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Singular maps on exotic 4–manifold pairs

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We show examples of pairs of smooth, compact, homeomorphic 4–manifolds, whose diffeomorphism types are distinguished by the topology of the singular sets of smooth stable maps defined on them. In this distinction we rely on results from Seiberg–Witten theory.

57R55; 57R45, 57M50, 57R15

1 Introduction

Different smooth structures on a given topological 4–manifold have been shown to exist by a rather delicate count of solutions of certain geometric PDEs associated to the smooth structure (and some further choices, such as a Riemannian metric and possibly a spin^c structure). This idea was the basis of the definition of Donaldson's polynomial invariant [4], as well as the Seiberg–Witten invariants; see Witten [22]. In these invariants specific connections (and sections of bundles associated to the further structures on the 4–manifold) have been counted for the potentially different smooth structures. By the pioneering work of Kronheimer and Mrowka [11] it was clear that, through the *adjunction inequalities*, the invariants provide strong restrictions on the topology of surfaces smoothly embedded in the 4–manifold representing some fixed homology classes. In a slightly different direction, work of Taubes [21] provided obstructions for the existence of symplectic structures compatible with the chosen smooth structure in terms of the Seiberg–Witten invariants. This idea then leads to a simple distinction between certain pairs of smooth structures: one of them (which is compatible with a symplectic structure) admits a Lefschetz fibration (or more generally a Lefschetz pencil) map (see Donaldson [5]), while the one which is not compatible with any symplectic structure does not admit such a map; see Gompf alone [8] and with the second author [9]. Similarly, for 4–manifolds with nonempty boundary there are topological examples with two smooth structures such that one admits a Lefschetz fibration with bounded fibers (and hence a Stein structure), cf Loi and Piergallini [14] and Akbulut and Ozbagci [2], while the other smooth structure does not carry a Stein

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structure, and therefore does not carry a Lefschetz fibration map either. Such a pair of smooth structures was found by Akbulut and Matveyev [1]; cf Theorem 2.1.

Further notable examples of manifolds distinguished by some properties of smooth maps defined on them are provided by "large" exotic $\mathbb{R}^4$, since these noncompact 4–manifolds do not admit embedding into the standard Euclidean 4–space $\mathbb{R}^4$; cf [9, Section 9]. Examples of similar kind are the smooth structures on certain connected sums of S^2–bundles over surfaces (cf the exotic structures described by J Park [18]), which have submersions with definite folds only for the standard structure; see Saeki and Sakuma [20].

In the present work we will find properties of stable/fold maps such that the geometry of their singular sets will distinguish exotic smooth structures on some appropriately chosen topological 4–manifolds. We will apply a result of Saeki from [19] (cf also Theorem 3.1) in constructing maps with the desired properties on some of our examples. We appeal to Seiberg–Witten theory (in particular, to the adjunction inequality and its consequences) in showing that maps with certain prescribed singular sets do not exist on our carefully chosen other examples. The first and most obvious pair of examples for such phenomena is provided by Akbulut and Matveyev [1] (cf Theorem 2.1); in the following we extend their idea to an infinite family of such exotic pairs (given in Theorem 1.6).

To start our discussion, suppose that X is a given smooth 4–manifold. For a smooth manifold Y the smooth map $f\colon X \to Y$ is called *stable* if for every smooth map g sufficiently close to f in $C^\infty(X,Y)$ there are diffeomorphisms $D_X\colon X \to X$ and $D_Y\colon Y \to Y$ such that $D_Y \circ f = g \circ D_X$. Considering the special case $Y = \mathbb{R}^3$, stable maps are dense in $C^\infty(X,\mathbb{R}^3)$ and the singular set of a stable map $f\colon X \to \mathbb{R}^3$ is an embedded (possibly nonorientable) surface $\Sigma_f \subset X$. A stable map is called a *fold* map if it has only fold singularities. Indeed, for a stable map $f\colon X \to \mathbb{R}^3$ a point $p \in X$ is a *fold singularity* if f can be written in some local charts around p and $f(p)$ as

$$(x,y,z,v) \longmapsto (x^2 \pm y^2, z, v).$$

A fold singularity with "$+$" in this formula is called a *definite fold singularity* and with "$-$" it is called an *indefinite fold singularity*. If X is a smooth 4–manifold with nonempty boundary, then for simplicity by a stable map of X into Y we mean a map of X into Y which can be extended to a stable map $f\colon \widetilde{X} \to Y$, where $\widetilde{X}$ is a smooth 4–manifold without boundary, $X \subset \widetilde{X}$ is a smooth submanifold and f has no singularities in a neighborhood of ∂X.

Let $\mathcal{M} \subset C^\infty(X,\mathbb{R}^3)$ be a fixed subset of stable maps with singular set consisting only of closed orientable surfaces. In the following we will define a smooth invariant of X denoted by $\mathrm{sg}_{\mathcal{M}}(X)$ in terms of the possible genera of the components of the singular sets of the maps in $\mathcal{M}$. More formally, fix an integer $k \geq 1$, take a map $f \in \mathcal{M}$ and write the singular set of f in the form $\bigcup_{i=1}^{n} \Sigma^i_f$, where $\Sigma^i_f \subset X$ are the connected components. For each k–element subset $I = \{i_1, \ldots, i_k\}$ of $\{1, \ldots, n\}$ denote by $g_I(f) \in \mathbb{N}$ the *maximal* genus of the surfaces $\Sigma^{i_1}_f, \ldots, \Sigma^{i_k}_f$. Define the set $G^k_{\max}(f)$ as

$$G^k_{\max}(f) = \{g_I(f) \mid I \subset \{1, \ldots, n\} \text{ and } |I| = k\}.$$

Finally, we define $\mathrm{sg}^k_{\mathcal{M}}(X)$ as

$$\min \bigcup_{f \in \mathcal{M}} G^k_{\max}(f).$$

This is a nonnegative integer or it is equal to ∞ if the set $\bigcup_{f \in \mathcal{M}} G^k_{\max}(f)$ is empty. In the next definition we consider only those components of the singular set of a map in $\mathcal{M}$, which represent a fixed homology class and consist only of a fixed set of singularity types. This leads us to:

Definition 1.1 For a smooth 4–manifold X (possibly with nonempty boundary) let $A \subseteq H_2(X; \mathbb{Z})$ be a set of second homology classes and let $\mathcal{S}$ be a fixed set of singularity types. For a stable map $f \in \mathcal{M}$ let $\bigcup_{i=1}^{n} \Sigma^i_f$ denote the union of those components of the singular set of f which have the property that Σ^i_f represents a homology class in A and contains singularities only from $\mathcal{S}$. As before, for a fixed integer $k \geq 1$ and $I \subset \{1, \ldots, n\}$ with $|I| = k$ let $g_I(f, A)$ denote the maximal genus of Σ^i_f with $i \in I$. Then $G^k_{\max}(f, A)$ is the set of all $g_I(f, A)$ (where I runs through the k–element subsets of $\{1, \ldots, n\}$), and $\mathrm{sg}^k(X, A) = \mathrm{sg}^k_{\mathcal{M},\mathcal{S}}(X, A)$ is the minimum of the union $\bigcup_f G^k_{\max}(f, A)$, where f runs over the stable maps in $\mathcal{M}$.

Remark 1 (1) For any $k \geq 1$ we have $\mathrm{sg}^k(X, A) \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$ since the minimum of the empty set is defined to be ∞.

