by

$$s(s-1)a_2^{s-2}h_{2,0}h_{1,1}h_{1,0}h_{1,n} \in E_1^{s+2,t,M},$$

where $M = 5s - 4 = 5s' + 5p - 4$. We need to prove that $E_2^{s+1,t,M+r} = 0\,(r \geqslant 1)$.

By using the method which was introduced in Section 2, the generator $g \in F_1^{s,t,*}$ can be represented by $a_{k_1}a_{k_2}\cdots a_{k_l}h_{i_1 j_1}\cdots h_{i_m j_m}$. For convenience, we write $g = x_1 \cdots x_l y_1 \cdots y_m \in F_1^{s,t,*}$, where $x_i = a_{k_i}$, $y_i = h_{i_m j_m}$, $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_l$,

$j_1 \leqslant j_2 \leqslant \cdots \leqslant j_m$, $i_m \geqslant i_{m+1}$ if $j_i = j_{i+1}$. Since $l = s'+p-2$ and $m \leqslant s+1$, then $g = x_1 \cdots x_{s'+p-2} y_1 \cdots y_m \in F_1^{s,t,*}$ and we have the following system of equations:

$$\text{(3-1)}\quad \begin{cases} x_{1,0}+\cdots+x_{s'+p-2,0}+y_{1,0}+\cdots+y_{m,0} = s'+k_1p = c_0, \\ x_{1,1}+\cdots+x_{s'+p-2,1}+y_{1,1}+\cdots+y_{m,1} = s'+1-k_1+k_2p = c_1, \\ x_{1,2}+\cdots+x_{s'+p-2,2}+y_{1,2}+\cdots+y_{m,2} = 1-k_2+k_3p = c_2, \\ x_{1,3}+\cdots+x_{s'+p-2,3}+y_{1,3}+\cdots+y_{m,3} = 0-k_3+k_4p = c_3, \\ \quad\vdots \\ x_{1,n-1}+\cdots+x_{s'+p-2,n-1}+y_{1,n-1}+\cdots+y_{m,n-1} \\ \qquad\qquad\qquad\qquad\qquad\qquad = 0-k_{n-1}+k_np = c_{n-1}, \\ x_{1,n}+\cdots+x_{s'+p-2,n}+y_{1,n}+\cdots+y_{m,n} = 1-k_n = c_n. \end{cases}$$

Case 1: $0 < s' < p-2$ From (3-1), the carry sequence $k = (k_1, k_2, \dots, k_n)$ can only be of the following forms:

$$\begin{gathered} (0,\dots,0), \\ (1,0,\dots,0), \\ (1,1,0,\dots,0), \\ (1,1,\dots,1). \end{gathered}$$

Subcase 1.1 When $k = (0,\dots,0)$, the corresponding $c = (s', s'+1, 1, 0, \dots, 0, 1)$. We see that the first two rows are:

$$\left(\begin{array}{cccccccc|cc} 1&1&\cdots&1&0&0&\cdots&0&0&0 \\ 1&1&\cdots&1&0&0&\cdots&0&1&0 \end{array}\right) \begin{array}{l} s' \\ s'+1 \end{array}$$

Then the possible third rows are:

$$\left(\begin{array}{cccccccc|cc} 1&1&\cdots&1&0&0&\cdots&0&0&0 \\ 1&1&\cdots&1&0&0&\cdots&0&1&0 \\ \hline 1&0&\cdots&0&0&0&\cdots&0&0&0 \\ 0&0&\cdots&0&0&0&\cdots&0&1&0 \end{array}\right) \begin{array}{lll} s' & & \\ s'+1 & & \\ \hline 1 & \cdots & (1) \\ & \cdots & (2) \end{array}$$

If we choose (1) as the third row, then we get the following solution:

$$\left(\begin{array}{cccccccc|cc} 1&1&\cdots&1&0&0&\cdots&0&0&0 \\ 1&1&\cdots&1&0&0&\cdots&0&1&0 \\ 1&0&\cdots&0&0&0&\cdots&0&0&0 \end{array}\right) \begin{array}{l} s' \\ s'+1 \\ 1 \end{array}$$

Thus, we can obtain the following matrix:

$$\left(\begin{array}{cccccccc|cc} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{array}\right) \begin{array}{l} s' \\ s'+1 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{array}$$

It detects the generator $a_3a_2^{s'-1}a_0^{p-2}h_{1,1}h_{1,n} \in F_1^{s,t,*}$, and by the replacement and resolution (2-9), we have

$$\left\{\begin{array}{cc} a_2^{s'-1}a_0^{p-1}h_{3,0}h_{1,1}h_{1,n} & a_2^{s'-1}a_1a_0^{p-2}h_{2,1}h_{1,1}h_{1,n} \\ a_2^{s'}a_0^{p-2}h_{1,2}h_{1,1}h_{1,n} & a_3a_2^{s'-2}a_0^{p-1}h_{2,0}h_{1,1}h_{1,n} \end{array}\right\} \in E_1^{s+1,t,M_1}$$

with May filtration $M_1 = 5s' + p + 1 < M$, and

$$\left\{a_3a_2^{s'-1}a_0^{p-2}b_{1,0}h_{1,n} \quad a_3a_2^{s'-1}a_0^{p-2}h_{1,1}b_{1,n-1}\right\} \in E_1^{s+1,t,M_2}$$

with May filtration $M_2 = 5s' + 2p + 1 < M$.

