

## Reidemeister torsion of Seifert fibered homology lens spaces and Dehn surgery

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We provide necessary conditions on the Alexander polynomial of a knot  $K$  in a homology sphere and on surgery coefficients  $p/q$  for the surgered manifold to be a Seifert fibered space over  $S^2$ . As an application, we show that no  $p/q$ -surgery with  $p > 3$  on a knot in a homology sphere with the same Alexander polynomial as the figure eight knot can produce a Seifert fibered space with base  $S^2$ . The main tool is the abelian Reidemeister torsion.

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### 1 Introduction

The Reidemeister torsion of manifolds has been studied since 1930's (see Turaev [19]). In particular, the Reidemeister torsion of Seifert fibered rational homology sphere is studied in Némethi and Nicolaescu [12]. In this paper, we study the Reidemeister torsion of Seifert fibered homology lens space over  $S^2$ . It is well-known that every Seifert fibered homology lens space has  $S^2$  or  $\mathbb{R}P^2$  as its base space, and every Seifert fibered space over  $S^2$  has a framed link presentation as in Figure 1, where  $p_i \geq 2$  for each  $i$ . We call  $p_i$  the *multiplicity* of the singular fiber. See Orlik [14] or Saveliev [16], for example. Thus the object of our study is such a 3-manifold that has the framed link presentation of Figure 1 and has a finite cyclic first homology group. We extract information on multiplicities of singular fibers from Reidemeister torsion by using the norm and order of homology lens spaces which we introduced in [6], and apply the information to the Seifert surgery problem to determine when Dehn surgery yields a Seifert fibered space.

Throughout this paper,  $\zeta_d$  denotes a primitive  $d$ -th root of unity,  $H_1(X)$  denotes the first homology group of  $X$  with integer coefficients, and the operation in  $H_1(X)$  is written multiplicatively unless otherwise stated. We also use the following notation:

- Let  $A$  and  $B$  be elements of  $\mathbb{Q}(\zeta_d)$ . Then  $A \doteq B$  means an equality of  $A$  and  $B$  up to multiplication of  $\pm \zeta_d^m$  ( $m \in \mathbb{Z}$ ).

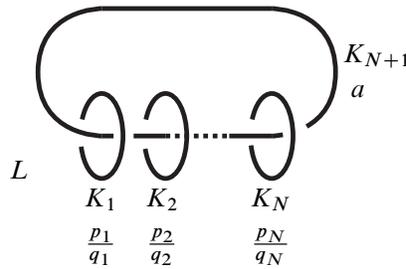


Figure 1: framed link presentation of a Seifert fibered space over  $S^2$

- Let  $\alpha$  be an element of  $\mathbb{Q}(\zeta_d)$ . Then  $N_d(\alpha)$  denotes the norm of  $\alpha$  associated to the algebraic extension  $\mathbb{Q}(\zeta_d)$  over  $\mathbb{Q}$ .
- Let  $x$  be an element of  $(\mathbb{Z}/p\mathbb{Z})^\times$ , the multiplicative group of the ring  $\mathbb{Z}/p\mathbb{Z}$ . Then  $\bar{x}$  denotes the inverse element of  $x$ .

Let  $\Sigma$  be a homology sphere, and  $K$  a knot in  $\Sigma$ . Then  $\Delta_K(t)$  is the *Alexander polynomial* of  $K$ , and  $\Sigma(K; p/q)$  is the result of  $p/q$ -surgery along  $K$ . More generally, let  $L = K_1 \cup \dots \cup K_n$  be an  $n$ -component link in  $\Sigma$ . Then  $\Delta_L(t_1, \dots, t_n)$  is the ( $n$ -variable) *Alexander polynomial* of  $L$ , and  $\Sigma(L; p_1/q_1, \dots, p_n/q_n)$  is the result of  $p_i/q_i$ -surgeries along  $K_i$  for all  $i = 1, \dots, n$ . We give a more precise description of  $\Sigma(L; p_1/q_1, \dots, p_n/q_n)$  in [Section 2](#).

Let  $M$  be a homology lens space with  $H_1(M) \cong \mathbb{Z}/p\mathbb{Z}$  ( $p \geq 2$ ), and  $T$  a generator of  $H_1(M)$ . Let  $d \geq 2$  be a divisor of  $p$ , and  $\psi_d: \mathbb{Z}[H_1(M)] \rightarrow \mathbb{Q}(\zeta_d)$  a ring homomorphism such that  $\psi_d(T) = \zeta_d$ . Then  $\tau^{\psi_d}(M) \in \mathbb{Q}(\zeta_d)$ , the Reidemeister torsion of  $M$  associated to  $\psi_d$ , is determined up to multiplication by  $\pm \zeta_d^m$  ( $m \in \mathbb{Z}$ ) (see Kadokami [\[7\]](#), Nicolaescu [\[13\]](#) or Turaev [\[18; 19\]](#) for details on Reidemeister torsion).

In [\[6\]](#), we introduced the *norm* and the *order* of polynomials and homology lens spaces: Let  $f(t)$  be a polynomial over  $\mathbb{Z}$ . We define the  $d$ -norm of  $f(t)$ , denoted by  $|f(t)|_d$ , by

$$|f(t)|_d = |N_d(f(\zeta_d))| = \left| \prod_{i \in (\mathbb{Z}/d\mathbb{Z})^\times} f(\zeta_d^i) \right|,$$

where  $d$  is a positive integer. We note that  $|f(t)|_d$  is a nonnegative integer. For an arbitrary homology lens space  $M$ , there exists a knot  $K$  in a homology sphere  $\Sigma$  such that  $M = \Sigma(K; p/q)$  ( $p \geq 2$ ) (see Boyer and Line [\[1, Lemma 2.1\]](#)). We define the  $d$ -norm and the  $d$ -order of  $M$ , denoted by  $|M|_d$  and  $\|M\|_d$  respectively, by

$$|M|_d = |\Delta_K(t)|_d \quad \text{and} \quad \|M\|_d = \prod_{d'|d} |M|_{d'},$$

where  $d$  is a divisor of  $p$ . As proved in [6], both  $|M|_d$  and  $\|M\|_d$  are invariants of a homology lens space  $M$ . We note that the norms and orders of a lens space are all 1 because a lens space is the result of surgery along the unknot.

We fix the following setting.

**Setting** (i)  $L = K_1 \cup \dots \cup K_{N+1}$  is the  $(N + 1)$ -component link in  $S^3$  of Figure 1.

(ii)  $M = S^3(L; p_1/q_1, \dots, p_N/q_N, a)$  is a homology lens space with  $H_1(M) \cong \mathbb{Z}/p\mathbb{Z}$  where  $p \geq 2$ ,  $p_i \geq 2$  ( $i = 1, \dots, N$ ),  $\gcd(p_i, q_i) = 1$  and  $a$  is an integer.

**Theorem 1.1** Let  $M = S^3(L; p_1/q_1, \dots, p_N/q_N, a)$  be a Seifert fibered homology lens space as in the setting above. Let  $d \geq 2$  be any divisor of  $p$  satisfying  $\gcd(d, p_i) = 1$  for all  $i = 1, \dots, N$ . Then we have the following:

(1) Let  $T$  be a generator of  $H_1(M)$  and  $\psi_d: \mathbb{Z}[H_1(M)] \rightarrow \mathbb{Q}(\zeta_d)$  a ring homomorphism such that  $\psi_d(T) = \zeta_d$ . Let  $u$  be an integer such that  $[m_{N+1}] = T^u$  in  $H_1(M)$ , where  $m_{N+1}$  is a meridian of  $K_{N+1}$ . Then we have  $\gcd(d, u) = 1$  and

$$\tau^{\psi_d}(M) \doteq (\zeta_d^u - 1)^{N-2} \prod_{i=1}^N (\zeta_d^{u\bar{p}_i} - 1)^{-1},$$

where  $\bar{p}_i$  is the inverse element of  $p_i$  in  $(\mathbb{Z}/d\mathbb{Z})^\times$ .

(2)  $|M|_d = 1$ .

(3)  $\|M\|_d = 1$ .

**Theorem 1.2** Let  $M = S^3(L; p_1/q_1, \dots, p_N/q_N, a)$  be a Seifert fibered homology lens space as in the setting above, and assume  $N \geq 3$ . Let  $d \geq 2$  be a common divisor of  $p_1$  and  $p_2$ . Then we have the following:

(1)  $d$  is a divisor of  $p$ ,  $\gcd(d, p_i) = 1$  ( $i = 3, \dots, N$ ), and  $\gcd(d, q_i) = 1$  ( $i = 1, 2$ ).

(2) Let  $T$  be a generator of  $H_1(M)$  and  $\psi_d: \mathbb{Z}[H_1(M)] \rightarrow \mathbb{Q}(\zeta_d)$  a ring homomorphism such that  $\psi_d(T) = \zeta_d$ . Let  $u_1$  be an integer such that  $[m_1] = T^{u_1}$  in  $H_1(M)$ , where  $m_1$  is a meridian of  $K_1$ . Then we have  $\gcd(d, u_1) = 1$  and

$$\tau^{\psi_d}(M) \doteq p_3 \cdots p_N (\zeta_d^{u_1 \bar{q}_1} - 1)^{-1} (\zeta_d^{u_1 \bar{q}_2} - 1)^{-1},$$

where  $\bar{q}_i$  ( $i = 1, 2$ ) is the inverse element of  $q_i$  in  $(\mathbb{Z}/d\mathbb{Z})^\times$ .