(2) The reason for the slightly complicated definition of the invariant sg^k is that in our applications we will find 4–manifold pairs with the property that in one 4–manifold k disjoint (-1)–spheres can be located, while in the other one we show that there are no k disjoint (-1)–spheres. By fixing the appropriate homology classes for A, the invariant $\mathrm{sg}^k(X, A)$ will distinguish these 4–manifolds; cf Theorems 1.2 and 1.3.

(3) In the case of $k = 1$ the value of $\mathrm{sg}^1(X, A)$ is just the minimum of all the genera of the possible allowed singular set components of the maps in $\mathcal{M}$.

(4) A fairly natural invariant of the same spirit is taking $\min_{f\in\mathcal{M}}\{g(f)\}$, where $g(f)$ denotes the maximal genus of *all* the components of Σ in Definition 1.1. We denote this by $\mathrm{sg}(X, A)$ and a slightly different version of it will play a role in Definition 1.4 and Theorem 1.5.

(5) For $1 \le k \le l$, we have $\mathrm{sg}^k(X, A) \le \mathrm{sg}^l(X, A)$ as it can be seen easily from the definition.

In the present paper we make two main choices for $\mathcal{M}$ and $\mathcal{S}$:

(i) Let $\mathcal{M}$ be the set of all the stable maps with singular set consisting only of closed orientable surfaces and let $\mathcal{S}$ be the set of all the singularities.

(ii) Let $\mathcal{M}$ be the set of all the stable fold maps with singular set consisting only of closed orientable surfaces and let $\mathcal{S}$ be the one element set of the definite fold singularity.

Our results work with both choices. By constructing specific stable maps, and estimating the invariant $\mathrm{sg}^k(X, A)$ for particular manifolds and homology classes using Seiberg–Witten theory, we prove:

Theorem 1.2 *There exist homeomorphic smooth 4–manifolds X_1 and X_2 with $H_2(X_1;\mathbb{Z}) \cong H_2(X_2;\mathbb{Z}) \cong \mathbb{Z}$ such that for the 2–element set of generators A in $H_2(X_1;\mathbb{Z})$ we have:*

(1) $\mathrm{sg}^1(X_1, A) = 0$ *and* $0 < \mathrm{sg}^1(X_2, A) < \infty$.

(2) $\mathrm{sg}^k(X_1, A) = \mathrm{sg}^k(X_2, A) = \infty$ *for* $k \ge 2$.

Remark 2 It follows easily from the proof of Theorem 1.2 that also $\mathrm{sg}(X_1, A) = 0$ and $0 < \mathrm{sg}(X_2, A) < \infty$.

Our example for the topological manifold X_1 in the above theorem is with nonempty boundary. For closed manifolds we show the following result:

Theorem 1.3 *There exist homeomorphic, smooth, closed 4–manifolds V and W and there exists $1 \le k \le 4$ such that if A denotes the set of homology classes in $H_2(V) \cong H_2(W)$ having self-intersection -1, then:*

(1) $\mathrm{sg}^k(V, A) = 0$ *and* $0 < \mathrm{sg}^k(W, A) < \infty$.

(2) $\mathrm{sg}^l(V, A) = \mathrm{sg}^l(W, A) = 0$ *for any* $1 \le l < k$.

Similar results can be derived by looking at the genera of singular sets of smooth maps satisfying some property related to the boundary 3–manifold of the source. To state this result, we need the following definition.

Definition 1.4 A stable map $f\colon X \to \mathbb{R}^3$ with $\partial X \neq \varnothing$ induces a stable framing ϕ on ∂X. Let $\mathcal{A}$ be the property that the stable framing induced on the boundary of the source is canonical, ie has minimal total defect in the sense of Kirby and Melvin [10]. (For discussions of these notions, see also Section 4.) For a 4–manifold X and a stable map $f\colon X \to \mathbb{R}^3$ with singular set consisting only of closed orientable surfaces let $\mathrm{g}(f)$ denote the maximal genus of the components of the definite fold singular set. Let $\mathrm{sg}(X, \mathcal{A})$ denote the minimum of the values $\mathrm{g}(f)$, where f runs over the stable fold maps of X into $\mathbb{R}^3$ with property $\mathcal{A}$ and with singular set consisting only of closed orientable surfaces.

With this notion at hand, now we can state our next result:

Theorem 1.5 *There exist homeomorphic smooth compact 4–manifolds X_1 and X_2 such that $\mathrm{sg}(X_1, \mathcal{A}) = 0$ and $0 < \mathrm{sg}(X_2, \mathcal{A}) < \infty$.*

The prominent example of a pair (X_1, X_2) of smooth 4–manifolds (with nonempty boundary) used in the above results was found by Akbulut and Matveyev [1]. In order to show that our method is applicable in further examples, we extend the examples of [1]:

Theorem 1.6 *There is an infinite family $(X_1(n), X_2(n))_{n\in\mathbb{N}}$ of homeomorphic, non-diffeomorphic compact 4–manifold pairs which are nonhomeomorphic for different n and Theorem 1.2 distinguishes the smooth structure of $X_1(n)$ from the smooth structure of $X_2(n)$.*

The paper is organized as follows. In Section 2 we show infinitely many examples for which Theorems 1.2, 1.3 and 1.5 apply, and hence provide a proof of Theorem 1.6. (A tedious computation within the proof of this theorem is deferred to an appendix.) In Sections 3 and 4 we provide the proofs of the results described in Section 1.

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2 An infinite family of exotic 4–manifold pairs

To make the proofs of Theorems 1.2, 1.3 and 1.5 of Section 1 more transparent, we start by describing the examples promised in Theorem 1.6. The following idea of constructing exotic pairs of 4–manifolds is due to Akbulut–Matveyev [1]. Suppose that K_1, K_2 are two given knots in S^3 with the following properties:

- The 3–manifold given by (-1)–surgery on K_1 is diffeomorphic to the 3–manifold given by (-1)–surgery on K_2
- K_1 is slice, that is, bounds a properly, smoothly embedded disk in D^4.
- The maximal Thurston–Bennequin number of K_2 is nonnegative, in particular, there is a Legendrian knot L (in the standard contact S^3) which is smoothly isotopic to K_2 and $\mathrm{tb}(L) = 0$. (For the definition of the Thurston–Bennequin number, see for example Ozbagci and the second author [17].)

An example for such a pair (K_1, K_2) was found by Akbulut and Matveyev in [1] (cf also Gompf and the second author [9, Theorem 11.4.8]). In the following, X_i will denote the smooth 4–manifold obtained by attaching a 4–dimensional 2–handle to D^4 along K_i with framing -1 for $i = 1, 2$. As it was shown in [1], the properties listed above allow us to prove that:

Theorem 2.1 [1] *The smooth 4–manifolds X_1, X_2 are homeomorphic but not diffeomorphic.*

For the convenience of the reader, we include a short outline of the proof of this theorem.

Proof Since both X_1 and X_2 are given as a single 2–handle attachment to D^4, both are simply connected. Since the surgery coefficient fixed on K_j in both cases is (-1), the boundaries ∂X_1 and ∂X_2 (which are assumed to be diffeomorphic) are integral homology spheres. Furthermore the intersection forms Q_{X_1} and Q_{X_2} can be represented by the 1×1 matrix $\langle -1 \rangle$, in particular, they are isomorphic. The extension of Freedman's fundamental result [7] on topological 4–manifolds to the case of 4–manifolds with integral homology sphere boundary (and trivial fundamental group) then implies that X_1 and X_2 are homeomorphic.

