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A TOPOLOGICAL PROOF OF CHEN'S ALTERNATIVE KNESER COLORING THEOREM

By

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Abstract. Johnson, Holroyd and Stahl [5] conjectured that the circular chromatic number of the Kneser graph is equal to the ordinary chromatic number. Chen completely confirmed the conjecture in [4]. Chen's alternative Kneser coloring theorem is a key lemma in his proof of Johnson-Holroyd-Stahl conjecture. Chen [4] and Chang, Liu and Zhu [3] proved the theorem by using Fan's lemma. In this paper, we prove Chen's alternative Kneser coloring theorem by using cohomology.

1. Introduction

Let $G = (V(G), E(G))$ be a graph and p, q integers with $1 \leqq q \leqq p$. We denote by $[p]$ the set $\{1, 2, \dots, p\}$. A (p, q)-*coloring* of G is a map $c : V(G) \to [p]$ such that $q \leqq |c(x) - c(y)| \leqq p - q$ for every edge xy of G. The *circular chromatic number* of G is

$$\chi_c(G) = \inf\left\{\frac{p}{q} \,\middle|\, \text{there exists a } (p,q)\text{-coloring of } G\right\}.$$

Because the ordinary chromatic number $\chi(G)$ is equal to

$$\min\{p \mid \text{there exists a } (p,1)\text{-coloring of } G\},$$

we see $\chi_c(G) \leqq \chi(G)$. It has been known that $\chi_c(G) > \chi(G) - 1$ (see [2], [11]).

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We denote by $\binom{[n]}{k}$ the collection of all k-subsets of $[n]$. The *Kneser graph* $KG_{n,k}$ for $n \geqq 2k > 0$, has vertex set $\binom{[n]}{k}$ and any two vertices $u, v \in \binom{[n]}{k}$ are adjacent if and only if $u \cap v = \emptyset$. Lovász [6] proved the chromatic number of the Kneser graph $KG_{n,k}$ is $n - 2k + 2$ by using a Borsuk-Ulam type theorem (see also [7]).

Johnson, Holroyd and Stahl [5] conjectured that the circular chromatic number $\chi_c(KG_{n,k})$ of the Kneser graph $KG_{n,k}$ is equal to the ordinary chromatic number $\chi(KG_{n,k})$. Meunier [8] and Simonyi and Tardos [10] proved independently that if n is even, then $\chi_c(KG_{n,k}) = \chi(KG_{n,k})$. Chen [4] completely proved Johnson-Holroyd-Stahl conjecture. Chang, Liu and Zhu gave a short proof of it in [3]. The following is a key lemma of the proofs of $\chi_c(KG_{n,k}) = \chi(KG_{n,k})$ in [3] and [4].

Chen's alternative Kneser coloring theorem ([3], [4]). *Let n and k be integers with $n \geqq 2k > 0$. If $c : \binom{[n]}{k} \to [n - 2k + 2]$ is a proper coloring of $KG_{n,k}$, then there exist two disjoint $(k-1)$-subsets S, T of $[n]$ and the integers of $[n] \setminus (S \cup T)$ are enumerated as $i_1, \dots, i_{n-2k+2}$ such that $c(S \cup \{i_j\}) = c(T \cup \{i_j\}) = j$ for $j = 1, 2, \dots, n - 2k + 2$.*

In [3] and [4], Fan's lemma was used to prove this theorem. In this paper, we prove Chen's alternative Kneser coloring theorem by using cohomology argument.

Let S^n denote the unit sphere in the $(n+1)$-dimensional Euclidean space. We denote by $\boldsymbol{Z}_2$ the cyclic group of order 2 and consider the antipodal $\boldsymbol{Z}_2$-action on S^n. The following Borsuk-Ulam type theorem is the key theorem in a topological proof of Chen's alternative Kneser coloring theorem in this paper.