(3)  $|M|_d = (p_3 \cdots p_N)^{\varphi(d)}$ , where  $\varphi(\cdot)$  is the Euler function.

If  $N \leq 2$ , then  $M$  is a lens space, and hence  $\|M\|_p = 1$ .

The next theorem gives a relation between multiplicities of singular fibers and the order of a Seifert fibered homology lens space.

**Theorem 1.3** *Let  $M = S^3(L; p_1/q_1, \dots, p_N/q_N, a)$  be a Seifert fibered homology lens space as in the setting above. We assume  $N \geq 3$ . Then we have the following:*

(1)  $\|M\|_p = 1$  holds if and only if  $\gcd(p_i, p_j) = 1$  for any pair  $\{i, j\}$  ( $1 \leq i < j \leq N$ ). Moreover it is equivalent to  $\gcd(p, p_i) = 1$  for all  $i = 1, \dots, N$ .

(2)  $\|M\|_p \neq 0, 1$  holds if and only if there uniquely exists a pair  $\{i, j\}$  ( $1 \leq i < j \leq N$ ) such that  $\gcd(p_i, p_j) \geq 2$ .

(3)  $\|M\|_p = 0$  holds if and only if there exist at least two distinct pairs  $\{i, j\}$  ( $1 \leq i < j \leq N$ ) and  $\{k, h\}$  ( $1 \leq k < h \leq N$ ) such that  $\gcd(p_i, p_j) \geq 2$  and  $\gcd(p_k, p_h) \geq 2$ .

As an application of the Reidemeister torsion to Seifert surgery problem, we have the following:

**Theorem 1.4** *Let  $K$  be a knot in a homology sphere  $\Sigma$  such that  $\Delta_K(t) = t^2 - 3t + 1$ . The only surgeries on  $K$  that may produce a Seifert fibered space with base  $S^2$  and with  $H_1 \neq \{0\}, \mathbb{Z}$ , have coefficients  $2/q$  and  $3/q$ , and produce Seifert fibered spaces with three singular fibers. Moreover if the coefficient is  $2/q$ , then the set of multiplicities is  $\{\alpha, \beta, 5\}$  where  $\gcd(\alpha, \beta) = 2$ , and if the coefficient is  $3/q$ , then the set of multiplicities is  $\{\alpha, \beta, 4\}$  where  $\gcd(\alpha, \beta) = 3$ .*

We note that the Alexander polynomial of the figure eight knot is  $t^2 - 3t + 1$ . Hence we may consider [Theorem 1.4](#) as an algebraic analogue of the fact shown by WP Thurston that all coefficients of Seifert surgery along the figure-eight knot are  $\pm 1, \pm 2$  and  $\pm 3$  [[17](#), Chapter 4]. We mention that the Reidemeister torsion cannot capture the case  $p = 1$  because Reidemeister torsion of a homology sphere is zero.

In [Section 2](#), we state surgery formulae for the Reidemeister torsion. In [Section 3](#), we prove a key lemma ([Lemma 3.4](#) (1)) derived from the cyclicity of the first homology group. In [Section 4](#) and [Section 5](#), we prove the theorems by using the surgery formula and lemmas in [Section 3](#).

For references on Dehn surgery including Seifert surgery, see Brittenham and Wu [[2](#)], Culler, Gordon, Luecke and Shalen [[3](#)], Fintushel and Stern [[5](#)], Kadokami [[6](#); [7](#)], Kadokami and Yamada [[8](#); [9](#)], Miyazaki and Motegi [[10](#)], Moser [[11](#)], Orlik [[14](#)], Saveliev [[16](#)] and Thurston [[17](#)]. For Reidemeister torsion, see Kadokami [[6](#); [7](#)], Kadokami and Yamada [[8](#); [9](#)], Nicolaescu [[13](#)], de Rham [[4](#)], Sakai [[15](#)] and Turaev [[18](#); [19](#)].

## 2 Surgery formula for Reidemeister torsion

[Lemma 2.5](#) below is used repeatedly in the later sections. It consists of special cases of a surgery formula for the Reidemeister torsion due to V Turaev, and follows from [Lemma 2.1](#), [Lemma 2.2](#), [Lemma 2.3](#) and [Lemma 2.4](#). We do not give any proofs in this section. For details see Turaev [18; 19], and also Sakai [15] for [Lemma 2.6](#).

Let  $R$  be a commutative ring with nonzero identity element. Then we denote the classical ring of quotients by  $Q(R)$ . Let  $X$  be a finite CW-complex. Then the *maximal abelian torsion* of  $X$  denoted by  $\tau(X)$  is an element of  $Q(\mathbb{Z}[H_1(X)])$ , which is defined from a chain complex  $\mathbf{C}_*$  induced by the maximal abelian covering of  $X$ . Let  $\psi: \mathbb{Z}[H_1(X)] \rightarrow R$  be a ring homomorphism. Then a chain complex is induced from  $\mathbf{C}_*$  and  $\psi$ . We denote it by  $\mathbf{C}_*^\psi$ . The *Reidemeister torsion* associated to  $\psi$ ,  $\tau^\psi(X)$ , is defined from  $\mathbf{C}_*^\psi$ . It is an element of  $Q(R)$  that is determined up to multiplication by an element in  $\pm\psi(H_1(X))$ . If  $\mathbf{C}_*^\psi$  is not acyclic, then we define  $\tau^\psi(X) = 0$ .

**Lemma 2.1** *Let  $X_1$  and  $X_2$  be subcomplexes of a finite CW-complex  $X$  such that  $X = X_1 \cup X_2$ , and  $Y = X_1 \cap X_2$ . Let  $\psi: \mathbb{Z}[H_1(X)] \rightarrow R$  be a ring homomorphism, and let  $\psi_i: \mathbb{Z}[H_1(X_i)] \rightarrow R$  ( $i = 1, 2$ ) and  $\psi': \mathbb{Z}[H_1(Y)] \rightarrow R$  be ring homomorphisms induced by  $\psi$ . Then we have*

$$\tau^\psi(X) \cdot \tau^{\psi'}(Y) \doteq \tau^{\psi_1}(X_1) \cdot \tau^{\psi_2}(X_2).$$

**Lemma 2.2** (1) *Let  $t$  be a generator of  $H_1(S^1 \times D^2)$ . Then we have*

$$\tau(S^1 \times D^2) \doteq (t - 1)^{-1}.$$

(2)  $\tau(S^1 \times S^1) \doteq 1$ .

(3) *Let  $L = K_1 \cup \cdots \cup K_n$  be an  $n$ -component link in a homology sphere,  $E$  the exterior of  $L$ , and  $t_i$  ( $i = 1, \dots, n$ ) the homology class of a meridian of  $K_i$  in  $H_1(E)$ . Then we have*

$$\tau(E) \doteq \begin{cases} \Delta_L(t_1)(t_1 - 1)^{-1} & (n = 1), \\ \Delta_L(t_1, \dots, t_n) & (n \geq 2). \end{cases}$$

**Lemma 2.3** *Let  $X$  be a finite CW-complex, and  $\psi: \mathbb{Z}[H_1(X)] \rightarrow R$  and  $\psi': R \rightarrow R'$  ring homomorphisms. If  $\tau^\psi(X) \neq 0$  and  $\psi'(\tau^\psi(X)) \neq 0$ , then we have*

$$\tau^{\psi' \circ \psi}(X) \doteq \psi'(\tau^\psi(X)).$$

**Lemma 2.4** *Let  $H$  be a finitely generated abelian group. If  $h$  is an element of  $H$  with infinite order, then  $h - 1$  is invertible in  $Q(\mathbb{Z}[H])$ .*

Let  $L = K_1 \cup \cdots \cup K_n$  be an  $n$ -component link in a homology sphere  $\Sigma$ . Let  $p_i$  and  $q_i$  ( $i = 1, \dots, n$ ) be integers satisfying  $p_i \geq 0, q_i \neq 0$  and  $\gcd(p_i, q_i) = 1$  for each  $i$ . Then  $\Sigma(L; p_1/q_1, \dots, p_n/q_n)$  is defined as follows:

(1) Let  $N(L)$  be a regular neighborhood of  $L$  in  $\Sigma$ . Note that  $N(L)$  is a disjoint union of  $N(K_i)$ 's;  $N(L) = \bigcup_{i=1}^n N(K_i)$ , where  $N(K_i)$  is a regular neighborhood of  $K_i$ . Let  $E_L$  denote the exterior of  $L$ :  $E_L = \Sigma \setminus \overline{N(L)}$ . Note that  $\partial E_L = \bigcup_{i=1}^n \partial N(K_i)$ .