Similarly, if we choose (2) as the third row, then we get the following solution:

$$\left(\begin{array}{cccccccc|cc} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \end{array}\right) \begin{array}{l} s' \\ s'+1 \\ 1 \end{array}$$

Thus we can obtain the following matrix:

$$\left(\begin{array}{cccccccc|cc} 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & 1 & \cdots & 1 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{array}\right) \begin{array}{l} s' \\ s'+1 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 1 \end{array}$$

It detects the generator $a_2^{s'}a_0^{p-2}h_{2,1}h_{1,n} \in F_1^{s,t,*}$, and by the replacement and resolution (2-9), we have

$$\left\{a_2^{s'}a_0^{p-2}h_{1,2}h_{1,1}h_{1,n}\ \ a_2^{s'-1}a_0^{p-1}h_{2,1}h_{2,0}h_{1,n}\ \ a_2^{s'-1}a_1a_0^{p-2}h_{2,1}h_{1,1}h_{1,n}\right\} \in E_1^{s+1,t,M_1}$$

with May filtration $M_1 = 5s' + p + 1 < M$,

$$\{a_2^{s'}a_0^{p-2}h_{2,1}b_{1,n-1}\} \in E_1^{s+1,t,M_2}$$

with May filtration $M_2 = 5s' + 2p + 1 < M$, and

$$\{a_3a_2^{s'-1}a_0^{p-2}h_{1,1}b_{1,n-1}\} \in E_1^{s+1,t,M_3}$$

with May filtration $M_3 = 5s' + 4p - 1 < M$.

Subcase 1.2 When $k = (1, 0, \dots, 0)$, the corresponding $c = (s' + p, s', 1, 0, \dots, 0, 1)$. Similar to Subcase 1.1, we have

$$a_3a_2^{s'-1}a_1^{p-2}h_{1,0}h_{1,0}h_{1,n} = 0 \quad \text{and} \quad a_3a_2^{s'-1}a_1^{p}h_{2,0}h_{2,0}h_{1,n} = 0.$$

Subcase 1.3 When $k = (1, 1, \dots, 0)$, the corresponding $c = (s'+p, s'+p, 1, 0, \dots, 0, 1)$. Then we have

$$a_2^{s'+p-2}h_{2,0}h_{2,0}h_{1,n} = 0.$$

Subcase 1.4 When $k = (1, 1, 1, \dots, 1)$, then we have that the corresponding $c = (s, s, p, p-1, p-1, \dots, p-1, 0)$. We can construct the matrix solutions of (3-1) as follows.

Note that the first two rows are:

$$\begin{array}{cl} \left(\begin{array}{cccccccc|cc} 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \end{array}\right) & \begin{array}{l} s \\ s \end{array} \end{array}$$

Then the possible third rows are:

$$\begin{array}{cl} \left(\begin{array}{cccccccc|cc} 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ \hline 1 & \cdots & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 1 & 0 \\ 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{array}\right) & \begin{array}{lcl} s & & \\ s & & \\ \hline p & \cdots & (1) \\ & \cdots & (2) \\ & \cdots & (3) \end{array} \end{array}$$

If we choose (1), (2) and (3) as the third row, respectively, then we get the following three solutions:

$$\left(\begin{array}{ccccccc|cc} 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \end{array}\right) \begin{array}{l} s \\ s \\ p, \end{array}$$

$$\left(\begin{array}{ccccccc|cc} 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 1 & 0 \end{array}\right) \begin{array}{l} s \\ s \\ p, \end{array}$$

$$\left(\begin{array}{ccccccc|cc} 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 & 1 & 1 \end{array}\right) \begin{array}{l} s \\ s \\ p. \end{array}$$

If we choose (1), then the four rows can be expressed as:

$$\left(\begin{array}{ccccccc|cc} 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ \hline 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \end{array}\right) \begin{array}{l} s \\ s \\ p \\ \hline p-1 \end{array}$$

We obtain the following matrix:

$$\left(\begin{array}{ccccccc|cc} 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\ 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{array}\right) \begin{array}{l} s \\ s \\ p \\ p-1 \\ \vdots \\ p-1 \\ 0 \end{array}$$

So we can get the generator $a_n^{p-1} a_3 a_2^{s'-2} h_{2,0} h_{2,0} \in F_1^{s,t,M_5}$ with May filtration $M_5 = (2n+1)(p-1) + 5s' + 3$.

If we choose (2), then the four rows can be expressed as:

$$\left(\begin{array}{ccccccc|cc} 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 \\ 1 & \cdots & 1 & 1 & 1 & 0 & \cdots & 0 & 1 & 0 \\ \hline 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & 1 & 0 \end{array}\right) \begin{array}{lcr} s & & \\ s & & \\ p & & \\ \hline p-1 & \cdots & (1) \\ & \cdots & (2) \end{array}$$

Similarly, we can get $a_n^{p-1}a_2^{s'-1}h_{3,0}h_{2,0}$, $a_n^{p-2}a_3a_2^{s'-1}h_{n,0}h_{2,0} \in F_1^{s,t,M_5}$ with May filtration $M_5=(2n+1)(p-1)+5s'+3$.

