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The topological Hawaiian earring group does not embed in the inverse limit of free groups

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Abstract   Endowed with natural topologies, the fundamental group of the Hawaiian earring continuously injects into the inverse limit of free groups. This note shows the injection fails to have a continuous inverse. Such a phenomenon was unexpected and appears to contradict results of another author.

1   Introduction

Quite generally the based fundamental group $\pi_1(X, p)$ of a space X becomes a topological group whose topology is invariant under the homotopy type of the underlying space X (Corollary 3.4 [1]). In the context of spaces complicated on the small scale the utility of this invariant is emerging. For example topological π_1 has the potential to distinguish spaces when the algebraic homotopy groups fail to do so [2]. Unfortunately even in the simplest cases the topological properties of $\pi_1(X, p)$ can be challenging to understand.

Consider the familiar Hawaiian earring $X = \cup_{n=1}^{\infty} S_n$, (the union of a null sequence of simple closed curves S_n joined at a common point) and the canonical homomorphism $\phi : \pi_1(X) \to \lim_{\leftarrow} \pi_1(\cup_{i=1}^{n} S_i)$.

The paper [1, page 370] seems to claim that ϕ is also a homeomorphism onto its image ("ψ^{-1} is surely continuous as well..."). The intent of this note is to show that such a claim is false. The monomorphism ϕ is not a homeomorphism onto its image, and thus ϕ fails to be a topological embedding (Theorem 2.1). To prove this we consider the sequence $[(y_1 * y_n * y_1^{-1} * y_n^{-1})^n]$ where y_i loops counterclockwise around the *ith* circle. The sequence diverges in $\pi_1(X, p)$ with the quotient topology but the sequence converges to the trivial element in the inverse limit space $\lim_{\leftarrow} \pi_1(\cup_{i=1}^{n} S_i)$.

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2 Main Result

Suppose X is a topological space and $p \in X$. Endowed with the compact open topology, let $C_p(X) = \{f : [0,1] \to X$ such that f is continuous and $f(0) = f(1) = p\}$. Then the *topological fundamental group* $\pi_1(X, p)$ is the quotient space of $C_p(X)$ obtained by treating the path components of $C_p(X)$ as points. Thus, letting $q : C_p([0,1], X) \to \pi_1(X, p)$ denote the canonical surjection, a set $A \subset \pi_1(X, p)$ is closed in $\pi_1(X, p)$ if and only if $q^{-1}(A)$ is closed in $C_p([0,1], X)$.

The space Y is said to be T_1 if the one point subsets of Y are closed.

If A_1, A_2, are topological spaces and $f_n : A_{n+1} \to A_n$ is a continuous surjection then, (endowing $A_1 \times A_2...$ with the product topology) the *inverse limit space* $\lim_\leftarrow A_n = \{(a_1, a_2, ...) \in (A_1 \times A_2...) | f_n(a_{n+1}) = a_n\}$.

The map $f : [0,1] \to Y$ *is of the form* $\alpha_1 * \alpha_2 ... * \alpha_n$ if there exists a partition $t_0 \le t_1 ... \le t_n$ of $[0,1]$ such that for each $i \ge 1$ we have $f_{[t_{i-1}, t_i]} = \alpha_i$.

For the remainder of the paper we use the following notation.

Let $X_n = \cup_{i=1}^n \{(x, y) \in R^2 | (x - \frac{1}{n})^2 + y^2 = \frac{1}{n^2}\}$. Note since X_n is locally contractible the path components of $C_p(X_n)$ are open in $C_p(X_n)$ and hence the topological group $\pi_1(X_n, p)$ has the discrete topology.

Let $r_n^* : \pi_1(X_n, p) \to \pi_1(X_{n-1}, p)$ denote the epimorphism induced by the retraction $r_n : X_n \to X_{n-1}$ collapsing the nth circle to the point $p = (0,0)$. Let $\lim_\leftarrow \pi_1(X_n, p)$ denote the inverse limit space under the maps r_n^*.

Let $X = \cup_{n=1}^\infty X_n$ denote the familiar Hawaiian and let $R_n : X \to X_n$ denote the retraction fixing X_n pointwise and collapsing $\cup_{i=n+1}^\infty X_i$ to the point p.