Theorem 1.1. *Let n be a positive integer and Y a connected regular cell complex such that $H^p(Y; \boldsymbol{Z}_2) = 0$ for $1 \leqq p \leqq n$. Let X be a subcomplex of Y which admits a free cellular $\boldsymbol{Z}_2$-action such that each cell of $Y \setminus X$ has dimension $(n+1)$. Then for each $\boldsymbol{Z}_2$-map $f : X \to S^n$, there exists an $(n+1)$-cell e in $Y \setminus X$ such that $(f|\partial e)^* : H^n(S^n; \boldsymbol{Z}_2) \to H^n(\partial e; \boldsymbol{Z}_2)$ is an isomorphism, where ∂e is the boundary of e and $f|\partial e$ is the restriction of f to ∂e.*

In Theorem 1.1, we remark that ∂e is homeomorphic to S^n because Y is a regular cell complex.

2. Proof of Theorem 1.1 and applications

In this section we prove Theorem 1.1 and give its applications. Throughout this section, the coefficient of cohomology is $\boldsymbol{Z}_2 = \boldsymbol{Z}/2\boldsymbol{Z}$ and is omitted in the notation unless otherwise stated.

First we prove the following lemma.

Lemma 2.1. *Let n be a positive integer and X a connected free $\boldsymbol{Z}_2$-CW-complex such that p-th cohomology $H^p(X)$ is zero for $1 \leqq p \leqq n-1$. If $f : X \to S^n$ is a $\boldsymbol{Z}_2$-map, then the induced map $f^* : H^n(S^n) \to H^n(X)$ is a non-zero homomorphism*

Proof. Let $w(S^n)$ and $w(X)$ be the Stiefel-Whitney class of $\pi_S : S^n \to \boldsymbol{RP}^n$ and $\pi_X : X \to X/\boldsymbol{Z}_2$ respectively. We consider the Gysin-Smith exact sequence (see [9, Corollary 12.3], [1, Chapter III]).

$$\cdots \xrightarrow{\pi_X^!} H^{p-1}(X/\boldsymbol{Z}_2) \xrightarrow{\cup w(X)} H^p(X/\boldsymbol{Z}_2) \xrightarrow{\pi_X^*} H^p(X) \xrightarrow{\pi_X^!} H^p(X/\boldsymbol{Z}_2) \xrightarrow{\cup w(X)} \cdots$$

Because $H^p(X) = 0$ for $1 \leqq p \leqq n-1$, we have $w(X)^n \neq 0$ from exactness of this sequence. Let $\bar{f} : X/\boldsymbol{Z}_2 \to \boldsymbol{RP}^n$ be a continuous map such that $\pi_S \circ f = \bar{f} \circ \pi_X$. Because $\bar{f}^*(w(S^n)^n) = \bar{f}^*(w(S^n))^n = w(X)^n \neq 0$, $\bar{f}^* : H^n(\boldsymbol{RP}^n) \to H^n(X/\boldsymbol{Z}_2)$ is a non-zero homomorphism. Note that transfer homomorphisms $\pi_X^! : H^n(X) \to H^n(X/\boldsymbol{Z}_2)$ and $\pi_S^! : H^n(S^n) \to H^n(\boldsymbol{RP}^n)$ satisfy $\pi_X^! \circ f^* = \bar{f}^* \circ \pi_S^!$. Because $\pi_S^! : H^n(S^n) \to H^n(S^n/\boldsymbol{Z}_2)$ is an isomorphism, $\pi_X^! \circ f^* = \bar{f}^* \circ \pi_S^! \neq 0 : H^n(S^n) \to H^n(X/\boldsymbol{Z}_2)$. Therefore $f^* : H^n(S^n) \to H^n(X)$ is a non-zero homomorphism. $\square$

Proof of Theorem 1.1. Let Y be a connected finite regular cell complex such that p-th cohomology $H^p(Y)$ is zero for $1 \leqq p \leqq n$. Let X be a subcomplex of Y such that every cell of $Y \backslash X$ is an $(n+1)$-cell and that there exists a free cellular $\boldsymbol{Z}_2$-action on X. Then $H^p(X) = 0$ for $1 \leqq p \leqq n-1$, because the n skelton of X and Y are the same. If there exisits a $\boldsymbol{Z}_2$-map $f : X \to S^n$, then $f^* : H^n(S^n) \to H^n(X)$ is a non-zero homomorphism by Lemma 2.1.