(2) Let  $m_i$  and  $l_i$  ( $i = 1, \dots, n$ ) be a meridian and a longitude of  $K_i$ , which lie on  $\partial N(K_i) \subset \partial E_L$ . We take a simple closed curve  $m'_i$  on  $\partial N(K_i)$  ( $i = 1, \dots, n$ ) such that  $[m'_i] = [m_i]^{p_i} [l_i]^{q_i}$  in  $H_1(\partial N(K_i))$ . Let  $r_i$  and  $s_i$  ( $i = 1, \dots, n$ ) be integers satisfying  $p_i s_i - q_i r_i = 1$ . We take a simple closed curve  $l'_i$  on  $\partial N(K_i)$  ( $i = 1, \dots, n$ ) such that  $[l'_i] = [m_i]^{r_i} [l_i]^{s_i}$  in  $H_1(\partial N(K_i))$ .

(3) Let  $V_1, \dots, V_n$  be  $n$ -copies of  $S^1 \times D^2$ . Then  $\Sigma(L; p_1/q_1, \dots, p_n/q_n)$  is defined as the 3-manifold obtained from  $E_L$  and  $\{V_1, \dots, V_n\}$  by identifying  $\partial N(K_i)$  and  $\partial V_i$  by a homeomorphism such that  $m'_i$  is identified with a meridian of  $V_i$  for each  $i$ . Note that  $l'_i$  is identified with a longitude of  $V_i$ . We sometimes express

$$\Sigma(L; p_1/q_1, \dots, p_n/q_n) = E_L \cup V_1 \cup \cdots \cup V_n$$

schematically.

(4) Let  $M = \Sigma(L; p_1/q_1, \dots, p_n/q_n)$  as above. We define  $M_k$  ( $k = 1, \dots, n$ ) by

$$M_k = \overline{M \setminus (V_1 \cup \cdots \cup V_k)} = E_L \cup V_{k+1} \cup \cdots \cup V_n.$$

Note that  $M \supset M_1 \supset \cdots \supset M_n = E_L$ .

We assume that  $M$  is a homology lens space with  $H_1(M) \cong \mathbb{Z}/p\mathbb{Z}$  ( $p \geq 2$ ). Let  $T$  be a generator of  $H_1(M)$ ,  $d \geq 2$  a divisor of  $p$  and  $\psi_d: \mathbb{Z}[H_1(M)] \rightarrow \mathbb{Q}(\zeta_d)$  a ring homomorphism such that  $\psi_d(T) = \zeta_d$ . We then define  $\psi_{k,d}: \mathbb{Z}[H_1(M_k)] \rightarrow \mathbb{Q}(\zeta_d)$  by  $\psi_{k,d} = \psi_d \circ \iota_k$ , where  $\iota_k: \mathbb{Z}[H_1(M_k)] \rightarrow \mathbb{Z}[H_1(M)]$  is a ring homomorphism induced from the natural inclusion  $M_k \hookrightarrow M$ . Then we have the following *surgery formula* for the Reidemeister torsion.

**Lemma 2.5** (1) *If  $n \geq 2$  and  $\psi_d([l'_i]) \neq 1$  ( $i = 1, \dots, n$ ), then we have*

$$\tau^{\psi_d}(M) \doteq \Delta_L(\xi_1, \dots, \xi_n) \prod_{i=1}^n (\psi_d([l'_i]) - 1)^{-1},$$

where  $\xi_i = \psi_d([m_i])$ .

(2) If  $n \geq 2$ , and  $[l'_i]$  ( $i = k + 1, \dots, n$ ) has infinite order in  $H_1(M_k)$ , then we have

$$\tau(M_k) \doteq \Delta_L([m_1], \dots, [m_n]) \prod_{i=k+1}^n ([l'_i] - 1)^{-1} \text{ in } \mathcal{Q}(\mathbb{Z}[H_1(M_k)]).$$

(3) If  $n \geq 1$ ,  $\tau(M_k) \neq 0$  and  $\psi_d([l'_i]) \neq 1$  ( $i = 1, \dots, k$ ), then we have

$$\tau^{\psi_d}(M) \doteq \psi_{k,d}(\tau(M_k)) \prod_{i=1}^k (\psi_d([l'_i]) - 1)^{-1}.$$

The following is an explicit form of Lemma 2.5 (3) in the case  $n = k = 1$ .

**Lemma 2.6** Let  $K$  be a knot in a homology sphere  $\Sigma$  and  $M = \Sigma(K; p/q)$  ( $p \geq 2$ ). Let  $T$  be a generator of  $H_1(M)$  that corresponds to a meridian of  $K$ , and  $\psi_d: \mathbb{Z}[H_1(M)] \rightarrow \mathbb{Q}(\zeta_d)$  a ring homomorphism such that  $\psi_d(T) = \zeta_d$ . Then we have

$$\tau^{\psi_d}(M) \doteq \Delta_K(\zeta_d)(\zeta_d - 1)^{-1}(\zeta_d^{\bar{q}} - 1)^{-1},$$

where  $\bar{q}$  is the inverse element of  $q$  in  $(\mathbb{Z}/d\mathbb{Z})^\times$ .

### 3 Conditions from the first homology group

We consider constraints on the multiplicities of singular fibers that come from the assumption that the first homology group is finite cyclic. Let  $M$  be a Seifert fibered homology lens space as in the setting in Section 1:

$$M = S^3(L; p_1/q_1, \dots, p_N/q_N, a) = E_L \cup V_1 \cup \dots \cup V_N \cup V_{N+1}.$$

We take  $m_i$  and  $l_i$  ( $i = 1, \dots, N + 1$ ) on  $\partial E_L$  as in Section 2. We also take  $m'_i$  and  $l'_i$  on  $\partial E_L$  as in Section 2 except  $m'_{N+1}$  and  $l'_{N+1}$  such that

$$[m'_{N+1}] = [m_{N+1}]^a [l_{N+1}] \quad \text{and} \quad [l'_{N+1}] = [m_{N+1}].$$

Then we have the following:

**Lemma 3.1** The first homology group  $H_1(M)$  is generated by  $[m_i], [l_i], [m'_i]$  and  $[l'_i]$  ( $i = 1, \dots, N + 1$ ) subject to the following relations:

$$\begin{aligned} [m'_i] &= [m_i]^{p_i} [l_i]^{q_i} = 1 \quad (i = 1, \dots, N), & [m'_{N+1}] &= [m_{N+1}]^a [l_{N+1}] = 1, \\ [l_i] &= [m_{N+1}] \quad (i = 1, \dots, N), & [m_1] \cdots [m_N] &= [l_{N+1}], \\ [l'_i] &= [m_i]^{r_i} [l_i]^{s_i} \quad (i = 1, \dots, N), & [l'_{N+1}] &= [m_{N+1}]. \end{aligned}$$

The presentation matrix  $A$  of  $H_1(M)$  with respect to  $[m_1], \dots, [m_{N+1}]$  is

$$A = \begin{pmatrix} p_1 & & 0 & q_1 \\ & \ddots & & \vdots \\ 0 & & p_N & q_N \\ 1 & \cdots & 1 & a \end{pmatrix}.$$

Here the operation is additive. We recall  $H_1(M) \cong \mathbb{Z}/p\mathbb{Z}$ .

**Lemma 3.2** Let  $\tau = p_1 \cdots p_N$  and  $\tau_i = \tau/p_i$  ( $i = 1, \dots, N$ ). Then

$$p = \left| \tau a - \sum_{i=1}^N q_i \tau_i \right|.$$

In particular, if  $d$  divides  $p_i$  and  $p_j$  ( $i \neq j$ ), then  $d$  divides  $p$ .

**Proof** It is well known that  $|\det A|$  is the order of  $H_1(M)$  and

$$\det A = \tau a - \sum_{i=1}^N q_i \tau_i$$

(cf Saviliev [16]). Since the order of  $H_1(M)$  is  $p$ , we have the result.  $\square$

**Lemma 3.3** Let  $A_{i,j}$  be the cofactor of  $(i, j)$ -entry of  $A$ . Then we have

$$\begin{aligned} A_{i,N+1} &= \pm \tau_i & (i = 1, \dots, N), \\ A_{i,i} &= \tau_i a - \sum_{j \neq i} q_j \tau_{ij} & (i = 1, \dots, N), \\ A_{i,j} &= \pm q_j \tau_{ij} & (i \neq j), \\ A_{N+1,i} &= \pm q_i \tau_i & (i = 1, \dots, N), \\ A_{N+1,N+1} &= \tau, \end{aligned}$$

where  $\tau_{ij} = \tau/p_i p_j$  ( $i \neq j$ ).

The following is a key to prove results stated in [Section 1](#).

**Lemma 3.4** (1) Let  $d \geq 2$  be a common divisor of  $p_i$  and  $p_j$  ( $i \neq j$ ). Then we have  $\gcd(d, p_k) = 1$  for  $k \neq i, j$ .

(2) Let  $\ell$  be a prime divisor of  $\gcd(p, p_i)$ . Then there exists  $j \neq i$  such that  $\ell$  is a divisor of  $p_j$ .

(3) If  $\gcd(p_i, p_j) = 1$  for every pair  $\{i, j\}$ , then we have  $\gcd(p, p_i) = 1$  for every  $i$ .