Since K_1 is a slice knot, the generator of $H_2(X_1; \mathbb{Z})$ can be represented by an embedded sphere. On the other hand, since K_2 is smoothly isotopic to a Legendrian knot with Thurston–Bennequin number $\mathrm{tb} = 0$, the famous theorem of Eliashberg [6] (cf also [3]) implies that X_2 admits a Stein structure. Since Stein manifolds embed into minimal surfaces of general type [13], and a minimal surface of general type does not contain a smoothly embedded sphere with homological square -1, we conclude that the generator of $H_2(X_2;\mathbb{Z})$ cannot be represented by a smoothly embedded sphere. (The statement about minimal surfaces of general type relies on a computational fact in Seiberg–Witten theory: minimal surfaces of general type have two basic classes $\pm c_1 \in H^2$, but $c_1^2 > 0$ for those surfaces.) This conclusion, however, shows that X_1 and X_2 are nondiffeomorphic. □

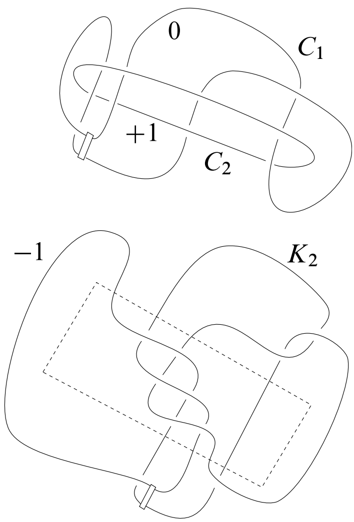

Figure 1: *The framed link* (C_1, C_2). In the upper diagram C_1 has framing 0 while the unknot C_2 has framing $+1$. The small box in the upper figure represents a tangle which will be specified later. The lower diagram shows the result of the blow-down of the $(+1)$–framed unknot C_2, and hence the framing of K_2 is equal to (-1). The dashed box shows the location of the blow-down.

The example of Akbulut–Matveyev can be generalized to an infinite sequence of pairs of knots which we describe presently.

Consider the 2–component link (C_1, C_2) given by the upper diagram of Figure 1. The small box intersecting C_1 in the lower left corner of the diagram contains a tangle which will be specified later. Equip C_1 with framing 0 and C_2 with framing $+1$. If we blow down the unknot C_2, we get a knot K_2, depicted by the lower picture of Figure 1. It is easy to see that the framing of K_2 is equal to -1. By isotoping the result slightly we get the front projection of a Legendrian knot isotopic to K_2 as in Figure 2.

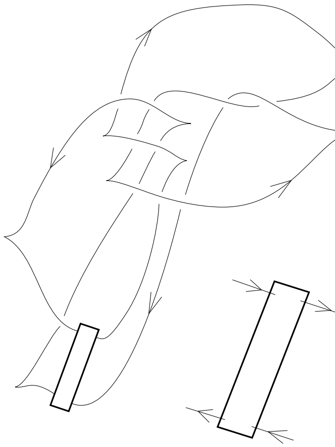

Figure 2: *The knot K_2 can be isotoped to be the front projection of a Legendrian knot.* Indeed, by considering an appropriate module in the box, the resulting Legendrian knot will have vanishing Thurston–Bennequin number.

Simple computation of the writhe and the number of cusps shows that for any module in the small box with nonnegative tb, the resulting knot K_2 will have nonnegative tb. In particular, if we use the module shown by Figure 3 then we get a Legendrian realization of K_2 with vanishing Thurston–Bennequin number. If we insert the module of Figure 3 with n full left twists into the box of Figure 2, the resulting knot will be denoted by $K_2(n)$. Notice that with this choice of the module, C_1 is an unknot. By isotoping C_1 together with the unknot C_2 (as it is shown by Figure 4) and then surgering out the 0–framed unknot C_1 to a 1–handle, we get Figure 5.

By blowing up C_2 at the dashed circle on the picture, and then pulling the resulting (-1)–framed unknot through the 1–handle, we get a slice knot S with framing (-1). Indeed, after the blow-up, the 1–handle resulting from the surgery along C_1 and the 2–handle attached along C_2 can be isotoped to have geometric linking one, hence these handles form a canceling pair. (We do not draw an explicit diagram of the slice knot S here; cf also Remark 3.) Now take K_1 to be equal to this knot S; more precisely, if we use the module of Figure 3 (with n full left twists in the module) in the small box of Figure 5, then denote the resulting knot by $K_1(n)$. Denote the 4–manifold we get by attaching a 2–handle to D^4 along $K_j(n)$ by $X_j(n)$ ($j \in \{1,2\}, n \in \mathbb{N}$). Notice that since we only blew up and down, isotoped and surgered a 0–framed 2–handle, the fact that the boundaries of the 4–manifolds $X_1(n)$ and $X_2(n)$ are diffeomorphic 3–manifolds is a simple fact (since both are diffeomorphic to the boundary of the 4–manifold we get from Figure 1). The original example of Akbulut and Matveyev is the pair $(K_1(n), K_2(n))$ for $n = 0$. The same argument as the one presented in

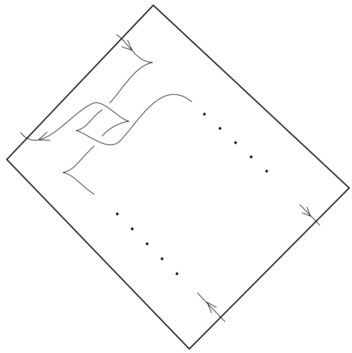

Figure 3: *The module with n full left twists provides knots $K_2(n)$ with vanishing Thurston–Bennequin numbers.* In the diagram the module is already in Legendrian position. Obviously, by adding a left and a right cusp to the diagram we get a Legendrian unknot with Thurston–Bennequin number -1.

the proof of Theorem 2.1 shows that $X_1(n)$ and $X_2(n)$ are homeomorphic but not diffeomorphic 4–manifolds.

Remark 3 It is somewhat tricky (but not difficult) to see that the knot S we chose for $K_1(n)$ is actually a slice knot for every $n \in \mathbb{N}$. Notice that, strictly speaking, we do not need this fact in our argument when showing that the 4–manifolds $X_1(n)$, $X_2(n)$ are nondiffeomorphic. The fact that the generator of $H_2(X_1(n); \mathbb{Z})$ can be represented by a sphere easily follows from the description of $X_1(n)$ we get after blowing up the clasp in the dashed circle in Figure 5: the exceptional sphere of the blow-up will represent the generator of the second homology, since the two other handles form a canceling pair in homology.

In fact, the same line of reasoning applies for all knot pairs we get by putting an appropriate module into the box of Figure 1. In general, we have:

Theorem 2.2 *If the module contains an oriented Legendrian diagram of the unknot after removing a left-most and a right-most cusps (with appropriately orientated arcs) with* $\mathrm{tb} = 0$*, then:*

1. K_0 *is the unknot.*
2. K_1 *is slice.*
3. K_2 *has a Legendrian realization with* $\mathrm{tb} = 0$.
4. *The 4–manifolds* X_1 *and* X_2 *are homeomorphic but not diffeomorphic.* □

It is unclear, however, whether in general the resulting exotic pairs will provide new examples. In order to prove that the pairs $(X_1(n), X_2(n))$ for the particular modules of Figure 3 do provide an infinite sequence of extensions of the example of [1], it is enough to show that the boundary integral homology spheres are different for $n \geq 0$.