If we choose (3), then the four rows can be expressed as:

$$
\begin{array}{cl}
\left(\begin{array}{ccccccccc|cc}
1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & & 1 & 1 \\
1 & \cdots & 1 & 1 & 1 & 1 & \cdots & 1 & & 1 & 1 \\
1 & \cdots & 1 & 1 & 1 & 0 & \cdots & 0 & & 1 & 1 \\
\hline
1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & & 1 & 0 \\
1 & \cdots & 1 & 1 & 0 & 0 & \cdots & 0 & & 1 & 1
\end{array}\right)
&
\begin{array}{l}
s \\
s \\
p \\
\hline
p-1 \ \cdots \ (1) \\
\qquad\ \ \cdots \ (2)
\end{array}
\end{array}
$$

Similarly, we can get $a_n^{p-2}a_2^{s'}h_{n,0}h_{3,0}$, $a_n^{p-3}a_3a_2^{s'}h_{n,0}h_{n,0} \in F_1^{s,t,M_5}$ with May filtration $M_5=(2n+1)(p-1)+5s'+3$.

From the above discussion, we get the following possible generators in F_1^{s,t,M_5}: $a_n^{p-1}a_3a_2^{s'-2}h_{2,0}h_{2,0}$, $a_n^{p-1}a_2^{s'-1}h_{3,0}h_{2,0}$, $a_n^{p-2}a_3a_2^{s'-1}h_{n,0}h_{2,0}$, $a_n^{p-2}a_2^{s'}h_{n,0}h_{3,0}$ and $a_n^{p-3}a_3a_2^{s'}h_{n,0}h_{n,0}$ which will be denoted by x_0,x_1,x_2,x_3 and x_4 respectively, where $x_0=0$ and $x_4=0$. It is easy to see that x_0,x_1,x_2,x_3 and x_4 belong to E_1^{s,t,M_5}. At the same time, we also get generators of E_1^{s+1,t,M_5} by (2-9). Then we list all the possibilities in Table 1.

Let

$$
\begin{aligned}
k_1 &= a_n^{p-1}a_3a_2^{s'-2}h_{2,0}h_{1,1}h_{1,0}, & k_2 &= a_n^{p-1}a_2^{s'-2}a_1h_{3,0}h_{2,0}h_{1,1}, \\
k_3 &= a_n^{p-1}a_2^{s'-1}h_{2,1}h_{2,0}h_{1,0}, & k_4 &= a_n^{p-1}a_3a_2^{s'-2}h_{2,0}h_{1,1}h_{1,0}, \\
k_5 &= a_n^{p-1}a_2^{s'-2}a_1h_{3,0}h_{2,0}h_{1,1}, & k_6 &= a_n^{p-1}a_2^{s'-1}h_{2,1}h_{2,0}h_{1,0}, \\
k_7 &= a_n^{p-2}a_3a_2^{s'-2}a_1h_{n,0}h_{2,0}h_{1,1}, & k_8 &= a_n^{p-2}a_2^{s'}h_{n,0}h_{2,0}h_{1,2}, \\
k_9 &= a_n^{p-2}a_2^{s'-1}a_1h_{n,0}h_{3,0}h_{1,1}, & k_{10} &= a_n^{p-2}a_2^{s'-1}a_1h_{n,0}h_{2,1}h_{2,0}, \\
k_{11} &= a_n^{p-2}a_2^{s'-1}a_0h_{n,0}h_{3,0}h_{2,0}, & k_{12} &= a_n^{p-2}a_3a_2^{s'-1}h_{n-1,1}h_{2,0}h_{1,0}, \\
k_{13} &= a_n^{p-2}a_2^{s'}h_{n-1,1}h_{3,0}h_{1,0}, & k_{14} &= a_n^{p-2}a_2^{s'-1}a_1h_{n-1,1}h_{3,0}h_{2,0}, \\
k_{15} &= a_n^{p-2}a_ia_2^{s'-1}h_{n-i,i}h_{3,0}h_{2,0}, & k_{16} &= a_n^{p-2}a_3a_2^{s'-1}h_{n-3,3}h_{3,0}h_{2,0}, \\
k_{17} &= a_n^{p-2}a_2^{s'}h_{n-2,2}h_{3,0}h_{2,0}, & k_{18} &= a_n^{p-2}a_3a_2^{s'-1}h_{n-i,i}h_{i,0}h_{2,0}, \\
k_{19} &= a_n^{p-2}a_2^{s'}h_{n-i,i}h_{i,0}h_{3,0}, & k_{20} &= a_n^{p-3}a_3a_2^{s'}h_{n,0}h_{n-1,1}h_{1,0}, \\
k_{21} &= a_n^{p-3}a_3a_2^{s'-1}a_1h_{n,0}h_{n-1,1}h_{2,0}, & k_{22} &= a_n^{p-3}a_2^{s'}a_1h_{n,0}h_{n-1,1}h_{3,0}, \\
k_{23} &= a_n^{p-3}a_3a_2^{s'}h_{n,0}h_{n-2,2}h_{2,0}, & k_{24} &= a_n^{p-3}a_2^{s'}a_2h_{n,0}h_{n-2,2}h_{3,0}, \\
k_{25} &= a_n^{p-3}a_3a_2^{s'}h_{n,0}h_{n-3,3}h_{3,0}, & k_{26} &= a_n^{p-3}a_3a_2^{s'-1}h_{n,0}h_{n-3,3}h_{2,0}, \\
k_{27} &= a_n^{p-3}a_ia_3a_2^{s'-1}h_{n,0}h_{n-i,i}h_{2,0}, & k_{28} &= a_n^{p-3}a_ia_2^{s'}h_{n,0}h_{n-i,i}h_{3,0}, \\
k_{29} &= a_n^{p-3}a_3a_2^{s'}h_{n,0}h_{n-i,i}h_{i,0},
\end{aligned}
$$