**Proof** (1) Let  $\delta$  be the greatest common divisor of all  $N$ -minors of  $A$ . Since  $H_1(M)$  is cyclic, we have  $\delta = 1$  by the elementary divisor theory. Suppose that there exists  $k \neq i, j$  such that  $\gcd(d, p_k) \geq 2$ . Then each  $A_{i,j}$  is divisible by  $\gcd(d, p_k)$  by Lemma 3.3, and hence  $\delta$  is divisible by  $\gcd(d, p_k)$ . This contradicts  $\delta = 1$ .

(2) By Lemma 3.2 and  $\gcd(p_i, q_i) = 1$ ,  $\ell$  divides  $\tau_i$ . Hence there exists  $j \neq i$  such that  $\ell$  divides  $p_j$ .

(3) Suppose there exists  $i$  such that  $\gcd(p, p_i) \geq 2$ . We take a prime divisor  $\ell$  of  $\gcd(p, p_i)$ . Then there exists  $j \neq i$  such that  $\ell$  divides  $p_j$  by (2). This contradicts  $\gcd(p_i, p_j) = 1$ . □

**Remark 3.5** We can show the following, although not needed here.

- (1)  $H_1(M)$  is cyclic if and only if  $\delta = 1$ .
- (2)  $\delta$  coincides with the greatest common divisor of  $\tau_{ij}$ 's.

### 4 Proofs of Theorem 1.1, Theorem 1.2 and Theorem 1.3

We need the following well known fact (cf Nicolaescu [13]).

**Lemma 4.1** Let  $L$  be the link in Figure 1. Then we have

$$\Delta_L(t_1, \dots, t_N, t_{N+1}) = (t_{N+1} - 1)^{N-1}.$$

Recall that

$$M = S^3(L; p_1/q_1, \dots, p_N/q_N, a) = E_L \cup V_1 \cup \dots \cup V_{N+1}.$$

When a generator  $T$  of  $H_1(M)$  and a ring homomorphism  $\psi_d: \mathbb{Z}[H_1(M)] \rightarrow \mathbb{Q}(\zeta_d)$  such that  $\psi_d(T) = \zeta_d$  are given, we set  $\xi_i = \psi_d([m_i])$  ( $i = 1, \dots, N + 1$ ). Then  $\xi_i$  is a  $d$ -th root of unity and  $\{\xi_1, \dots, \xi_{N+1}\}$  generates  $\psi_d(H_1(M)) \cong \mathbb{Z}/d\mathbb{Z}$ . The next lemma follows from Lemma 3.1.

**Lemma 4.2** The following relations hold:

- (1)  $\xi_i^{p_i} \xi_{N+1}^{q_i} = 1$  ( $i = 1, \dots, N$ ),
- (2)  $\left( \prod_{i=1}^N \xi_i \right) \xi_{N+1}^a = 1$ ,
- (3)  $\psi_d([l'_i]) = \xi_i^{r_i} \xi_{N+1}^{s_i}$  ( $i = 1, \dots, N$ ) and  $\psi_d([l'_{N+1}]) = \xi_{N+1}$ .

**Proof of Theorem 1.1** (1) Let  $d \geq 2$  be as in the statement of Theorem 1.1. By Lemma 4.2 (1) and  $\gcd(d, p_i) = 1$  ( $i = 1, \dots, N$ ), we have  $\xi_i = \xi_{N+1}^{-q_i \bar{p}_i}$ , where  $\bar{p}_i$  is the inverse element of  $p_i$  in  $(\mathbb{Z}/d\mathbb{Z})^\times$ . Hence  $\xi_{N+1}$  generates  $\psi_d(H_1(M)) \cong \mathbb{Z}/d\mathbb{Z}$ . Since a generator of  $\psi_d(H_1(M))$  is a primitive  $d$ -th root of unity,  $\xi_{N+1} = \zeta_d^u$  is also a primitive  $d$ -th root of unity. Therefore we have  $\gcd(d, u) = 1$ .

By Lemma 4.2 (3),  $\xi_i = \xi_{N+1}^{-q_i \bar{p}_i}$  and  $p_i s_i - q_i r_i = 1$ , we have

$$\psi_d([l'_i]) = \xi_i^{r_i} \xi_{N+1}^{s_i} = \xi_{N+1}^{-q_i r_i \bar{p}_i + s_i} = \xi_{N+1}^{\bar{p}_i (p_i s_i - q_i r_i)} = \xi_{N+1}^{\bar{p}_i} \quad (i = 1, \dots, N)$$

and  $\psi_d([l'_{N+1}]) = \xi_{N+1}$ . Hence  $\psi_d([l'_i])$  ( $i = 1, \dots, N + 1$ ) is a primitive  $d$ -th root of unity.

By Lemma 2.5 (1) and Lemma 4.1, we have

$$\begin{aligned} \tau^{\psi_d}(M) &\doteq \Delta_L(\xi_1, \dots, \xi_{N+1}) \prod_{i=1}^{N+1} (\psi_d([l'_i]) - 1)^{-1} \\ &= (\xi_{N+1} - 1)^{N-1} (\xi_{N+1} - 1)^{-1} \prod_{i=1}^N (\xi_{N+1}^{\bar{p}_i} - 1)^{-1} \\ &= (\zeta_d^u - 1)^{N-2} \prod_{i=1}^N (\zeta_d^{u \bar{p}_i} - 1)^{-1}. \end{aligned}$$

(2) Let  $M$  be the result of  $p/q$ -surgery along a knot  $K$  in a homology sphere  $\Sigma$ :  $M = \Sigma(K; p/q)$ . We take the homology class of a meridian for  $K$  as  $T$ . By (1) and Lemma 2.6, we have

$$(4-1) \quad \Delta_K(\zeta_d)(\zeta_d - 1)^{-1} (\zeta_d^{\bar{q}} - 1)^{-1} \doteq (\zeta_d^u - 1)^{N-2} \prod_{i=1}^N (\zeta_d^{u \bar{p}_i} - 1)^{-1}$$

By noting  $\gcd(d, q) = \gcd(d, u) = \gcd(d, p_i) = 1$ , we have

$$|N_d(\zeta_d - 1)| = |N_d(\zeta_d^{\bar{q}} - 1)| = |N_d(\zeta_d^u - 1)| = |N_d(\zeta_d^{u \bar{p}_i} - 1)| \neq 0.$$

Therefore by taking the norms of both sides of (4-1), we have  $|N_d(\Delta_K(\zeta_d))| = 1$  and hence  $|M|_d = 1$ .

(3) If  $d'$  divides  $d$ , then we have  $\gcd(d', p_i) = 1$  for all  $i$ . Hence  $|M|_{d'} = 1$  by (2). Thus we have

$$\|M\|_d = \prod_{d'|d} |M|_{d'} = 1. \quad \square$$

**Lemma 4.3** For  $1 \leq j < i \leq N$ , let  $M_j = E_L \cup V_{j+1} \cup \dots \cup V_{N+1}$  be a 3-manifold as in Section 2. Then we have

$$([m_{N+1}] - 1)([l'_i] - 1)^{-1} = \sum_{k=0}^{p_i-1} [l'_i]^k \text{ in } Q(\mathbb{Z}[H_1(M_j)]).$$

**Proof** We prove the lemma in steps.

**Step 1**  $[m_{N+1}]$  in  $H_1(M_j)$  has infinite order.

**Proof** In this step, we write operation additively, and consider  $H_1(M_j; \mathbb{Q})$  as a vector space over  $\mathbb{Q}$ . It is sufficient to prove  $[m_{N+1}] \neq 0$  in  $H_1(M_j; \mathbb{Q})$ .

Since  $M$  is a rational homology sphere and  $M_j$  is the exterior of a  $j$ -component link in  $M$  with  $\{m'_1, \dots, m'_j\}$  as meridians, we have  $\dim H_1(M_j; \mathbb{Q}) = j$  and that  $\{[m'_1], \dots, [m'_j]\}$  is a basis.

Assume that  $[m_{N+1}] = 0$  in  $H_1(M_j; \mathbb{Q})$ . Then by the relations  $[l_i] = [m_{N+1}]$  ( $i = 1, \dots, N$ ) etc, we have  $[m'_1] = p_1[m_1], \dots, [m'_j] = p_j[m_j]$  and  $[m_1] + \dots + [m_j] = 0$ . This contradicts the fact that  $\{[m'_1], \dots, [m'_j]\}$  is a basis.  $\square$

**Step 2**  $[m_{N+1}] = [l'_i]^{p_i}$  ( $i > j$ ) in  $H_1(M_j)$ .

**Proof** By the relations

$$[m_i]^{p_i}[m_{N+1}]^{q_i} = 1, [m_i]^{r_i}[m_{N+1}]^{s_i} = [l'_i] \quad \text{and} \quad p_i s_i - q_i r_i = 1$$

that hold in  $H_1(M_j)$  for  $i \geq j + 1$ , we have

$$[m_{N+1}] = [m_{N+1}]^{p_i s_i - q_i r_i} = [m_{N+1}]^{p_i s_i} [m_i]^{p_i r_i} = ([m_i]^{r_i} [m_{N+1}]^{s_i})^{p_i} = [l'_i]^{p_i}. \quad \square$$

**Step 3**  $[l'_i]$  ( $i > j$ ) in  $H_1(M_j)$  has infinite order.

**Proof** If there exists an index  $i$  ( $j + 1 \leq i \leq N$ ) such that  $[l'_i]$  has finite order, then  $[m_{N+1}]$  has finite order by Step 2. This contradicts Step 1.  $\square$

By Step 3 and Lemma 2.4,  $[l'_i] - 1$  is invertible in  $Q(\mathbb{Z}[H_1(M_j)])$  for  $i > j$ . Hence we have the result by Step 2.  $\square$

**Proof of Theorem 1.2** (1) By Lemma 3.2,  $d$  divides  $p$ . By Lemma 3.4 (1),  $\gcd(d, p_i) = 1$  ( $i = 3, \dots, N$ ). Since  $\gcd(p_1, q_1) = 1$  and  $d$  divides  $p_1$ , we have  $\gcd(d, q_1) = 1$ . Similarly we have  $\gcd(d, q_2) = 1$ .