Proposition 2.3 *The integral homology spheres obtained by -1 surgery along $K_2(n)$ are pairwise not homeomorphic for $n \geq 0$.*

Proof The *Ohtsuki invariants* $\lambda_k(Y)$ of an integral homology sphere Y (extracted from the quantum invariant of Reshetikin–Turaev of the 3–manifold) can be used to distinguish integral homology spheres. These invariants were introduced by Ohtsuki [16],

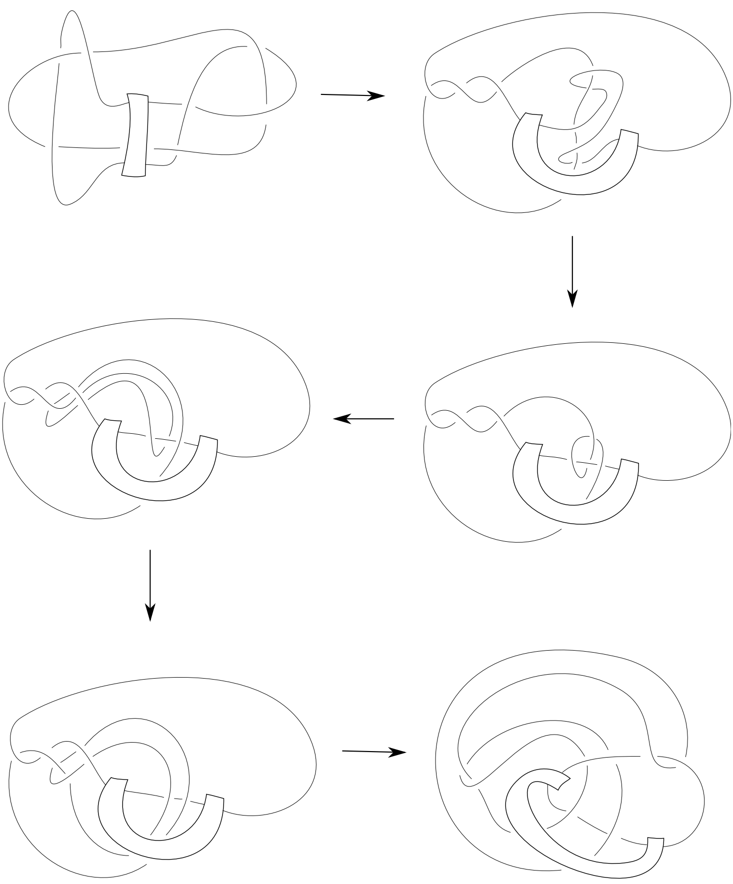

Figure 4: *The isotopy on the link of Figure 1.* The diagrams show the intermediate stages of the isotopy transforming the link of Figure 1 into the link of Figure 5.

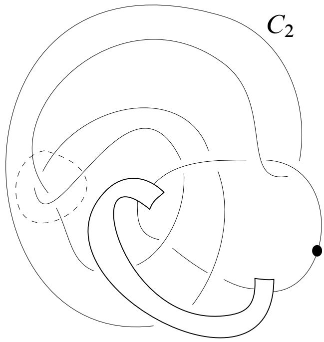

Figure 5: After blowing up the clasp in the dashed circle, the two handles will form a canceling pair, showing that the unknot introduced by the blow-up gives a slice knot in S^3.

and a more computable derivation of the invariants was given in Lin and Wang [12]. $\lambda_1(Y)$ is determined by the Casson invariant of the 3–manifold Y, and in case Y is given as integral surgery along a knot K, the invariant $\lambda_2(Y)$ can be computed from the Jones polynomial and the Conway polynomial of K. More precisely, if Y is given as (-1)–surgery along a knot $K \subset S^3$ then by [12, Theorem 5.2]

$$\lambda_2(Y) = \tfrac{1}{2}v_2(K) + \tfrac{1}{3}v_3(K) + \tfrac{5}{3}v_2^2(K) - 60c_4(K),$$

where $c_4(K)$ is the coefficient of z^4 in the Conway polynomial of K and $v_i(K) = \partial^i V(K,e^h)/\partial h^i(0)$, where $V(K,t)$ is the Jones polynomial of K. (The Conway and Jones polynomials are defined by skein theory and normalization as given in [12].) A somewhat tedious computation (postponed to an appendix; cf Lemmas A.1 and A.3) shows that for the knot $K_2(n)$ the value of c_4 is $-n$, the value of v_2 is -12, while v_3 is $36n+108$. This shows that λ_2 of the 3–manifold $S^3_{-1}(K_2(n))$ we get by (-1)–surgery along $K_2(n)$ is equal to $72n+270$. Therefore the Ohtsuki invariants λ_2 of the manifolds $S^3_{-1}(K_2(n))$ are all different, implying that the 3–manifolds are pairwise nondiffeomorphic. □

Proof of Theorem 1.6 The examples $(K_1(n), K_2(n))$ found above verify the theorem: the same proof as the one given in Theorem 2.1 applies and shows that the 4–manifolds corresponding to a pair $(K_1(n), K_2(n))$ are homeomorphic but nondiffeomorphic, and by Proposition 2.3 for $n \geq 0$ these examples are pairwise distinct. By the proof of Theorem 1.2, we see that for each $n \geq 0$ the smooth 4–manifolds corresponding to $K_1(n)$ and $K_2(n)$ are distinguished by the sg–invariants given in Definition 1.1. (The proof of Theorem 1.2 is given in the next section.) □

3 Maps on 4–manifolds

One of the main ingredients of our arguments below is derived from a construction of Saeki. This construction (under some specific restrictions on the topology of the 4–manifold) provides stable maps on 4–manifolds with strong control on their singular sets. (In fact, in our applications we will only use the existence part of the equivalence.)

Theorem 3.1 [19, Theorem 3.1] *Suppose that X is a closed, oriented, connected, smooth 4–manifold with embedded nonempty (not necessarily connected) surfaces $F = F_0 \cup F_1$. There exists a fold map $f\colon X \to \mathbb{R}^3$ with F_0 the definite and F_1 the indefinite fold singular locus if and only if:*

(1) *The Euler characteristics satisfy $\chi(X) = \chi(F_0) - \chi(F_1)$.*

(2) *The Poincaré dual of the class $[F]$ represented by the surface F (in mod 2 homology) coincides with $w_2(X)$.*

(3) *F_0 is orientable.*

(4) *The self-intersection of every component of F_1 is zero.*

(5) *The self-intersection of F_0 is equal to $3\sigma(X)$, where $\sigma(X)$ is the signature of X.* □

Let K be a knot in S^3 and denote by $X = X_K$ the 4–manifold obtained by gluing a 2–handle to D^4 determined by K and framing -1. The Euler characteristic $\chi(X)$ of X is equal to 2. An embedded orientable surface S coming from the core of the 2–handle and a surface in D^4 bounding K represents the generator of $H_2(X;\mathbb{Z}) = \mathbb{Z}$. It follows that $S \cdot S = -1$. If K is slice, then S can be chosen to be a sphere.