The generators of F_1^{s,t,M_5}	The generators of E_1^{s+1,t,M_5}
$x_0 = a_n^{p-1}a_3a_2^{s'-2}h_{2,0}h_{2,0}$	$a_n^{p-1}a_3a_2^{s'-2}h_{2,0}h_{1,1}h_{1,0}$
$x_1 = a_n^{p-1}a_2^{s'-1}h_{3,0}h_{2,0}$	$a_n^{p-1}a_2^{s'-1}h_{3,0}h_{1,1}h_{1,0}$ $a_n^{p-1}a_2^{s'-1}h_{2,1}h_{2,0}h_{1,0}$ $a_n^{p-1}a_2^{s'-1}a_1h_{3,0}h_{2,0}h_{1,1}$ $a_n^{p-2}a_2^{s'-1}a_0h_{n,0}h_{3,0}h_{2,0}$ $a_n^{p-2}a_2^{s'-1}a_1h_{n-1,1}h_{3,0}h_{2,0}$ $a_n^{p-2}a_2^{s'-1}a_2h_{n-2,2}h_{3,0}h_{2,0}$ $a_n^{p-2}a_3a_2^{s'-1}h_{n-3,3}h_{3,0}h_{2,0}$ $a_n^{p-2}a_ia_2^{s'-1}h_{n-i,i}h_{3,0}h_{2,0} \quad (4 \leqslant i \leqslant n-1)$
$x_2 = a_n^{p-2}a_3a_2^{s'-1}h_{n,0}h_{2,0}$	$a_n^{p-2}a_3a_2^{s'-1}h_{n,0}h_{1,1}h_{1,0}$ $a_n^{p-2}a_2^{s'-1}a_0h_{n,0}h_{3,0}h_{2,0}$ $a_n^{p-2}a_2^{s'-1}a_1h_{n,0}h_{2,1}h_{2,0}$ $a_n^{p-2}a_2^{s'}h_{n,0}h_{2,0}h_{1,2}$ $a_n^{p-2}a_3a_2^{s'-2}a_1h_{n,0}h_{2,0}h_{1,1}$ $a_n^{p-2}a_3a_2^{s'-1}h_{n-1,1}h_{2,0}h_{1,0}$ $a_n^{p-2}a_3a_2^{s'-1}h_{n-3,3}h_{3,0}h_{2,0}$ $a_n^{p-2}a_3a_2^{s'-1}h_{n-i,0}h_{i,0}h_{2,0} \quad (4 \leqslant i \leqslant n-1)$ $a_n^{p-3}a_3a_2^{s'-1}a_1h_{n,0}h_{n-1,1}h_{2,0}$ $a_n^{p-3}a_3a_2^{s'-1}a_2h_{n,0}h_{n-2,2}h_{2,0}$ $a_n^{p-3}a_3^2a_2^{s'-1}h_{n,0}h_{n-3,3}h_{2,0}$ $a_n^{p-3}a_ia_3a_2^{s'-1}h_{n,0}h_{n-i,i}h_{2,0} \quad (4 \leqslant i \leqslant n-1)$
$x_3 = a_n^{p-2}a_2^{s'}h_{n,0}h_{3,0}$	$a_n^{p-2}a_2^{s'}h_{n,0}h_{2,1}h_{1,0}$ $a_n^{p-2}a_2^{s'}a_0h_{n,0}h_{2,0}h_{1,2}$ $a_n^{p-2}a_2^{s'-1}a_0h_{n,0}h_{3,0}h_{2,0}$ $a_n^{p-2}a_2^{s'-1}a_1h_{n,0}h_{3,0}h_{1,1}$ $a_n^{p-2}a_2^{s'}h_{n-1,1}h_{3,0}h_{1,0}$ $a_n^{p-2}a_2^{s'}h_{n-2,2}h_{3,0}h_{2,0}$ $a_n^{p-2}a_2^{s'}h_{n-i,0}h_{i,0}h_{3,0} \quad (4 \leqslant i \leqslant n-1)$ $a_n^{p-3}a_2^{s'}a_1h_{n,0}h_{n-1,1}h_{3,0}$ $a_n^{p-3}a_2^{s'}a_2h_{n,0}h_{n-2,2}h_{3,0}$ $a_n^{p-3}a_3a_2^{s'}h_{n,0}h_{n-3,3}h_{3,0}$ $a_n^{p-3}a_ia_3a_2^{s'}h_{n,0}h_{n-i,i}h_{3,0} \quad (4 \leqslant i \leqslant n-1)$
$x_4 = a_n^{p-3}a_3a_2^{s'}h_{n,0}h_{n,0}$	$a_n^{p-3}a_3a_2^{s'}h_{n,0}h_{n-1,1}h_{1,0}$ $a_n^{p-3}a_3a_2^{s'}h_{n,0}h_{n-2,2}h_{2,0}$ $a_n^{p-3}a_3a_2^{s'}h_{n,0}h_{n-3,3}h_{3,0}$ $a_n^{p-3}a_3a_2^{s'}h_{n,0}h_{n-i,i}h_{i,0} \quad (4 \leqslant i \leqslant n-1)$

Table 1: The generators of E_1^{s+1,t,M_5}

where $4 \leqslant i \leqslant n-1$, and then consider the first May differential of x_1, x_2, x_3, $d_1(x_1) = -k_1 + \cdots$, $d_1(x_2) = -k_2 + \cdots$, $d_1(x_3) = -k_3 + \cdots$. We can see that the leading terms k_1, k_2, k_3 are not contained in the first May differential of the other generators and are also not equal to $a_2^{s'+p-2} h_{2,0} h_{1,1} h_{1,0} h_{1,n}$ up to sign. From the above results we know that x_1, x_2, x_3 in $E_r^{s,t,*} (r \geq 2)$ is not bounded.