(2) We prove this part in steps.

**Step 1**  $\xi_2 = \xi_1^{-1}$  and  $\xi_i = 1$  ( $i = 3, \dots, N + 1$ ).

**Proof** Since  $\xi_1^d = 1$  and  $d$  divides  $p_1$ , we have  $\xi_1^{p_1} = 1$ . Since  $\xi_1^{p_1} = 1$ ,  $\gcd(d, q_1) = 1$  and  $\xi_1^{p_1} \xi_{N+1}^{q_1} = 1$  in Lemma 4.2 (1), we have  $\xi_{N+1} = 1$ . Since  $\xi_{N+1} = 1$ ,  $\gcd(d, p_i) = 1$  ( $i = 3, \dots, N$ ) and  $\xi_i^{p_i} \xi_{N+1}^{q_i} = 1$ , we have  $\xi_i = 1$  ( $i = 3, \dots, N$ ), and hence  $\xi_1 \xi_2 = 1$  by Lemma 4.2 (2).  $\square$

**Step 2**  $\xi_1 = \zeta_d^{u_1}$ ,  $\xi_2 = \zeta_d^{-u_1}$  and  $\gcd(d, u_1) = 1$ .

**Proof** We have  $\xi_1 = \psi_d([m_1]) = \psi_d(T^{u_1}) = \zeta_d^{u_1}$ . By Step 1,  $\xi_2 = \xi_1^{-1} = \zeta_d^{-u_1}$ , and  $\xi_1 = \zeta_d^{u_1}$  generates  $\psi_d(H_1(M)) \cong \mathbb{Z}/d\mathbb{Z}$ . Hence we have  $\gcd(d, u_1) = 1$ .  $\square$

We set  $M_2 = E_L \cup V_3 \cup \dots \cup V_{N+1}$  as in Section 2.

**Step 3**  $\tau(M_2) \doteq \prod_{i=3}^N \left( \sum_{k=0}^{p_i-1} [l'_i]^k \right)$  in  $Q(\mathbb{Z}[H_1(M_2)])$ .

**Proof** By Lemma 2.5 (2), Lemma 4.1 and Lemma 4.3, we have

$$\tau(M_2) \doteq ([m_{N+1}] - 1)^{N-2} \prod_{i=3}^N ([l'_i] - 1)^{-1} = \prod_{i=3}^N \left( \sum_{k=0}^{p_i-1} [l'_i]^k \right). \quad \square$$

By Lemma 4.2 (3), Step 1 and Step 2, we have  $\psi_d([l'_i]) = 1$  ( $i \geq 3$ ),  $\psi_d([l'_1]) = \zeta_d^{-u_1 \bar{q}_1}$  and  $\psi_d([l'_2]) = \zeta_d^{u_1 \bar{q}_2}$ . Hence we have the result by Lemma 2.5 (3) and Step 3.

(3) Let  $M$  be the result of  $p/q$ -surgery along a knot  $K$  in a homology sphere  $\Sigma$ . We take the homology class of a meridian for  $K$  in  $H_1(M)$  as  $T$ . By (2) and Lemma 2.6, we have

$$(4-2) \quad \Delta_K(\zeta_d)(\zeta_d - 1)^{-1}(\zeta_d^{\bar{q}} - 1)^{-1} \doteq p_3 \cdots p_N (\zeta_d^{u_1 \bar{q}_1} - 1)^{-1} (\zeta_d^{u_1 \bar{q}_2} - 1)^{-1}$$

By noting  $\gcd(d, q) = \gcd(d, u_1) = \gcd(d, q_1) = \gcd(d, q_2) = 1$ , we have

$$|N_d(\zeta_d - 1)| = |N_d(\zeta_d^{\bar{q}} - 1)| = |N_d(\zeta_d^{u_1 \bar{q}_1} - 1)| = |N_d(\zeta_d^{u_1 \bar{q}_2} - 1)| \neq 0.$$

By taking the norms of both sides of (4-2), we obtain  $|M|_d = |N_d(\Delta_K(\zeta_d))| = (p_3 \cdots p_N)^{\varphi(d)}$ .  $\square$

**Proof of Theorem 1.3** It is sufficient to prove (1) and (3).

(1) Suppose  $\|M\|_p = 1$ . By Theorem 1.2 (2), we have  $\gcd(p_i, p_j) = 1$  for every pair  $\{i, j\}$ .

Suppose  $\gcd(p_i, p_j) = 1$  for every pair  $\{i, j\}$ . By Lemma 3.4 (3), we have that  $\gcd(p, p_i) = 1$  for every  $i$ .

Suppose  $\gcd(p, p_i) = 1$  for every  $i = 1, \dots, N$ . By [Theorem 1.1](#) (3), we have  $\|M\|_p = 1$ .

(3) Let  $M$  be the result of  $p/q$ -surgery along a knot  $K$  in a homology sphere  $\Sigma$ . We first prove “if” part. Without loss of generality, we may assume that  $\{i, j\} = \{1, 2\}$ , and  $\{k, h\} = \{1, 3\}$  or  $\{3, 4\}$ . Hence it is sufficient to prove Step 1 and Step 2 below.

**Step 1** If  $\gcd(p_1, p_2) \geq 2$  and  $\gcd(p_1, p_3) \geq 2$ , then there exists a divisor  $d \geq 2$  of  $p$  such that  $\Delta_K(\zeta_d) = 0$ .

**Proof** Take a divisor  $d_2 \geq 2$  of  $\gcd(p_1, p_2)$  and a divisor  $d_3 \geq 2$  of  $\gcd(p_1, p_3)$ , and set  $d = d_2 d_3$ . By [Lemma 3.4](#) (1), we have  $\gcd(d_2, d_3) = 1$ , and hence  $d$  is a divisor of  $p_1$ . By [Lemma 3.2](#), both  $d_2$  and  $d_3$  are divisors of  $p$ , and hence  $d$  is a divisor of  $p$  because  $\gcd(d_2, d_3) = 1$ .

Let  $\psi_d: \mathbb{Z}[H_1(M)] \rightarrow \mathbb{Q}(\zeta_d)$  be a ring homomorphism such that  $\psi_d(T) = \zeta_d$ , where  $T$  is a generator of  $H_1(M)$  that corresponds to a meridian of  $K$ . We set  $\psi_d([m_i]) = \xi_i$  ( $i = 1, \dots, N + 1$ ), where  $m_i$  is a meridian of  $K_i$ . Since  $d$  is a divisor of  $p_1$ ,  $\gcd(d, q_1) = 1$  and  $\xi_1^{p_1} \xi_{N+1}^{q_1} = 1$  in [Lemma 4.2](#) (1), we have  $\xi_{N+1} = 1$ . Hence  $\xi_i^{p_i} = 1$  for all  $i$  by [Lemma 4.2](#) (1). By [Lemma 3.4](#) (1),  $\gcd(d_2, p_i) = 1$  for  $i \neq 1, 2$ , and  $\gcd(d_3, p_i) = 1$  for  $i \neq 1, 3$ . Hence  $\gcd(d, p_i) = 1$  for  $i \geq 4$ . Since  $\xi_i^{p_i} = 1$ , we have  $\xi_i = 1$  for  $i \geq 4$ , and  $\xi_1 \xi_2 \xi_3 = 1$  by [Lemma 4.2](#) (2). Therefore  $\psi_d(H_1(M))$  is generated by  $\xi_2$  and  $\xi_3$ . We note that  $\gcd(d_2, p_3) = \gcd(d_3, p_2) = 1$  are used in the next paragraph.