A fold map is called a *definite fold map* if it has only definite fold singularities.[1]

Proposition 3.2 *Let X be a 4–manifold given by attaching a single 2–handle to D^4. There is a definite fold map $f\colon X \to \mathbb{R}^3$ such that S is equal to the singular set of f.*

Proof Double X along its boundary. It is easy to see that the resulting closed 4–manifold is diffeomorphic to $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$; cf [9]. Apply [19, Theorem 3.1] of Saeki quoted in Theorem 3.1 as follows. Let F_0 be $S \cup \overline{S}$ in $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ and let F_1 be a standardly embedded surface of genus $1 + 2g(S)$ in a small local chart in the second copy of X such that $(S \cup \overline{S}) \cap F_1 = \varnothing$. Then conditions (1)–(5) of [19, Theorem 3.1] are satisfied: $\chi(\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2) = \chi(S \cup \overline{S}) - \chi(F_1) = 4$; the Poincaré dual of the homology class $[S \cup \overline{S} \cup F_1]$ is characteristic, hence reduces to $w_2(\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2)$ mod 2;

[1] Another terminology for a definite fold map is *special generic map* or *submersion with definite folds*.

$S \cup \bar{S}$ is orientable; $F_1 \cdot F_1 = 0$ and $S \cdot S + \bar{S} \cdot \bar{S} = 0$. Hence there is a fold map $f' \colon \mathbb{CP}^2 \# \overline{\mathbb{CP}}^2 \to \mathbb{R}^3$ such that S is a component of the definite fold singular set. Thus restricting f' to X gives a definite fold map $f \colon X \to \mathbb{R}^3$ such that S is equal to the singular set of f. □

This construction provides the essential tool to prove Theorem 1.2.

Proof of Theorem 1.2 Let $X_j = X_{K_j(n)}$ for some $n \in \mathbb{N}$ and $j \in \{1, 2\}$, where $K_j(n)$ are the knots found in the text preceeding the proof of Theorem 1.6. By Proposition 3.2 both X_1 and X_2 have definite fold maps into $\mathbb{R}^3$ such that the singular set components represent a generator for the second homology group H_2, thus $\mathrm{sg}^1(X_i, A) \neq \infty$, $i = 1, 2$. We have that $\mathrm{sg}^1(X_1, A) = 0$ since X_1 is the manifold obtained by handle attachment along a slice knot. On the other hand, as the proof of Theorem 2.1 showed, an embedded sphere cannot represent a generator of $H_2(X_2; \mathbb{Z})$, hence $\mathrm{sg}^1(X_2, A) > 0$. Finally, clearly each of two disjoint surfaces cannot represent a generator, hence we get the result for sg^k, where $k \geq 2$. □

Maps on closed 4–manifolds

With a little bit of additional work, we can prove a similar statement for closed 4–manifolds, eventually leading us to the proof of Theorem 1.3.

Proof of Theorem 1.3 Let $K_1 = K_1(n)$ and $K_2 = K_2(n)$ be one of the pairs of examples provided by Theorem 1.6. The property of K_2 being smoothly isotopic to a Legendrian knot with vanishing Thurston–Bennequin invariant implies that the 4–manifold with boundary X_2 we get by attaching a 4–dimensional 2–handle along K_2 with framing -1 admits a Stein structure. By a result of Lisca–Matić [13] the 4–manifold X_2 therefore embeds into a minimal complex surface Z of general type (which we can always assume to have $b_2^+ > 1$). Since X_2 has odd intersection form, it follows that the intersection form Q_Z of Z is also odd. Indeed, we can also assume that the intersection form of $Z - \mathrm{int}\, X_2$ is also odd. Therefore Q_Z can be written in an appropriate basis $B_Z = \{e, a, b, f_1, \ldots, f_n, g_1, \ldots, g_m\}$ of $H_2(Z; \mathbb{Z})/\mathit{Torsion}$ as

$$\langle -1 \rangle \oplus H \oplus n\langle -1 \rangle \oplus m\langle 1 \rangle$$

for some n, m, where the first summand (generated by e) corresponds to the generator of $H_2(X_2; \mathbb{Z})$ and H denotes a hyperbolic pair with basis elements a, b. Blow up Z j times (with $j = 0, 1, 2$ or 3) in order to achieve that the resulting complex surface W has signature $\sigma(W)$ divisible by 4: $\sigma(W) = 4k$. The basis elements

$h_1, \dots, h_j \in B_W = B_Z \cup \{h_1, \dots, h_j\}$ correspond to the exceptional divisors of the (possible) blow-ups. Consider now the homology class

$$\Sigma = e + 2 \cdot a + 2k \cdot b + \sum f_i + \sum g_j + \sum h_l .$$

Since by definition $-1 - n + m - j = \sigma(W) = 4k$, it is easy to see that:

Lemma 3.3 *The homology class Σ is a characteristic element in the sense that the Poincaré dual $PD(\Sigma)$ reduced mod 2 is equal to the second Stiefel–Whitney class $w_2(W)$, and the self-intersection of Σ is equal to $-1 + 8k - n + m - j = 3\sigma(W)$.* □

As any second homology class of a smooth 4–manifold, the class Σ can be represented by a (not necessarily connected) oriented surface $F_0 \subset W$. Indeed, we can assume that the part $\sum h_l$ of Σ is represented by j disjoint embedded spheres of self-intersection -1. Notice also that W does not contain $j+1$ disjoint (-1)-spheres: Since Z has two Seiberg–Witten basic classes $\pm c_1(Z)$ with $c_1^2(Z) > 0$, by the blow-up formula (and since it is of simple type) W has 2^{j+1} basic classes. If W had $j+1$ disjoint (-1)–spheres, then it could be written as $W = Y \#_{j+1} \overline{\mathbb{CP}}^2$, hence by the blow-up formula again Y has a unique basic class, which is therefore equal to 0, implying that $c_1^2(Z) = -1$, a contradiction. Since $b_2^+(W) > 1$, there is no homologically nontrivial embedded sphere in the complex surface W with nonnegative self-intersection.

Now let X_1 denote the 4–manifold with boundary we get by attaching a 4–dimensional 2–handle to D^4 along K_1. Since ∂X_1 is diffeomorphic to ∂X_2, we can consider the smooth 4–manifold $V = X_1 \cup (W - X_2)$. Notice that it is homeomorphic to W (since X_1 is homeomorphic to X_2). Consider the homology class $\Sigma' \in H_2(V;\mathbb{Z})$ corresponding to $\Sigma \in H_2(W;\mathbb{Z})$. It can be represented by an orientable embedded surface F_0' which has $j+1$ disjoint spherical components (all with self-intersection (-1)) and some further components. This follows from the fact that the j exceptional divisors of $W - X_2$ can be represented by such spheres in $W - X_2 = V - X_1$ and, in addition, the generator of $H_2(X_1;\mathbb{Z})$ also can be represented by an embedded sphere (of self-intersection (-1)), since K_1 is a slice knot.

Note that (since the signature $\sigma(W)$ is divisible by 4) the Euler characteristics $\chi(W)$ and $\chi(V)$ are even. Let F_1 be a closed, orientable surface embedded in W such that $F_0 \cap F_1 = \varnothing$, $\chi(F_1) = \chi(F_0) - \chi(W)$, $[F_1] = 0$ and $F_1^r \cdot F_1^r = 0$ for each connected component F_1^r of F_1. (For example, F_1 can be chosen to be standardly embedded in a local coordinate chart of W.) Similarly, let F_1' be a closed, orientable surface in V such that $F_0' \cap F_1' = \varnothing$, $\chi(F_1') = \chi(F_1') - \chi(V)$, $[F_1'] = 0$ and $F_1'^r \cdot F_1'^r = 0$ for each connected component $F_1'^r$ of F_1'.