Since $M_5 = (2n+1)(p-1) + 5s' + 3$, we take

$$
\begin{aligned}
g_4 &= a_n^{p-1} a_2^{s'-2} a_0 h_{3,0} h_{2,0} h_{1,1} h_{1,0}, & g_5 &= a_n^{p-1} a_2^{s'-2} a_1 h_{2,1} h_{2,0} h_{1,1} h_{1,0},\\
g_6 &= a_n^{p-1} a_2^{s'-1} h_{2,0} h_{1,2} h_{1,1} h_{1,0}, & g_7 &= a_n^{p-2} a_3 a_2^{s'-2} a_0 h_{n,0} h_{2,0} h_{1,1} h_{1,0},\\
g_8 &= a_n^{p-2} a_2^{s'-1} a_1 h_{n,0} h_{2,0} h_{1,2} h_{1,1}, & g_9 &= a_n^{p-2} a_2^{s'-1} a_0 h_{n,0} h_{3,0} h_{1,1} h_{1,0},\\
g_{10} &= a_n^{p-2} a_2^{s'-1} a_1 h_{n,0} h_{2,1} h_{1,1} h_{1,0}, & g_{11} &= a_n^{p-2} a_2^{s'-1} a_0 h_{n,0} h_{2,1} h_{2,0} h_{1,0},\\
g_{12} &= a_n^{p-2} a_2^{s'-1} a_0 h_{n-1,1} h_{3,0} h_{2,0} h_{1,0}, & g_{13} &= a_n^{p-2} a_2^{s'} h_{n-1,1} h_{2,0} h_{1,2} h_{1,0},\\
g_{14} &= a_n^{p-2} a_2^{s'-1} a_1 h_{n-1,1} h_{3,0} h_{1,1} h_{1,0}, & g_{15} &= a_n^{p-2} a_i a_2^{s'-1} h_{n-i,i} h_{2,1} h_{2,0} h_{1,0},\\
g_{16} &= a_n^{p-2} a_3 a_2^{s'-1} h_{n-3,3} h_{3,0} h_{1,1} h_{1,0}, & g_{17} &= a_n^{p-2} a_2^{s'} h_{n-2,2} h_{2,1} h_{2,0} h_{1,0},\\
g_{18} &= a_n^{p-2} a_3 a_2^{s'-1} h_{n-i,i} h_{i,0} h_{1,1} h_{1,0}, & g_{19} &= a_n^{p-2} a_2^{s'} h_{n-i,i} h_{i,0} h_{2,1} h_{1,0},\\
g_{20} &= a_n^{p-3} a_2^{s'+1} h_{n,0} h_{n-1,1} h_{1,2} h_{1,0}, & g_{21} &= a_n^{p-3} a_3 a_2^{s'-1} a_0 h_{n,0} h_{n-1,1} h_{2,0} h_{1,0},\\
g_{22} &= a_n^{p-3} a_2^{s'-1} a_1 a_0 h_{n,0} h_{n-1,1} h_{3,0} h_{2,0}, & g_{23} &= a_n^{p-3} a_3 a_2^{s'-1} a_1 h_{n,0} h_{n-2,2} h_{2,0} h_{1,1},\\
g_{24} &= a_n^{p-3} a_3 a_2^{s'} a_0 h_{n,0} h_{n-2,2} h_{3,0} h_{2,0}, & g_{25} &= a_n^{p-3} a_3 a_2^{s'} h_{n,0} h_{n-3,3} h_{2,1} h_{1,0},\\
g_{26} &= a_n^{p-3} a_3 a_2^{s'-1} h_{n,0} h_{n-3,3} h_{1,1} h_{1,0}, & g_{27} &= a_n^{p-3} a_i a_3 a_2^{s'-1} h_{n,0} h_{n-i,i} h_{1,1} h_{1,0},\\
g_{28} &= a_n^{p-3} a_i a_2^{s'} h_{n,0} h_{n-i,i} h_{2,0} h_{1,0}, & g_{29} &= a_n^{p-3} a_3 a_2^{s'-1} a_0 h_{n,0} h_{n-i,i} h_{i,0} h_{2,0},
\end{aligned}
$$

where $4 \leqslant i \leqslant n-1$. By (2-2), an easy computation shows that

$$
\begin{aligned}
d_1(k_4) &= g_4 + g_5 - g_6 + \cdots,\\
d_1(k_5) &= -g_4 + g_5 + \cdots,\\
d_1(k_6) &= g_5 - g_6 + \cdots,\\
d_1(k_7) &= -g_7 + \cdots,\\
d_1(k_8) &= -g_8 + \cdots,\\
d_1(k_9) &= g_9 - g_{10} + \cdots,\\
d_1(k_{10}) &= g_{10} - g_{11} + \cdots,\\
d_1(k_{11}) &= g_9 + g_{11} + \cdots,\\
d_1(k_{12}) &= -g_{12} - g_{13} + g_{14} \cdots,\\
d_1(k_{13}) &= -g_{12} + g_{13} + g_{14} + \cdots,\\
d_1(k_{14}) &= g_{13} + g_{14} + \cdots,
\end{aligned}
$$