Since  $\xi_2^{p_2} = 1$ , we have  $(\xi_2^{d_2})^{p_2} = 1$ . On the other hand  $(\xi_2^{d_2})^{d_3} = \xi_2^d = 1$ . Hence  $\xi_2^{d_2} = 1$  since  $\gcd(d_3, p_2) = 1$ . Similarly  $\xi_3^{d_3} = 1$  since  $\gcd(d_2, p_3) = 1$ . If one of  $\xi_2$  and  $\xi_3$  is not “primitive”, then  $\xi_2$  and  $\xi_3$  do not generate  $\psi_d(H_1(M))$ . Hence  $\xi_2$  and  $\xi_3$  are primitive  $d_2$ -th and  $d_3$ -th root of unities respectively, and hence  $\xi_1$  is a primitive  $d$ -th root of unity since  $\xi_1 \xi_2 \xi_3 = 1$  and  $\gcd(d_2, d_3) = 1$ . Since  $\gcd(p_1, q_1) = \gcd(p_2, q_2) = 1$  and  $d_2$  divides  $p_1$  and  $p_2$ , we have  $\gcd(d_2, q_1) = \gcd(d_2, q_2) = 1$ . Similarly we have  $\gcd(d_3, q_1) = \gcd(d_3, q_3) = 1$ . Since  $\gcd(d_2, q_1) = \gcd(d_3, q_1) = 1$ , we have  $\gcd(d, q_1) = 1$ . Hence we have

$$\psi_d([l'_i]) = \xi_i^{-\bar{q}_i} \neq 1 \quad (i = 1, 2, 3),$$

where  $\bar{q}_1, \bar{q}_2$  and  $\bar{q}_3$  are integers satisfying  $q_1 \bar{q}_1 \equiv 1 \pmod{d}$ ,  $q_2 \bar{q}_2 \equiv 1 \pmod{d_2}$  and  $q_3 \bar{q}_3 \equiv 1 \pmod{d_3}$  respectively.

We set  $M_3 = E_L \cup V_4 \cup \dots \cup V_{N+1}$  and  $\psi_{3,d}: \mathbb{Z}[H_1(M_3)] \rightarrow \mathbb{Q}(\zeta_d)$  as in [Section 2](#). By [Lemma 4.3](#), we have

$$\tau(M_3) \doteq ([m_{N+1}] - 1)^{N-2} \prod_{i=4}^N ([l'_i] - 1)^{-1} = ([m_{N+1}] - 1) \prod_{i=4}^N \left( \sum_{k=0}^{p_i-1} [l'_i]^k \right),$$

and hence  $\tau^{\psi^{3,d}}(M_3) = 0$ . By Lemma 2.5 (3), we have

$$\tau^{\psi^d}(M) \doteq \tau^{\psi^{3,d}}(M_3) \prod_{i=1}^3 (\xi_i^{\bar{q}_i} - 1)^{-1} = 0,$$

and hence  $\Delta_K(\zeta_d) = 0$  by Lemma 2.6.  $\square$

**Step 2** If  $\gcd(p_1, p_2) \geq 2$  and  $\gcd(p_3, p_4) \geq 2$  ( $N \geq 4$ ), then there exists a divisor  $d \geq 2$  of  $p$  such that  $\Delta_K(\zeta_d) = 0$ .

**Proof** Take a divisor  $d_1 \geq 2$  of  $\gcd(p_1, p_2)$  and a divisor  $d_3 \geq 2$  of  $\gcd(p_3, p_4)$ , and set  $d = d_1 d_3$ . By Lemma 3.4 (1), we have  $\gcd(d_1, d_3) = 1$ . By Lemma 3.2, both  $d_1$  and  $d_3$  are divisors of  $p$ . Hence  $d$  is a divisor of  $p$  since  $\gcd(d_1, d_3) = 1$ .

Let  $\psi_d: \mathbb{Z}[H_1(M)] \rightarrow \mathbb{Q}(\zeta_d)$  be a ring homomorphism such that  $\psi_d(T) = \zeta_d$ , where  $T$  is a generator of  $H_1(M)$  that corresponds to a meridian of  $K$ . We set  $\psi_d([m_i]) = \xi_i$  ( $i = 1, \dots, N+1$ ), where  $m_i$  is a meridian of  $K_i$ . Since  $d_1$  divides  $p_1$ , we have  $(\xi_1^{p_1})^{d_3} = 1$  and also  $\gcd(d_1, q_1) = 1$  since  $\gcd(p_1, q_1) = 1$ . Hence  $(\xi_{N+1}^{d_3})^{q_1} = 1$  since  $\xi_1^{p_1} \xi_{N+1}^{q_1} = 1$ . On the other hand  $(\xi_{N+1}^{d_3})^{d_1} = 1$ . Thus  $\xi_{N+1}^{d_3} = 1$  since  $\gcd(d_1, q_1) = 1$ . Similarly we have  $\xi_{N+1}^{d_1} = 1$ . Hence we have  $\xi_{N+1} = 1$  since  $\gcd(d_1, d_3) = 1$ , and we have  $\xi_i^{p_i} = 1$  for all  $i$  by Lemma 4.2 (1). By Lemma 3.4 (1),  $\gcd(d_1, p_i) = 1$  for  $i \neq 1, 2$ , and  $\gcd(d_3, p_i) = 1$  for  $i \neq 3, 4$ . Hence  $\gcd(d, p_i) = 1$  for  $i \geq 5$ . Therefore we have  $\xi_i = 1$  for  $i \geq 5$ , and  $\xi_1 \xi_2 \xi_3 \xi_4 = 1$ . We note that  $\gcd(d_1, p_3) = \gcd(d_1, p_4) = \gcd(d_3, p_1) = \gcd(d_3, p_2) = 1$  are used in the next paragraph.

Since  $\gcd(d_3, p_1) = 1$  and  $d_1$  divides  $p_1$ , we have  $\gcd(d, p_1) = d_1$ . Hence  $\xi_1^d = 1$  and  $\xi_1^{p_1} = 1$  imply  $\xi_1^{d_1} = 1$ . Similarly we have  $\xi_2^{d_1} = 1$ ,  $\xi_3^{d_3} = 1$  and  $\xi_4^{d_3} = 1$ . Thus  $\xi_1 \xi_2$  and  $\xi_3 \xi_4$  are  $d_1$ -th and  $d_3$ -th root of unities respectively. Hence we have  $\xi_1 \xi_2 = \xi_3 \xi_4 = 1$  since  $\xi_1 \xi_2 \xi_3 \xi_4 = 1$  and  $\gcd(d_1, d_3) = 1$ . Hence  $\psi_d(H_1(M)) \cong \mathbb{Z}/d\mathbb{Z}$  is generated by  $\xi_1$  and  $\xi_3$ . For the similar reason as Step 1, both  $\xi_1$  and  $\xi_3$  are “primitive”. Thus  $\xi_1$  and  $\xi_2$  (resp.  $\xi_3$  and  $\xi_4$ ) are primitive  $d_1$ -th (resp.  $d_3$ -th) root of unities. Since  $\gcd(p_1, q_1) = 1$  and  $d_1$  divides  $p_1$ , we have  $\gcd(d_1, q_1) = 1$ . Similarly we have  $\gcd(d_1, q_2) = \gcd(d_3, q_3) = \gcd(d_3, q_4) = 1$ . Hence we have

$$\psi_d([l'_i]) = \xi_i^{-\bar{q}_i} \neq 1 \quad (i = 1, 2, 3, 4),$$

where  $\bar{q}_1, \bar{q}_2, \bar{q}_3$  and  $\bar{q}_4$  are integers satisfying  $q_1 \bar{q}_1 \equiv 1 \pmod{d_1}$ ,  $q_2 \bar{q}_2 \equiv 1 \pmod{d_1}$ ,  $q_3 \bar{q}_3 \equiv 1 \pmod{d_3}$  and  $q_4 \bar{q}_4 \equiv 1 \pmod{d_3}$  respectively.

We set  $M_4 = E_L \cup V_5 \cup \dots \cup V_{N+1}$  and  $\psi_{4,d}: \mathbb{Z}[H_1(M_4)] \rightarrow \mathbb{Q}(\zeta_d)$  as in Section 2. By Lemma 4.3, we have

$$\tau(M_4) \doteq ([m_{N+1}] - 1)^{N-2} \prod_{i=5}^N ([l'_i] - 1)^{-1} = ([m_{N+1}] - 1)^2 \prod_{i=5}^N \left( \sum_{k=0}^{p_i-1} [l'_i]^k \right),$$

and hence  $\tau^{\psi_{4,d}}(M_4) = 0$ . By Lemma 2.5 (3), we have

$$\tau^{\psi_d}(M) \doteq \tau^{\psi_{4,d}}(M_4) \prod_{i=1}^4 (\xi_i^{\bar{q}_i} - 1)^{-1} = 0,$$

and hence  $\Delta_K(\zeta_d) = 0$  by Lemma 2.6. □

We next prove “only if” part of Theorem 1.3 (3). Suppose  $\|M\|_p = 0$ . Then there exists a divisor  $d \geq 2$  of  $p$  such that  $\Delta_K(\zeta_d) = 0$ , and there exists a pair  $\{i, j\}$  such that  $\gcd(p_i, p_j) \geq 2$  by (1). We may set  $\{i, j\} = \{1, 2\}$  without loss of generality. Let  $\psi_d: \mathbb{Z}[H_1(M)] \rightarrow \mathbb{Q}(\zeta_d)$  be a ring homomorphism such that  $\psi_d(T) = \zeta_d$ , where  $T$  is a generator of  $H_1(M)$  that corresponds to a meridian of  $K$ . We set  $\psi_d([m_i]) = \xi_i$  ( $i = 1, \dots, N + 1$ ), where  $m_i$  is a meridian of  $K_i$ .