Then conditions (1)–(5) of [19, Theorem 3.1] are satisfied for W, F_0 and F_1: (1) is obvious by the choice of F_1; (2) follows from Lemma 3.3; (3) and (4) are obvious and (5) follows from Lemma 3.3 as well. Similarly, conditions (1)–(5) of [19, Theorem 3.1] are also satisfied for V, F_0' and F_1'. So there exist fold maps on W and V such that their singular sets are the surfaces $F_0 \cup F_1$ and $F_0' \cup F_1'$, respectively. By construction, V contains $j+1$ disjoint (-1)–spheres, hence by Theorem 3.1, with the choice $k = j+1$, we have $\mathrm{sg}^k(V, A) = 0$. For W the argument following Lemma 3.3 shows that with the same choice of k we have $\mathrm{sg}^k(W, A) > 0$. It is easy to see that for any $1 \le l \le j$ we have $\mathrm{sg}^l(V, A) = \mathrm{sg}^l(W, A) = 0$. This completes the proof of Theorem 1.3. □

4 Stable maps and defects

Let us start by recalling the notions of total defect, canonical framing and stable framing [10]. A *stable framing* of a 3–manifold M is a homotopy class of a trivialization (ie a maximum number of linearly independent sections) of the trivial vector bundle $TM \oplus \varepsilon^1$. The degree $d(\phi)$ of a stable framing ϕ is the degree of the map $\nu\colon M \to S^3$, where ν is the framing of ε^1. The *Hirzebruch defect* $h(\phi)$ of ϕ is defined to be $p_1(X, \phi) - 3\sigma(X)$, where X is a compact oriented 4–manifold bounded by M. The *total defect* $H(\phi)$ of ϕ is the pair $(d(\phi), h(\phi))$ and $H\colon \mathbb{F}_{\mathfrak{s}} \to \mathbb{Z} \oplus \mathbb{Z}$ is an embedding of the set of homotopy classes of stable framings $\mathbb{F}_{\mathfrak{s}}$ extending a fixed spin structure $\mathfrak{s}$ on M. Finally, a stable framing ϕ is *canonical* for a spin structure $\mathfrak{s}$ if it is compatible with $\mathfrak{s}$ and $|H(\phi)| \le |H(\psi)|$ for any stable framing ψ which is compatible with $\mathfrak{s}$. It also follows that the invariant $2|d| + |h|\colon \mathbb{F}_{\mathfrak{s}} \to \mathbb{N}$ takes its minimum on a canonical ϕ. A canonical framing may be not unique. For details see [10].

A smooth map $f\colon X \to \mathbb{R}^3$ of a 4–manifold without singularities near the boundary induces a framing ϕ on the complement of a neighborhood $N(\Sigma)$ of the singular set Σ such that ϕ also gives a stable framing $\phi_{\partial X}$ on the boundary 3–manifold. We get $\phi = (\xi_0, \xi_1, \xi_2, \xi_3)$ by taking the 1–dimensional kernel ξ_0 of df in the tangent space of $X - N(\Sigma)$ and by pulling back the standard 1–forms dx_1, dx_2, dx_3 in $\mathbb{R}^3$ via the differential of the submersion $f|_{X - N(\Sigma)}$. Then a chosen Riemannian metric on $X - N(\Sigma)$ gives ξ_i as the dual to dx_i, $i = 1, 2, 3$.

Lemma 4.1 *Let X be a compact oriented 4–manifold with boundary and $\Sigma^0 = \bigcup_{i=1}^{u} \Sigma_i$ and $\Sigma^1 = \bigcup_{i=1}^{v} \Sigma_{u+i}$ unions of closed, oriented, connected, nonempty disjoint surfaces embedded in X. Assume $\Sigma^0 \cup \Sigma^1$ is disjoint from a neighborhood of ∂X. If there exists a fold map $X \to \mathbb{R}^3$ with Σ^0 and Σ^1 as definite and indefinite fold singular sets, respectively, then:*

(1) *The Hirzebruch defect* $h(\phi_{\partial X}) = \Sigma^0 \cdot \Sigma^0 - 3\sigma(X)$.

(2) $\Sigma^1 \cdot \Sigma^1 = 0$.

(3) *The Poincaré dual to the mod 2 homology class* $[\Sigma^0 \cup \Sigma^1]$ *is* $w_2(X)$.

(4) *The degree* $d(\phi_{\partial X}) = \chi(X) - \chi(\Sigma^0) + \chi(\Sigma^1)$.

(5) $\phi_{\partial X}$ *is compatible with a spin structure on the complement* $X - N(\Sigma^0 \cup \Sigma^1)$ *of a tubular neighborhood of* $\Sigma^0 \cup \Sigma^1$.

Remark 4 Our proof implies that if there exists such a fold map but $\Sigma^1 = \varnothing$, then (1)–(5) still hold (if we define $\varnothing \cdot \varnothing = 0$ and $\chi(\varnothing) = 0$).

Proof Suppose that there exists such a fold map $f\colon X \to \mathbb{R}^3$.

(2) holds because the normal disk bundle of Σ^1 is trivial, since the symmetry group of the indefinite fold singularity germ $(x, y) \mapsto x^2 - y^2$ can be reduced to a finite 2–primary group.

(3) holds because the map f restricted to $X - (\Sigma^0 \cup \Sigma^1)$ is a submersion into $\mathbb{R}^3$ hence the tangent bundle of $X - (\Sigma^0 \cup \Sigma^1)$ has a framing.

For (5) let ϕ denote the induced framing on $X - N(\Sigma)$. Since the Poincaré dual $PD[\Sigma^0 \cup \Sigma^1] \equiv w_2(X)$ (mod 2), ϕ is compatible with a spin structure on $X - N(\Sigma)$ and this proves (5).

For (4), we know by [10, Lemma 2.3(b)] that $d(\phi_{\partial(X-N(\Sigma))}) = \chi(X - N(\Sigma))$ since the stable framing $\phi_{\partial(X-N(\Sigma))}$ on $\partial(X - N(\Sigma))$ given by f extends to a framing on $X - N(\Sigma)$. We have $d(\phi_{\partial(X-N(\Sigma))}) = d(\phi_{\partial X}) - 2\chi(\Sigma^1)$ by [19, Lemma 3.2]. Hence we have $d(\phi_{\partial X}) = 2\chi(\Sigma^1) + \chi(X - N(\Sigma)) = \chi(\Sigma^1) + \chi(X) - \chi(\Sigma^0)$, which proves (4).

For (1), by [10, Lemma 2.3(b)] we have $p_1(X - N(\Sigma), \phi_{\partial(X-N(\Sigma))}) = 0$. Also we have that $\sum_j h(\phi_j) = p_1(X - N(\Sigma), \phi) - 3\sigma(X - N(\Sigma))$, where j runs over the boundary components of $X - N(\Sigma)$ and ϕ_j denotes the corresponding stable framing on that boundary component. Hence $h(\phi_{\partial X}) = -3\sigma(X - N(\Sigma)) - \sum_{i=1}^{u+v} h(\phi_i)$, where $\phi_1, \dots, \phi_{u+v}$ are the stable framings on $\partial N(\Sigma_i)$, $i = 1, \dots, u+v$, respectively. From the proofs of [19, Theorem 3.1 and Lemma 3.4], we know that $h(\phi_i) = -\Sigma_i \cdot \Sigma_i + 3\,\mathrm{sgn}(\Sigma_i \cdot \Sigma_i)$ if Σ_i is a definite fold component of Σ, otherwise $h(\phi_i) = 0$. Note that $\Sigma_i \cdot \Sigma_i = 0$ for indefinite fold singular set components. Thus

$$h(\phi_{\partial X}) = -3\sigma(X - N(\Sigma)) + \Sigma \cdot \Sigma - 3 \sum_{i=1}^{u+v} \mathrm{sgn}(\Sigma_i \cdot \Sigma_i)$$

and

$$h(\phi_{\partial X}) = \Sigma \cdot \Sigma - 3\sigma(X),$$

proving (1). □

Let X be the 4–manifold obtained by attaching a 2–handle to D^4 along a p–framed knot in S^3, where $p \in \mathbb{Z}$. We can express the total defect of the induced stable framing $\phi_{\partial X}$ as follows.