$$
\begin{aligned}
d_1(k_{15}) &= g_{15}+\cdots,\\
d_1(k_{16}) &= g_{16}+\cdots,\\
d_1(k_{17}) &= g_{17}+\cdots,\\
d_1(k_{18}) &= g_{18}+\cdots,\\
d_1(k_{19}) &= g_{19}+\cdots,\\
d_1(k_{20}) &= g_{20}+\cdots,\\
d_1(k_{21}) &= -g_{21}+\cdots,\\
d_1(k_{22}) &= -s'g_{22}+\cdots,\\
d_1(k_{23}) &= g_{23}+\cdots,\\
d_1(k_{24}) &= -(s'+1)g_{24}+\cdots,\\
d_1(k_{25}) &= g_{25}+\cdots,\\
d_1(k_{26}) &= g_{26}+\cdots,\\
d_1(k_{27}) &= g_{27}+\cdots,\\
d_1(k_{28}) &= -g_{28}+\cdots,\\
d_1(k_{29}) &= -s'g_{29}+\cdots.
\end{aligned}
$$

Through the computation of the first May differentials of the generators and the rank of its coefficient matrix, we see that their first May differentials of generators are linearly independent.

Case 2: $s'=p-2$ From (3-1), we can get the carry sequence $k=(0,0,1,\dots,1)$, the corresponding $c=(s',s'+1,p+1,p-1,\dots,p-1,0)$. Similar to Subcase 1.4, we can get the five generators in the $E_1^{s+1,t,*}$ as follows:

$$
\begin{gathered}
a_4^{s'}a_0^{p-2}h_{3,1}h_{1,2}h_{1,2}=0, \quad a_4^{s'}a_0^{p-2}h_{2,2}h_{2,1}h_{1,2},\\
a_4^{s'-1}a_3a_0^{p-2}h_{2,2}h_{2,2}h_{2,1}=0, \quad a_4^{s'-1}a_3a_0^{p-2}h_{3,1}h_{2,2}h_{1,2},\\
a_4^{s'-2}a_3^2a_0^{p-2}h_{3,1}h_{2,2}h_{2,2}=0.
\end{gathered}
$$

The first May differential of the above generators are as follows:

$$
\begin{aligned}
d_1(a_4^{s'}a_0^{p-2}h_{2,2}h_{2,1}h_{1,2}) &= s'a_4^{s'-1}a_0^{p-1}h_{4,0}h_{2,2}h_{2,1}h_{1,2}+\cdots\neq 0,\\
d_1(a_4^{s'-1}a_3a_0^{p-2}h_{3,1}h_{2,2}h_{1,2}) &= -a_4^{s'-1}a_0^{p-1}h_{3,1}h_{3,0}h_{2,2}h_{1,2}+\cdots\neq 0.
\end{aligned}
$$

From the above results, we get $E_2^{s+1,t,M+r}=0$ $(r\geqslant 1)$ in the MSS, so it follows that $\widetilde{\beta}_s h_0 h_n \neq 0\in \mathrm{Ext}_A^{s+2,t,*}(\mathbb{Z}_p,\mathbb{Z}_p)$. □

Theorem 3.2 *Let $p \geqslant 5$, $n > 4$ and $p+1 < s < 2p-1$. Then we have that* $\mathrm{Ext}_A^{s+2-r,t'-r+1,*}(\mathbb{Z}_p,\mathbb{Z}_p) = 0$, *where $t' = q(s+sp+p^n)+s$ and $2 \leqslant r < s+2$.*

Proof We need to prove $E_2^{s+2-r,t'-r+1,*} = 0$. Let $s = s'+p$, then $t'-r+1 = q(p^n+p^2+(s'+1)p+s')+s'+p-r-1$. We claim that $s'+p-r-1 \geqslant 0$. Otherwise, if $s'+p-r-1 < 0$ and $p \geqslant 5$, then $p > q+(s'+p-r-1) \geqslant p$, it is a contradiction. Consider $g = x_1 \cdots x_{s'+p-r-1} y_1 \cdots y_m \in F_1^{s'+p-r+2,t'-r-1,*}$, where $x_i = a_{k_i}$, $y_i = h_{i_m j_m}$, $k_1 \geqslant k_2 \geqslant \cdots \geqslant k_l$, $j_1 \leqslant j_2 \leqslant \cdots \leqslant j_m$, $i_m \geqslant i_{m+1}$ if $j_m = j_{m+1}$.

Case 1: $0 < s' < p-2$

Subcase 1.1 When $k = (0,\dots,0)$, the corresponding $c = (s', s'+1, 1, 0, \dots, 0, 1)$. Then we get that the generators are

$$a_3 a_2^{s'-1} a_0^{p-r-1} h_{1,1} h_{1,n} \in F_1^{s'+p-r+1,t'-r-1,M},$$
$$a_2^{s'} a_0^{p-r-1} h_{2,1} h_{1,n} \in F_1^{s'+p-r+1,t'-r-1,M},$$