We denote by $e_1, \ldots, e_k$ $(n+1)$-cells in $Y \backslash X$. Consider the following commutative diagram.

$$\begin{array}{ccccc}
H^n(Y) & \longrightarrow & H^n(X) & \xrightarrow{\quad\delta^*\quad} & H^{n+1}(Y, X) \\
 & & \Big\downarrow{\scriptstyle \oplus j_i^*} & & \Big\downarrow{\scriptstyle \cong} \\
 & & \bigoplus_{i=1}^k H^n(\partial e_i) & \xrightarrow{\cong} & \bigoplus_{i=1}^k H^{n+1}(\overline{e_i}, \partial e_i)
\end{array}$$

where the first row is a part of the cohomology exact sequence of the pair (Y, X), $\overline{e_i}$ denotes the closure of e_i and $j_i : \partial e_i \to X$ denotes the inclusion map. We see that $\delta^* : H^n(X) \to H^{n+1}(Y, X)$ is injective because $H^n(Y) = 0$. From the above diagram, we see that $\bigoplus j_i^* : H^n(X) \to \bigoplus_{i=1}^k H^n(\partial e_i)$ is injective. Therefore $(f|\partial e_i)^* = j_i^* \circ f^* : H^n(S^n) \to H^n(\partial e_i)$ is a non-zero homomorphism for some i. Because $H^n(S^n) \cong H^n(\partial e_i) \cong \boldsymbol{Z}_2$, $(f|\partial e_i)^* : H^n(S^n) \to H^n(\partial e_i)$ is an isomorphism. $\square$

The following is an application of Theorem 1.1 to combinatorics.

Proposition 2.2. *Let n be a positive integer and Y a connected finite regular cell complex such that $H^p(Y) = 0$ for $1 \leqq p \leqq n$. Let X be a subcomplex of Y which admits a simplicial subdivision with a free simplicial $\boldsymbol{Z}_2$-action, such that each cell of $Y \backslash X$ has dimension $(n+1)$. Let $V(X)$ be the vertex set of a simplicial subdivision of X and g the generator of $\boldsymbol{Z}_2$. Let $f : V(X) \to \{\pm 1, \pm 2, \dots, \pm(n+1)\}$ be a map satisfying $f(g \cdot u) = -f(u)$ for $u \in V(X)$ and $f(u) \neq -f(v)$ for each edge $\{u, v\}$. Then there exists a cell e in $Y \backslash X$ such that $f|V(\partial e) : V(\partial e) \to \{\pm 1, \pm 2, \dots, \pm(n+1)\}$ is surjective.*

Proof. Let $\boldsymbol{e}_1, \dots, \boldsymbol{e}_{n+1}$ be the vectors of the standard orthonormal basis in $\boldsymbol{R}^{n+1}$. The $(n+1)$-dimensional crosspolytope is the convex hull of the points $\boldsymbol{e}_1, -\boldsymbol{e}_1, \dots, \boldsymbol{e}_{n+1}, -\boldsymbol{e}_{n+1}$. We denote by Γ^n the boundary of the $(n+1)$-dimensional crosspolytope. Note that Γ^n is homeomorphic to S^n. There is a simplicial complex structure on Γ^n such that vertices are $\boldsymbol{e}_1, -\boldsymbol{e}_1, \dots, \boldsymbol{e}_{n+1}, -\boldsymbol{e}_{n+1}$ and a subset F of $\{\boldsymbol{e}_1, -\boldsymbol{e}_1, \dots, \boldsymbol{e}_{n+1}, -\boldsymbol{e}_{n+1}\}$ is a face if and only if there is no $i \in [n]$ with both $\boldsymbol{e}_i \in F$ and $-\boldsymbol{e}_i \in F$. We identify the vertex set $V(\Gamma^n)$ of Γ^n with $\{\pm 1, \pm 2, \dots, \pm(n+1)\}$.