Proposition 4.2 *Let $f\colon X \to \mathbb{R}^3$ be a fold map with $\Sigma^0 = \bigcup_{i=1}^{u} \Sigma_i$ and $\Sigma^1 = \bigcup_{i=1}^{v} \Sigma_{u+i}$ as nonempty definite and indefinite fold singular sets, respectively, both consisting of closed connected orientable surfaces. Then:*

1. *The total defect $H(\phi_{\partial X}) = (\chi(\Sigma^1) + 2 - \chi(\Sigma^0), -3\,\mathrm{sgn}(p) + p\sum_{i=1}^{u} k_i^2)$, where each component Σ_i of the singular set represents $\pm k_i$ times the generator of $H_2(X;\mathbb{Z})$.*
2. *If ∂X is an integral homology sphere, then $H(\phi_{\partial X})$ is of the form $(2s, 4r+2)$ or $(2s, 4r)$ for some $r, s \in \mathbb{Z}$.*

Proof We get that $h(\phi_{\partial X}) = p\sum_i k_i^2 - 3\,\mathrm{sgn}(p)$, where i runs over the definite fold components of Σ. We also have $d(\phi_{\partial X}) = \chi(\Sigma^1) + 2 - \chi(\Sigma^0)$. This gives that if all the singular set components are orientable, then $d(\phi_{\partial X})$ is even. The total defect $H(\phi_{\partial X})$ is equal to

$$\left(\chi(\Sigma^1) + 2 - \chi(\Sigma^0), -3\,\mathrm{sgn}(p) + p\sum_{i=1}^{u} k_i^2\right).$$

If ∂X is a homology sphere, then by [10, Theorem 2.6] we obtain that $H(\phi_{\partial X})$ should be in the coset $\Lambda_0 + (0,k)$, where $k = 0$ or $k = 2$, and Λ_0 is the subgroup of $\mathbb{Z} \oplus \mathbb{Z}$ generated by $(0,4)$ and $(-1,2)$. Thus $h(\phi_{\partial X})$ is of the form $4r+2$ for $r \in \mathbb{Z}$ or of the form $4r$ for $r \in \mathbb{Z}$ (this depends on the μ–invariant of ∂X; see [10]). □

Let X be the 4–manifold obtained by attaching a 2–handle to D^4 along a (-1)–framed knot in S^3.

Proposition 4.3 *There is a fold map $f\colon X \to \mathbb{R}^3$ such that the total defect of the stable framing induced by f on ∂X is canonical.*

Proof Double X along its boundary. As before, the resulting closed 4–manifold is diffeomorphic to $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ (cf [9]). Let S be an embedded orientable surface in X

coming from the core of the 2–handle and a surface in D^4 bounding K and hence representing the generator of $H_2(X;\mathbb{Z}) = \mathbb{Z}$.

Apply [19, Theorem 3.1] as follows. Let F_0 be $S \cup \bar{S}$ in $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ and F_1 be a union of two disjoint copies of a small closed orientable surface of Euler characteristic $\chi(S) - 2$ such that each component of F_1 is null-homologous (and so its self-intersection is equal to 0). One component of F_1 is embedded into X, the other component into $\bar{X}$ and suppose $(S \cup \bar{S}) \cap F_1 = \varnothing$. Then conditions (1)–(5) of [19, Theorem 3.1] are satisfied, so there exists a fold map on $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$ with $F_0 \cup F_1$ as its singular set. Restricting this map to X, we get a fold map such that (by Proposition 4.2) the total defect of the induced stable framing on ∂X is equal to $(0, 2)$. Hence this stable framing is canonical. □

Note that if S is a sphere, then a similar construction to Proposition 3.2 gives us a definite fold map with the claimed property.

Now we are in the position to prove Theorem 1.5.

Proof of Theorem 1.5 Once again, let (K_1, K_2) be a pair of knots provided by Theorem 1.6, and let $X_j = D^4_{-1}(K_j)$ be the 2–handlebody we get by attaching a single 2–handle to D^4 along K_j with framing (-1). Clearly both X_1 and X_2 have fold maps as given in the proof of Proposition 4.3. These stable maps satisfy property $\mathcal{A}$, so $\mathrm{sg}(X_i, \mathcal{A}) < \infty$, $i = 1, 2$. Obviously $\mathrm{sg}(X_1, \mathcal{A}) = 0$. If X_2 has a fold map giving $\mathrm{sg}(X_2, \mathcal{A}) = 0$, then all the singular set components are spheres and by Proposition 4.2 the total defect $H(\phi_{\partial X_2}) = (2s, 3 - \sum_i k_i^2)$ for some $s \in \mathbb{Z}$. Since by assumption $\phi_{\partial X_2}$ is canonical and ∂X_2 is a homology sphere, by [10] and Proposition 4.2 its total defect should be equal to $(0, \pm 2)$ or $(0, 0)$. Hence $\sum_i k_i^2$ is equal to 1, 3 or 5, which implies that some $k_i = \pm 1$, which is impossible for X_2 since the generator of its second homology cannot be represented by a smoothly embedded sphere. □

Remark 5 It is not difficult to obtain results similar to those of Sections 3 and 4 in the case of $\mathcal{M}$ being the set of all the definite fold maps and $\mathcal{S}$ being the one element set of the definite fold singularity. However, a 4–manifold X typically does not admit any definite fold map into $\mathbb{R}^3$ (cf [20]); in this case $\mathrm{sg}^k_{\mathcal{M},\mathcal{S}}(X, A) = \infty$.

Appendix A: Calculation of Conway and Jones polynomials

In this appendix we give the details of the computation of the Ohtsuki invariants of the 3–manifolds $S^3_{-1}(K_j(n))$ from Section 2. For simplicity let L_n denote $K_2(n)$, the knot we get from the knot K_2 of the lower part of Figure 1 after inserting the module of n full twists in the box.

Remark 6 From the knot tables we get that L_0 is the 5_2 knot, $L_1 = 9_{45}$ and $L_2 = 11n_{63}$.

We define the Conway and Jones polynomials of oriented links using the conventions (and, in particular, the skein relations and normalizations) as they are given by Lin and Wang [12, page 299].

Lemma A.1 *The Conway polynomial of L_n is equal to*

$$\nabla(L_n) = 1 + 2z^2 - nz^4.$$

In particular, the coefficient of the z^4–term is $-n$.

Proof Recall that the Conway polynomial of an oriented knot/link satisfies the skein relation

$$\nabla(K_+) - \nabla(K_-) = -z \cdot \nabla(K_0)$$

and is normalized as $\nabla(U) = 1$, where U denotes the unknot and K_+, K_-, K_0 admit projections identical away from a crossing, where K_+ has a positive, K_- a negative crossing and K_0 is the oriented resolution.