with May filtration $M = 5s'+p-r+3$. By (2-9), we have

$$\left\{\begin{matrix} a_2^{s'-1} a_0^{p-r} h_{3,0} h_{1,1} h_{1,n} & a_2^{s'-1} a_1 a_0^{p-r-1} h_{2,1} h_{1,1} h_{1,n} \\ a_2^{s'} a_0^{p-r-1} h_{1,2} h_{1,1} h_{1,n} & a_3 a_2^{s'-2} a_0^{p-r} h_{2,0} h_{1,1} h_{1,n} \\ a_3 a_2^{s'-1} a_0^{p-r-1} b_{1,0} h_{1,n} & a_3 a_2^{s'-1} a_0^{p-r-1} h_{1,1} b_{1,n-1} \\ a_2^{s'} a_0^{p-r} h_{1,2} h_{1,1} h_{1,n} & a_2^{s'-1} a_0^{p-r} h_{2,1} h_{2,0} h_{1,n} \\ a_2^{s'-1} a_1 a_0^{p-r} h_{2,1} h_{1,1} h_{1,n} & a_2^{s'} a_0^{p-r-1} h_{2,1} b_{1,n-1} \\ a_3 a_2^{s'-1} a_0^{p-2} h_{1,1} b_{1,n-1} & \end{matrix}\right\} \in F_1^{s'+p-r+2,t'-r-1,M'}.$$

Subcase 1.2 When $k = (1,0,\dots,0)$ or $k = (1,1,0,\dots,0)$, it is easy to show that such a g cannot exist.

Subcase 1.3 When we have $k = (1,\dots,1)$, then we have that the corresponding $c = (s'+p, s'+p, p, p-1, \dots, p-1, 0)$. The generators exist if and only if $r = 2$. We list all the possibilities in the following:

$$a_n^{p-2} a_2^{s'-1} h_{4,0} h_{3,0} h_{2,0} \in F_1^{s'+p-r+2,t'-r+1,*},$$
$$a_n^{p-1} a_3 a_2^{s'-3} h_{2,0} h_{2,0} h_{2,0} = 0, \quad a_n^{p-1} a_2^{s'-2} h_{3,0} h_{2,0} h_{2,0} = 0,$$
$$a_n^{p-2} a_3 a_2^{s'-2} h_{4,0} h_{2,0} h_{2,0} = 0, \quad a_n^{p-1} a_2^{s'-2} h_{3,0} h_{3,0} h_{2,0} = 0,$$
$$a_n^{p-4} a_3 a_2^{s'} h_{3,0} h_{3,0} h_{3,0} = 0, \quad a_n^{p-5} a_3^2 a_2^{s'} h_{4,0} h_{3,0} h_{3,0} = 0,$$
$$a_n^{p-6} a_3^3 a_2^{s'} h_{4,0} h_{4,0} h_{3,0} = 0, \quad a_n^{p-7} a_3^4 a_2^{s'} h_{4,0} h_{4,0} h_{4,0} = 0.$$

Furthermore,

$$d_1(a_2^{s'-1}a_0^{p-r}h_{3,0}h_{1,1}h_{1,n}) = -a_2^{s'-1}a_0^{p-r}h_{2,1}h_{1,1}h_{1,0}h_{1,n}+\cdots\neq 0,$$
$$d_1(a_2^{s'-1}a_1a_0^{p-r-1}h_{2,1}h_{1,1}h_{1,n}) = -(s'-2)a_2^{s'-2}a_0^{p-r+1}h_{2,1}h_{1,1}h_{2,0}h_{1,n}+\cdots\neq 0,$$
$$d_1(a_2^{s'}a_0^{p-r-1}h_{1,2}h_{1,1}h_{1,n}) = (s'-1)a_2^{s'-1}a_0^{p-r}h_{2,0}h_{1,2}h_{1,1}h_{1,n}+\cdots\neq 0,$$
$$d_1(a_3a_2^{s'-2}a_0^{p-r}h_{2,0}h_{1,1}h_{1,n}) = a_2^{s'-2}a_0^{p-r+1}h_{3,0}h_{2,0}h_{1,1}h_{1,n}+\cdots\neq 0,$$
$$d_1(a_3a_2^{s'-1}a_0^{p-r-1}b_{1,0}h_{1,n}) = a_2^{s'-1}a_0^{p-r}h_{3,0}b_{1,0}h_{1,n}+\cdots\neq 0,$$
$$d_1(a_3a_2^{s'-1}a_0^{p-r-1}h_{1,1}b_{1,n-1}) = a_2^{s'-1}a_0^{p-r}h_{3,0}h_{1,1}b_{1,n-1}+\cdots\neq 0,$$
$$d_1(a_2^{s'}a_0^{p-r}h_{1,2}h_{1,1}h_{1,n}) = (s'-1)a_2^{s'-1}a_0^{p-r+1}h_{2,0}h_{1,2}h_{1,1}h_{1,n}+\cdots\neq 0,$$
$$d_1(a_2^{s'-1}a_0^{p-r}h_{2,1}h_{2,0}h_{1,n}) = a_2^{s'-1}a_0^{p-r}h_{2,0}h_{1,2}h_{1,1}h_{1,n}+\cdots\neq 0,$$
$$d_1(a_2^{s'-1}a_1a_0^{p-r}h_{2,1}h_{1,1}h_{1,n}) = a_2^{s'-1}a_0^{p-r+1}h_{2,1}h_{1,1}h_{1,0}h_{1,n}+\cdots\neq 0,$$
$$d_1(a_2^{s'}a_0^{p-r-1}h_{2,1}b_{1,n-1}) = a_2^{s'}a_0^{p-r-1}h_{1,2}h_{1,1}b_{1,n-1}+\cdots\neq 0,$$
$$d_1(a_3a_2^{s'-1}a_0^{p-2}h_{1,1}b_{1,n-1}) = a_2^{s'-1}a_0^{p-1}h_{3,0}h_{1,1}b_{1,n-1}+\cdots\neq 0,$$
$$d_1(a_n^{p-2}a_2^{s'-1}h_{4,0}h_{3,0}h_{2,0}) = a_n^{p-2}a_2^{s'-1}h_{3,1}h_{3,0}h_{2,0}h_{1,0}+\cdots\neq 0.$$

Obviously, the first May differential of every generator contains a term which is not contained in the first May differential of the other generators. This implies that all the first May differentials of the generator are linearly independent.