Suppose that $f : V(X) \to \{\pm 1, \pm 2, \dots, \pm(n+1)\}$ satisfies $f(g \cdot u) = -f(u)$ for $u \in V(X)$ and $f(u) \neq -f(v)$ for each edge $\{u, v\}$. Then we have a $\boldsymbol{Z}_2$ simplicial map $f : X \to \Gamma^n$. By Theorem 1.1, there exists an $(n+1)$-cell e such that $(f|\partial e)^* : H^n(\Gamma^n) \to H^n(\partial e)$ is an isomorphism. Therefore we see that $f|V(\partial e) : V(\partial e) \to V(\Gamma^n)$ is surjective. $\square$

We give examples of Theorem 1.1 and Proposition 2.2.

Let n and k be integers with $n > k$. We put $I = [-1, 1]$, $[n] = \{1, 2, \dots, n\}$ and

$$\Lambda_n = \{(S, T) \mid S \subset [n], T \subset [n], S \cap T = \varnothing\}.$$

For an integer k with $0 \leqq k \leqq n$, we define a subset $\Lambda_{n,k}$ of Λ_n by

$$\Lambda_{n,k} = \{(S,T) \in \Lambda_n \mid |S| + |T| = n - k\},$$

where $|S|$ and $|T|$ denotes the number of elements of S and T respectively. For $(S,T) \in \Lambda_n$, we define a subset $e_{S,T}$ of I^n by

$$e_{S,T} = \left\{ (x_1, \dots, x_n) \in I^n \;\middle|\; \begin{array}{c} x_i = 1 \text{ for } i \in S,\ x_j = -1 \text{ for } j \in T, \\ -1 < x_l < 1 \text{ for } l \notin S \cup T \end{array} \right\}.$$

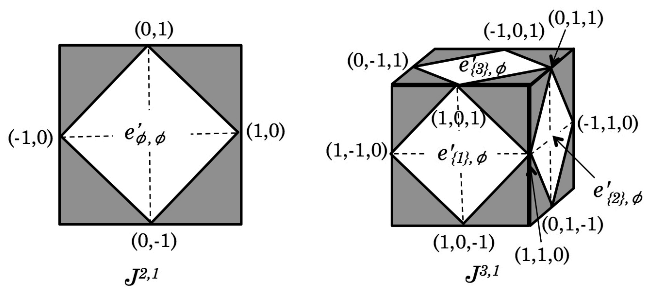
We have a cell decomposition of I^n such that the set of k-cells is $\{e_{S,T} \mid (S,T) \in \Lambda_{n,k}\}$. We denote by $(I^n)^{(k)}$ the k-skelton of I^n. We define a $\boldsymbol{Z}_2$-action on $(I^n)^{(k)}$ by $g \cdot x = -x$, where g is the generator of $\boldsymbol{Z}_2$. If $k < n$, this action on $(I^n)^{(k)}$ is a cellular free action. It is easily seen that $H^l((I^n)^{(k+1)}) = 0$ for $1 \leqq l \leqq k$. Therefore we have the following from Theorem 1.1.

Proposition 2.3. *Let $f : (I^n)^{(k)} \to S^k$ be a $\boldsymbol{Z}_2$-map. Then there exists an $(S,T) \in \Lambda_{n,k+1}$ such that $(f|\partial e_{S,T})^* : H^k(S^k) \to H^k(\partial e_{S,T})$ is an isomorphism.*

We define a subset $e'_{S,T}$ of $e_{S,T}$ by

$$e'_{S,T} = \left\{ (x_1, \dots, x_n) \in I^n \;\middle|\; x_i = 1 \text{ for } i \in S,\ x_j = -1 \text{ for } j \in T,\ \sum_{l \in [n] \setminus (S \cup T)} |x_l| < 1 \right\}.$$

Set $J^{n,k} = (I^n)^{(k+1)} \setminus \bigcup_{(S,T) \in \Lambda_{n,k+1}} e'_{S,T}$. It is seen that $(I^n)^{(k)}$ is a strong deformation retract of $J^{n,k}$.