The skein relation applied to any of the crossing in the module of L_n shows that

$$\nabla(L_n) - \nabla(L_{n-1}) = -z \cdot \nabla(J_0),$$

where J_0 is the 2–component link we get by replacing the module with Figure 6.

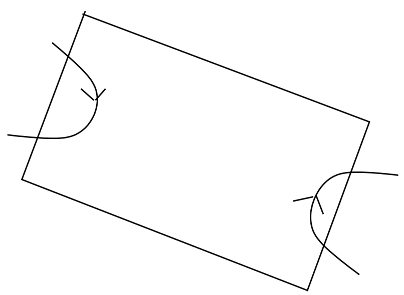

Figure 6: *The module giving the link J_0.* After inserting this module, we get a two-component link, one component being the right handed trefoil knot, the other one the unknot. The two components link geometrically twice, but with vanishing linking number.

This identity shows that $\nabla(L_n) = \nabla(L_0) - nz \cdot \nabla(J_0)$. Further repeated application of the skein relation computes $\nabla(L_0) = 1 + 2z^2$.

The link J_0 has a trefoil component and an unknot component (linking it twice, with linking number zero). The repeated application of the skein relation shows that $\nabla(J_0) = z^3$. This computation then proves the lemma. □

Remark 7 It is not hard to see that L_0 is isotopic to the knot 5_2 in the usual knot tables, while the 2–component link J_0 can be identified with the link $L7n2$ in Thistlethwaite's Link Table.

Lemma A.2 *For $n \geq 1$ the Jones polynomial $V(L_n, t)$ of L_n is*

$$(1 + t^{-2} + \cdots + (t^{-2})^{n-1})\widetilde{V}(t) + t^{-2n} V(L_0, t),$$

where

$$\begin{aligned}\widetilde{V}(t) &= t^{-1}(t^{1/2} - t^{-1/2})V(J_0, t)\\ &= 2t^{-1} - 3t^{-2} + 3t^{-3} - 3t^{-4} + 2t^{-5} - 2t^{-6} + t^{-7},\\ V(L_0, t) &= t^{-1} - t^{-2} + 2t^{-3} - t^{-4} + t^{-5} - t^{-6}.\end{aligned}$$

Proof We compute the Jones polynomial using the skein relation

$$tV(K_+, t) - t^{-1}V(K_-, t) = (t^{1/2} - t^{-1/2})V(K_0),$$

where the Jones polynomial of the unknot is defined to be 1 and K_+, K_-, K_0 are as in the previous lemma. As before, we apply the skein relation to any of the crossing in the module of L_n. (The two further links in the relation are L_{n-1} and J_0 again.) Therefore induction for $n \geq 0$ shows that

$$V(L_n, t) = \big(1 + t^{-2} + \cdots + (t^{-2})^{n-1}\big)t^{-1}\big(t^{1/2} - t^{-1/2}\big)V(J_0, t) + (t^{-2})^n V(L_0, t),$$

By computing $V(J_0, t)$ and $V(L_0, t)$ using the same skein relation we get the statement. (cf also Remark 7 regarding the polynomials of L_0 and J_0.) □

Lemma A.3 *For the Jones polynomial of L_n we have*

$$\frac{\partial^2 V(L_n, e^h)}{\partial h^2}(0) = -12, \qquad \frac{\partial^3 V(L_n, e^h)}{\partial h^3}(0) = 36n + 108.$$

Proof Simple differentiation and substitution gives that

$$\text{(A-1)}\qquad \begin{aligned} &\widetilde{V}(e^h)|_{h=0} = 0, & &\frac{\partial \widetilde{V}(e^h)}{\partial h}\Big|_{h=0} = 2,\\ &\frac{\partial^2 \widetilde{V}(e^h)}{\partial h^2}\Big|_{h=0} = -4, & &\frac{\partial^3 \widetilde{V}(e^h)}{\partial h^3}\Big|_{h=0} = -28.\end{aligned}$$

The identities

$$\frac{\partial V(L_n,e^h)}{\partial h}(h) = \big(-2e^{-2h}-4e^{-4h}-\cdots-(2n-2)e^{(-2n+2)h}\big)\widetilde{V}(e^h) + \big(1+e^{-2h}+\cdots+e^{(-2n+2)h}\big)\frac{\partial \widetilde{V}(e^h)}{\partial h}(h) + \sum_{i=1}^{6}(-1)^{i+1}(-2n-i)e^{(-2n-i)h}+(-2n-3)e^{(-2n-3)h},$$

$$\frac{\partial^2 V(L_n,e^h)}{\partial h^2}(h) = \big(2^2e^{-2h}+4^2e^{-4h}+\cdots+(2n-2)^2e^{(-2n+2)h}\big)\widetilde{V}(e^h) + 2\big(-2e^{-2h}-4e^{-4h}-\cdots-(2n-2)e^{(-2n+2)h}\big)\frac{\partial \widetilde{V}(e^h)}{\partial h}(h) + \big(1+e^{-2h}+\cdots+e^{(-2n+2)h}\big)\frac{\partial^2 \widetilde{V}(e^h)}{\partial h^2}(h) + \sum_{i=1}^{6}(-1)^{i+1}(-2n-i)^2e^{(-2n-i)h}+(-2n-3)^2e^{(-2n-3)h},$$

$$\frac{\partial^3 V(L_n,e^h)}{\partial h^3}(h) = \big(-2^3e^{-2h}-4^3e^{-4h}-\cdots-(2n-2)^3e^{(-2n+2)h}\big)\widetilde{V}(e^h) + 3\big(2^2e^{-2h}+4^2e^{-4h}+\cdots+(2n-2)^2e^{(-2n+2)h}\big)\frac{\partial \widetilde{V}(e^h)}{\partial h}(h) + 3\big(-2e^{-2h}-4e^{-4h}-\cdots-(2n-2)e^{(-2n+2)h}\big)\frac{\partial^2 \widetilde{V}(e^h)}{\partial h^2}(h) + \big(1+e^{-2h}+\cdots+e^{(-2n+2)h}\big)\frac{\partial^3 \widetilde{V}(e^h)}{\partial h^3}(h) + \sum_{i=1}^{6}(-1)^{i+1}(-2n-i)^3e^{(-2n-i)h}+(-2n-3)^3e^{(-2n-3)h},$$

together with the values determined in Equation (A-1) now give

$$\frac{\partial^2 V(L_n,e^h)}{\partial h^2}(0) = -4n(n-1)-4n+\sum_{i=1}^{6}(-1)^{i+1}(2n+i)^2+(2n+3)^2 = -4n(n-1)-4n-12n-21+(2n+3)^2 = -12.$$

This computation proves the first claim of the lemma, and it also shows (by [15]) that the Casson invariant of $S^3_{-1}(L_n)$ is equal to -2 (and, in particular, is independent of n). Furthermore

$$\begin{aligned}\frac{\partial^3 V(L_n, e^h)}{\partial h^3}(0) &= 24(1^2+\cdots+(n-1)^2)+24(1+\cdots+(n-1))-28n \\ &\qquad + \sum_{i=1}^{6}(-1)^i(2n+i)^3-(2n+3)^3 \\ &= 4(n-1)n(2n-1)+12n(n-1)-28n \\ &\qquad + \sum_{i=1}^{6}(-1)^i(2n+i)^3-(2n+3)^3 \\ &= 36n+108,\end{aligned}$$

verifying the second claim of the lemma. □

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