Case 2: $s'=p-2$ In this case, we have that the carry sequence $k=(0,0,1,\dots,1)$ and $c=(s',s'+1,p+1,p-1,\dots,p-1,0)$. Similar to Subcase 1.3, we get the five generators in the $E_1^{s-r+2,t,*}$ as follows:

$$a_4^{s'}a_0^{p-r-1}h_{3,1}h_{1,2}h_{1,2}=0,\quad a_4^{s'}a_0^{p-r-1}h_{2,2}h_{2,1}h_{1,2},$$
$$a_4^{s'-1}a_3a_0^{p-r-1}h_{2,2}h_{2,2}h_{2,1}=0,\quad a_4^{s'-1}a_3a_0^{p-r-1}h_{3,1}h_{2,2}h_{1,2},$$
$$a_4^{s'-2}a_3^2a_0^{p-r-1}h_{3,1}h_{2,2}h_{2,2}=0.$$

Consider the first May differential,

$$d_1(a_4^{s'}a_0^{p-r-1}h_{2,2}h_{2,1}h_{1,2}) = s'a_4^{s'-1}a_0^{p-r}h_{4,0}h_{2,2}h_{2,1}h_{1,2}+\cdots\neq 0,$$
$$d_1(a_4^{s'-1}a_3a_0^{p-r-1}h_{3,1}h_{2,2}h_{1,2}) = -a_4^{s'-1}a_0^{p-r}h_{3,1}h_{3,0}h_{2,2}h_{1,2}+\cdots\neq 0.$$

Similarly, the first May differentials of the generator are linearly independent.

From the above results, it is easy to see that $E_2^{s+2-r,t'-r+1,*}=0$ for $r \geqslant 2$. It follows that $\mathrm{Ext}_A^{s+2-r,t'-r+1,*}(\mathbb{Z}_p,\mathbb{Z}_p)=0$. $\square$

4 Proof of the main theorem

In this section, we give the proof of the main theorem.

Proof of Theorem 1.1 From [2], $(i_1)_*(h_0h_n) \in \mathrm{Ext}_A^{2,p^nq+q}(H^*M,\mathbb{Z}_p)$ is a permanent cycle in the ASS and converges to a nontrivial element $\xi_n \in \pi_{p^nq+q-2}M$. Consider the composite

$$\Sigma^{p^nq+q-2}S \xrightarrow{\xi_n} M \xrightarrow{i_2} V(1) \xrightarrow{\beta^s} \Sigma^{-s(p+1)q}V(1) \xrightarrow{j_1j_2} \Sigma^{-s(p+1)q+q+2}S.$$

Since ξ_n is represented by $(i_1)_*(h_0h_n) \in \mathrm{Ext}_A^{2,p^nq+q}(H^*M,\mathbb{Z}_p)$ in the ASS, then $\widetilde{f}$ is represented by

$$\widetilde{c}=(j_1j_2)_*(\beta^s)_*(i_2)_*(i_1)_*(h_0h_n)=(j_1j_2\beta^si_2i_1)_*(h_0h_n)$$

in the ASS.

By using the Yoneda products, we know that the composite

$$\mathrm{Ext}_A^{0,0}(\mathbb{Z}_p,\mathbb{Z}_p) \xrightarrow{(i_2i_1)_*} \mathrm{Ext}_A^{0,0}(H^*M,\mathbb{Z}_p) \xrightarrow{(j_1j_2)_*(\beta^s)_*} \mathrm{Ext}_A^{s,spq+(s-1)q+s-2}(\mathbb{Z}_p,\mathbb{Z}_p)$$

is a multiplication by

$$\widetilde{\beta}_s \in \mathrm{Ext}_A^{s,spq+(s-1)q+s-2}(\mathbb{Z}_p,\mathbb{Z}_p).$$

Hence $\widetilde{f}$ is represented by

$$\widetilde{\beta}_sh_0h_n \in \mathrm{Ext}_A^{s+2,p^nq+spq+sq+s-2}(\mathbb{Z}_p,\mathbb{Z}_p)$$

in the ASS.

From Theorem 3.1, we see that $\widetilde{\beta}_sh_0h_n \neq 0$. Moreover, from Theorem 3.2 it follows that $\widetilde{\beta}_sh_0h_n$ can not be hit by any differential in the ASS. Thus the $\widetilde{\beta}_sh_0h_n$ survives nontrivially to a homotopy element of π_*S. $\square$

Acknowledgements The authors would like to express their deep thanks to the referee, who gave the authors many helpful comments and suggestions on this paper. The authors would also like to thank Professor Xiangjun Wang for introducing them to the problem considered in this paper. The research was supported by the National Natural Science Foundation of China (grants numbers 11301386, 11261062, 11071125 and 11026197) and the corresponding author of this paper is Yuyu Wang.

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Received: 28 September 2011 Revised: 27 September 2012