The formula $\phi([f]) = ([R_1(f)], [R_2(f)], ...)$ determines a canonical homomorphism $\phi : \pi_1(X, p) \to \lim_\leftarrow \pi_1(X_n, p)$.

Remark The homomorphism $\phi : \pi_1(X, p) \to \lim_\leftarrow \pi_1(X_n, p)$ is continuous (Proposition 3.3 [1]) and one to one (Theorem 4.1 [3]). Since $\pi_1(X_n, p)$ is discrete the space $\Pi_{n=1}^\infty \pi_1(X_n, p)$ is metrizable and in particular the subspace $\lim_\leftarrow \pi_1(X_n, p)$ is a T_1 space. Consequently $\pi_1(X, p)$ is a T_1 space since ϕ is continuous and one to one. Thus the path components of $C_p(X)$ are closed in $C_p(X)$.

Theorem 2.1 *The injection $\phi : \pi_1(X, \{p\}) \hookrightarrow \lim_\leftarrow \pi_1(X_n, p)$ is not a topological embedding.*

Proof Let $q = (2,0)$ in X_1. For a loop $f : [0,1] \to \cup_{i=1}^{\infty} X_i$ with base point $p = (0,0)$ define the oscillation number $O_q(f)$ as the maximal n such that there exist $0 = t_0 < t_1 \cdots t_{2n-1} < t_{2n} = 1$ with $f(t_{2i}) = p$ and $f(t_{2i-1}) = q$. Let $y_i \in C_p(X)$ loop once counterclockwise around the *ith* circle and let $y_i^{-1} \in C_p(X)$ loop once clockwise around the *ith* circle.

First note that if $f \in C_p(\cup_{i=1}^{\infty} X_i)$ is path homotopic to a map of the form $(y_1^{-1} * y_n^{-1} * y_1 * y_n)^n$ then $O_q(f) \geq 2n$ for $n \geq 2$. To see this first observe $O_q(f) = O_q(R_n f)$. Now recall $\pi_1(X_n, p)$ is canonically isomorphic to the free group on generators $\{y_1, ...y_n\}$. Thus if w is an (unreduced) word corresponding to $R_n f$ then each step of the algebraic reduction of w to $(y_1^{-1} y_n^{-1} y_1 y_n)^n$ never raises the oscillation number of the corresponding path in X_n. Hence $O_q(f) \geq O_q((y_1^{-1} * y_n^{-1} * y_1 * y_n)^n) = 2n$.

To prove ϕ is not an embedding consider the set $A \subset \pi_1(X, p)$ defined as $A = \{[f_2], [f_3]...\}$ where f_n is of the form $(y_1^{-1} * y_n^{-1} * y_1 * y_n)^n$. To see that A is closed in $\pi_1(X, p)$ consider the union of (closed) path components $B = \cup_{n=2}^{\infty} [f_n] \subset C_p(X)$. Observe if $f \in C_p(X)$ there exists an open neighborhood $U \subset C_p(X)$ such that $O_q(f) \geq O_q(g)$ for each $g \in U$. Thus $U \cap [f_n] \neq \emptyset$ for at most finitely many of the closed sets $[f_n]$. Hence B is closed in $C_p(X)$ and consequently A is closed in $\pi_1(X, p)$. On the other hand $\phi(A)$ is not closed in the image of ϕ since the sequence $\{\phi([f_n])\}$ converges to the trivial element in $\lim_{\leftarrow} \pi_1(X_n, p)$. Hence ϕ is not a homeomorphism from $\pi_1(X, p)$ onto the image of ϕ. $\qquad\square$

References

[1] **D K Biss**, *The topological fundamental group and generalized covering spaces*, Topology Appl. 124 (2002) 355–371   MathReview

[2] **P Fabel** *Topological fundamental groups can distinguish spaces with isomorphic homotopy groups*, `arXiv:math.AT/0502402`

[3] **J W Morgan**, **I Morrison**, *A van Kampen theorem for weak joins*, Proc. London Math. Soc. (3) 53 (1986) 562–576   MathReview

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