**Step 3**  $\xi_{N+1} = 1$ .

**Proof** If  $\psi_d([l'_i]) = 1$  for some  $i$ , then we have  $\xi_{N+1} = 1$  by Lemma 4.2 (1) and (3), and  $p_i s_i - q_i r_i = 1$ . Otherwise, since  $\Delta_K(\zeta_d) = 0$  and by Lemma 2.6, we have  $\tau^{\psi_d}(M) = 0$ . Hence by Lemma 2.5 (1),

$$\Delta_L(\xi_1, \dots, \xi_{N+1}) = (\xi_{N+1} - 1)^{N-1} = 0.$$

Thus we have  $\xi_{N+1} = 1$ . □

**Step 4** If  $\gcd(d, p_k) \geq 2$ , there is an index  $h \neq k$  such that  $\gcd(p_k, p_h) \geq 2$ .

**Proof** Take a prime divisor  $\ell$  of  $\gcd(d, p_k)$ . By Lemma 3.4 (2), there is an index  $h \neq k$  such that  $\ell$  is a divisor of  $p_h$ . Then  $\ell$  is a divisor of  $\gcd(p_k, p_h)$ . □

**Step 5**  $d$  is not a common divisor of  $p_1$  and  $p_2$ .

**Proof** If  $d$  is a common divisor of  $p_1$  and  $p_2$ , then  $|M|_d \neq 0$  by Theorem 1.2 (2). This contradicts  $\Delta_K(\zeta_d) = 0$ . □

**Step 6** There are indices  $k \neq h$  such that  $\{k, h\} \neq \{1, 2\}$  and  $\gcd(p_k, p_h) \geq 2$ .

**Proof** By Step 3 and Lemma 4.2 (1), we have  $\xi_k^{p_k} = 1$  for all  $k$ . Suppose  $\gcd(d, p_k) = 1$  for all  $k \geq 3$ . Then we have  $\xi_k = 1$  for all  $k \geq 3$ . By Lemma 4.2 (2), we have  $\xi_1 \xi_2 = 1$ . Thus  $\xi_1$  generates  $\psi_d(H_1(M)) \cong \mathbb{Z}/d\mathbb{Z}$ , and hence both  $\xi_1$  and  $\xi_2 = \xi_1^{-1}$  are primitive  $d$ -th root of unities. Since  $\xi_1^{p_1} = \xi_2^{p_2} = 1$ ,  $d$  is a common divisor of  $p_1$  and  $p_2$ . This contradicts Step 5. Therefore there exists an index  $k \geq 3$  such that  $\gcd(d, p_k) \geq 2$ . By Step 4, there exists an index  $h \neq k$  such that  $\gcd(p_k, p_h) \geq 2$ , and  $\{k, h\} \neq \{1, 2\}$  by  $k \geq 3$ .  $\square$

This concludes the proof of Theorem 1.3.  $\square$

### 5 Proof of Theorem 1.4

We set  $f(t) = t^2 - 3t + 1$  in this section. Let  $\alpha$  and  $\beta$  ( $\alpha > \beta$ ) be the roots of  $f(t) = t^2 - 3t + 1 = 0$ . Then we need the following lemmas on the norms of  $f(t)$ .

**Lemma 5.1**  $|f(t)|_d = |\Phi_d(\alpha)\Phi_d(\beta)|$ , where

$$\Phi_d(x) = \prod_{i \in (\mathbb{Z}/d\mathbb{Z})^\times} (x - \zeta_d^i),$$

the  $d$ -th cyclotomic polynomial.

**Proof**

$$|f(t)|_d = \left| \prod_{i \in (\mathbb{Z}/d\mathbb{Z})^\times} (\zeta_d^i - \alpha)(\zeta_d^i - \beta) \right| = |\Phi_d(\alpha)\Phi_d(\beta)|. \quad \square$$

**Lemma 5.2** Let  $\ell$  be a prime number. Then we have the following:

- (1)  $|f(t)|_\ell = \alpha^\ell + \beta^\ell - 2$ .
- (2)  $|f(t)|_\ell > 2$ .

**Proof** (1) Since  $\ell$  is prime,  $\Phi_\ell(x) = (x^\ell - 1)/(x - 1)$ . Note that  $\alpha + \beta = 3$  and  $\alpha\beta = 1$ . Hence by Lemma 5.1, we have

$$|f(t)|_\ell = \left| \frac{\alpha^\ell - 1}{\alpha - 1} \cdot \frac{\beta^\ell - 1}{\beta - 1} \right| = \alpha^\ell + \beta^\ell - 2.$$

- (2) Since  $\alpha = (3 + \sqrt{5})/2 = 2.618\dots > 2$ , we have  $|f(t)|_\ell = \alpha^\ell + \beta^\ell - 2 > 2^\ell - 2 = 2$ .  $\square$

**Proof of Theorem 1.4** Let  $\ell$  be a prime divisor of  $p$ . By Lemma 5.2, we have  $|M|_\ell > 2$ . Hence  $\|M\|_p \neq 1$  and  $N \geq 3$  because  $M$  is not a lens space. Since  $f(t) = t^2 - 3t + 1$  is irreducible over  $\mathbb{Q}$  and is not a cyclotomic polynomial, we have  $f(\zeta_d) \neq 0$  for every positive integer  $d$ , and hence we have  $\|M\|_p \neq 0$ . By Theorem 1.3, without loss of generality we may assume that

$$(*) \quad \gcd(p_1, p_2) \geq 2 \quad \text{and} \quad \gcd(p_i, p_j) = 1 \quad \text{for } \{i, j\} \neq \{1, 2\}.$$

**Step 1** Let  $\ell$  be a prime divisor of  $p$ . Then  $\ell$  is a divisor of  $\gcd(p_1, p_2)$ .

**Proof** Suppose  $\ell$  divides none of  $p_i$ 's. Then we have  $|M|_\ell = |f(t)|_\ell = 1$  by Theorem 1.1 (2). This contradicts Lemma 5.2 (2). Therefore  $\ell$  is a divisor of some  $p_k$ . By Lemma 3.4 (2), there exists  $h \neq k$  such that  $\ell$  is a divisor of some  $p_h$ . By the assumption (\*), we have  $\{k, h\} = \{1, 2\}$ , and hence  $\ell$  is a divisor of  $\gcd(p_1, p_2)$ .  $\square$

We first show that  $\gcd(p_1, p_2) = 2$  or  $3$ .

**Step 2**  $\gcd(p_1, p_2)$  is not divisible by 4, 6 nor 9.

**Proof** Suppose that  $\gcd(p_1, p_2)$  is divisible by 4. Then  $p$  is also divisible by 4. By Theorem 1.2 (2),  $|f(t)|_2 = p_3 \cdots p_N$  and  $|f(t)|_4 = (p_3 \cdots p_N)^2$ . By computing concretely, we have  $|f(t)|_2 = 5$  and  $|f(t)|_4 = 3^2$ . This is a contradiction. Therefore  $\gcd(p_1, p_2)$  is not divisible by 4. Other cases are shown in a similar way.  $\square$

**Step 3** For any prime number  $\ell \geq 5$ ,  $\ell$  is not a divisor of  $\gcd(p_1, p_2)$ .

**Proof** Suppose  $\ell$  is a divisor of  $\gcd(p_1, p_2)$ . Then  $|M|_\ell = (p_3 \cdots p_N)^{\ell-1}$  by Theorem 1.2 (2). Hence by Lemma 5.2 (2), there exists an integer  $m \geq 2$  such that

$$(5-1) \quad \alpha^\ell + \beta^\ell - 2 = m^{\ell-1}$$

Since  $\alpha^x + \alpha^{-x} - 2^{x-1} - 2 > 2^{x-1} - 2$ ,  $\alpha^x + \alpha^{-x} - 2 = 2^{x-1}$  does not hold for  $x \geq 5$ . Hence we have  $m \neq 2$ .

By the equation (5-1), we have

$$m^{\ell-1} < \alpha^\ell \quad \text{and} \quad m < \alpha^{\frac{\ell}{\ell-1}}.$$

Suppose  $\ell \geq 11$ . Since

$$\alpha^{11} < (2.62)^{11} = 39931.4 \cdots < 3^{10} = 59049,$$

we have  $m < \alpha^{\frac{\ell}{\ell-1}} \leq \alpha^{\frac{11}{10}} < 3$ . By  $m \neq 2$ , this case does not occur.

Suppose  $\ell = 5$  or  $7$ . Since

$$\alpha^5 < (2.62)^5 = 123.454 \dots < 4^4 = 256,$$

we have  $m < \alpha^{\frac{7}{6}} < \alpha^{\frac{5}{4}} < 4$  and  $m = 3$ .

If  $\ell = 5$ , then we have  $\alpha^5 + \alpha^{-5} - 2 = 121 \neq 3^4 = 81$ .

If  $\ell = 7$ , then  $\alpha^7 + \alpha^{-7} - 2 = 841 \neq 3^6 = 729$ , and we have Step 3. □

**Step 4**  $\gcd(p_1, p_2) = 2$  or  $3$ .

**Proof** By Step 2 and Step 3, we have the result. □

We prove that  $p = 2$  or  $3$ .

**Step 5**  $p$  is a power of  $2$  or  $3$ .

**Proof** By Step 1 and Step 4, we have the result. □

**Step 6**  $p$  is not divisible by  $4$ .

**Proof** Recall  $|f(t)|_4 = 9$  (see Step 2). Suppose that  $p$  is divisible by  $4$ . Then  $|f(t)|_4 = 1$  as shown below, and this is a contradiction:

By Step 1 and Step 4, we have  $\gcd(p_1, p_2) = 2$ . Hence without loss of generality we may assume that  $p_1$  is of the form  $p_1 = 2p'_1$  where  $p'_1$  is odd. By Lemma 3.4 (1),  $\gcd(p_i, 2) = 1$  for  $i \geq 3$ , and hence we have the following claim:

**Claim A**  $\gcd(p_i, 4) = 1$  ( $3 \leq i \leq N$ ).

We compute  $\tau^{\psi_4}(M)$  by two ways. We use the same notation as in Section 4. By Lemma 2.6, we have

$$(5-2) \quad \tau^{\psi_4}(M) \doteq f(\zeta_4)(\zeta_4 - 1)^{-1}(\zeta_4^{\bar{q}} - 1)^{-1}.$$

We note that at least one of  $\xi_1, \dots, \xi_{N+1}$  is a primitive 4–th root of unity because they generate  $\psi_4(H_1(M)) \cong \mathbb{Z}/4\mathbb{Z}$ . Suppose  $\xi_1$  is not a primitive 4–th root of unity. Then we have  $\xi_1^{p_1} = \xi_1^{2p'_1} = 1$ . Hence  $\xi_{N+1} = 1$  since  $\gcd(4, q_1) = 1$  and  $\xi_1^{p_1} \xi_{N+1}^{q_1} = 1$ . Hence  $\xi_i = 1$  ( $3 \leq i \leq N$ ) since  $\xi_i^{p_i} \xi_{N+1}^{q_i} = 1$  ( $3 \leq i \leq N$ ),  $\xi_{N+1} = 1$  and Claim A holds. By Lemma 4.2 (2), we have  $\xi_1 \xi_2 = 1$ . Thus  $\xi_2$  is also not a primitive 4–th root of unity. This is a contradiction. Therefore  $\xi_1$  is a primitive 4–th root of unity. For the same reason,  $\xi_2$  is also a primitive 4–th root of unity.

Since  $\xi_1^{p_1} \xi_{N+1}^{q_1} = 1$  ( $p_1 = 2p'_1$ ,  $p'_1$  : odd,  $q_1$  : odd), we have  $\xi_{N+1} = -1$ . Since  $\xi_{N+1} = -1$  and Claim A holds, we have

$$\psi_d([l'_i]) = \xi_i^{r_i} \xi_{N+1}^{s_i} = \xi_{N+1}^{-\bar{p}_i} = -1 \quad (3 \leq i \leq N)$$

(see the proof of [Theorem 1.1](#) (1)). Since  $r_1$  and  $r_2$  are odd, both  $\psi_d([l'_1])$  and  $\psi_d([l'_2])$  are primitive 4–th root of unities. We set  $\psi_d([l'_i]) = \zeta_4^{v_i}$  ( $i = 1, 2$ ) where  $\gcd(v_i, 4) = 1$ .

By [Lemma 2.5](#) (1), we have

$$\begin{aligned} \tau^{\psi_d}(M) &\doteq (-1 - 1)^{N-1} (\zeta_4^{v_1} - 1)^{-1} (\zeta_4^{v_2} - 1)^{-1} (-1 - 1)^{-(N-1)} \\ (5-3) \quad &\doteq (\zeta_4^{v_1} - 1)^{-1} (\zeta_4^{v_2} - 1)^{-1}. \end{aligned}$$

By (5-2) and (5-3), we have  $|f(t)|_4 = 1$  as in the proof of [Theorem 1.1](#) (2). □

**Step 7**  $p$  is not divisible by 9.

**Proof** By an easy computation, we have  $|f(t)|_9 = 19^2$ . Suppose that  $p$  is divisible by 9. Then  $|f(t)|_9 = 1$  as shown below, and this is a contradiction:

Suppose that  $p$  is divisible by 9. By Step 1 and Step 4, we have  $\gcd(p_1, p_2) = 3$ . Hence without loss of generality we may assume that  $p_1$  is of the form  $p_1 = 3p'_1$  where  $\gcd(p'_1, 3) = 1$ . By [Lemma 3.4](#) (1),  $\gcd(p_i, 3) = 1$  for  $i \geq 3$ , and hence we have the following claim:

**Claim B**  $\gcd(p_i, 9) = 1$  ( $3 \leq i \leq N$ ).

We compute  $\tau^{\psi_9}(M)$  by two ways. By [Lemma 2.6](#), we have

$$(5-4) \quad \tau^{\psi_9}(M) \doteq f(\zeta_9)(\zeta_9 - 1)^{-1} (\zeta_9^{\bar{q}} - 1)^{-1}.$$

We note that at least one of  $\xi_1, \dots, \xi_{N+1}$  is a primitive 9–th root of unity because they generate  $\psi_9(H_1(M)) \cong \mathbb{Z}/9\mathbb{Z}$ . Suppose  $\xi_1$  is not a primitive 9–th root of unity. Then we have  $\xi_1^{p_1} = \xi_1^{3p'_1} = 1$ . Hence  $\xi_{N+1} = 1$  since  $\gcd(9, q_1) = 1$  and  $\xi_1^{p_1} \xi_{N+1}^{q_1} = 1$ . Hence  $\xi_i = 1$  ( $3 \leq i \leq N$ ) since  $\xi_i^{p_i} \xi_{N+1}^{q_i} = 1$  ( $3 \leq i \leq N$ ),  $\xi_{N+1} = 1$  and Claim B holds. By [Lemma 4.2](#) (2), we have  $\xi_1 \xi_2 = 1$ . Thus  $\xi_2$  is also not a primitive 9–th root of unity. This is a contradiction. Therefore  $\xi_1$  is a primitive 9–th root of unity. For the same reason,  $\xi_2$  is also a primitive 9–th root of unity.

Since  $\xi_1^{p_1} \xi_{N+1}^{q_1} = 1$ , we have that  $\xi_{N+1}$  is a primitive 3–rd root of unity. Since  $\xi_{N+1}$  is a primitive 3–rd root of unity and Claim B holds,  $\psi_d([l'_i]) = \xi_i^{r_i} \xi_{N+1}^{s_i} = \xi_{N+1}^{-\bar{p}_i}$  ( $3 \leq i \leq N$ ) is a primitive 3–rd root of unity (see the proof of [Theorem 1.1](#) (1)). Since  $\gcd(r_i, 3) = 1$  ( $i = 1, 2$ ),  $\psi_d([l'_i])$  ( $i = 1, 2$ ) is a primitive 9–th root of unity. We set

$\psi_d([l'_i]) = \zeta_9^{v_i}$  ( $i = 1, 2$ ),  $\psi_d([l'_i]) = \zeta_3^{w_i}$  ( $3 \leq i \leq N$ ) and  $\psi_d([l'_{N+1}]) = \zeta_3^w$  where  $\gcd(v_i, 9) = 1$ ,  $\gcd(w_i, 3) = 1$  and  $\gcd(w, 3) = 1$ .

By Lemma 2.5 (1), we have

$$\begin{aligned} \tau^{\psi_d}(M) &\doteq (\zeta_3^w - 1)^{N-1} (\zeta_9^{v_1} - 1)^{-1} (\zeta_9^{v_2} - 1)^{-1} (\zeta_3^w - 1)^{-1} \prod_{i=3}^N (\zeta_3^{w_i} - 1)^{-1} \\ (5-5) \quad &\doteq (\zeta_3^w - 1)^{N-2} (\zeta_9^{v_1} - 1)^{-1} (\zeta_9^{v_2} - 1)^{-1} \prod_{i=3}^N (\zeta_3^{w_i} - 1)^{-1}. \end{aligned}$$

By (5-4) and (5-5), we have  $|f(t)|_9 = 1$  as in the proof of Theorem 1.1 (2), which concludes the proof of Step 7.  $\square$

We show finally the rest, concluding the proof of Theorem 1.4.

(i) By Theorem 1.2 (3) and  $|f(t)|_2 = 5$ , we have  $p_3 \cdots p_N = 5$ . Hence  $N = 3$  and  $p_3 = 5$ .

(ii) By Theorem 1.2 (3) and  $|f(t)|_3 = 4^2$ , we have  $p_3 \cdots p_N = 4$ . Then the case (a)  $N = 3$  and  $p_3 = 4$  or the case (b)  $N = 4$  and  $p_3 = p_4 = 2$  occur. Since  $\gcd(p_i, p_j) = 1$  for  $\{i, j\} \neq \{1, 2\}$ , the case (b) does not occur. Hence  $N = 3$  and  $p_3 = 4$ .  $\square$

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