We give a triangulation of $J^{n,k}$ as follows.

v is a vertex of $J^{n,k}$ if and only if $v \in \{-1,0,1\}^n \cap J^{n,k}$. For $v = (v_1 \dots, v_n) \in \{-1,0,1\}^n$, we define subsets v_+ and v_- of $[n]$ by $v_+ = \{i \in [n] \mid v_i = 1\}$ and $v_- = \{i \in [n] \mid v_i = -1\}$ respectively. In what follows, we identify the set $\{-1,0,1\}^n$

with the set Λ_n by the bijection

$$\varphi : \{-1, 0, 1\}^n \to \Lambda_n, \quad v \mapsto (v_+, v_-)$$

and note that v is in e'_{v_+, v_-}. For $v \in \{-1, 0, 1\}^n$, we put $|v| = |v_+| + |v_-|$, where $|v_+|$ and $|v_-|$ denote the number of elements of v_+ and v_- respectively. For $u = (u_+, u_-)$, $v = (v_+, v_-)$, we write $u \leqq v$ if $u_+ \subset v_+$ and $u_- \subset v_-$ and define $u \cup v$ and $u \cap v$ by $(u_+ \cup v_+, u_- \cup v_-)$ and $(u_+ \cap v_+, u_- \cap v_-)$ respectively. Under these notations, each vertex v of $J^{n,k}$ is identified with a pair (v_+, v_-) such that $|v| \geqq n - k$.

Let $v_1, v_2 \dots v_t$ be mutually distinct vertices of $J^{n,k}$ regarded as elements of Λ_n by the above identification. Since $|v_j| \geqq n - k$ for each j, we may rearrange these vertices so that $|v_1| = \cdots = |v_s| = n - k$ and $|v_{s+1}| > n - k, \dots, |v_t| > n - k$. the set $\{v_1, v_2, \dots, v_t\}$ is a simplex in $J^{n,k}$ if and only if $v_1, v_2, \dots, v_t$ satisifies the following conditions.

(1) $v_1 \cup v_2 \cup \cdots \cup v_s \leqq v_{s+1} \leqq \cdots \leqq v_t$.

(2) If $s \geqq 2$, then $|v_1 \cap v_2 \cap \cdots \cap v_s| = n - k - 1$ and $v_{i+} \cap v_{j-} = \emptyset$ for $1 \leqq i, j \leqq s$.

When $s = 0$ or $s = 1$, $\{v_1, v_2, \dots, v_t\}$ is a simplex in $J^{n,k}$ if and only if $v_1 \leqq v_2 \leqq \cdots \leqq v_t$.

In this way, we have a triangulation of $J^{n,k}$. We give a cell decomposition of $(I^n)^{(k+1)}$ by simplexes of $J^{n,k}$ and $e'_{S,T}$ for $(S, T) \in \Lambda_{n,k+1}$.

We define a $\boldsymbol{Z}_2$-action on $J^{n,k}$ by $g \cdot x = -x$. This action is simplicial and free. We denote by $V(J^{n,k})$ the vertex set of $J^{n,k}$. Then we have the following.

PROPOSITION 2.4. *If $f : V(J^{n,k}) \to \{\pm 1, \pm 2, \dots, \pm(k+1)\}$ satisfies $f(g \cdot u) = -f(u)$ for any veretex u of $J^{n,k}$ and $f(u) \neq -f(v)$ for any edge $\{u, v\}$ of $J^{n,k}$, then there exists an $(S, T) \in \Lambda_{n,k+1}$ such that $f|V(\partial e'_{S,T}) : V(\partial e'_{S,T}) \to \{\pm 1, \pm 2, \dots, \pm(k+1)\}$ is bijective.*

PROOF. By Proposition 2.2, there exists an (S, T) in $\Lambda_{n,k+1}$ such that $f|V(\partial e'_{S,T})$ is surjective. Because $|V(\partial e'_{S,T})| = |V(\Gamma^k)| = 2(k+1)$, $f|V(\partial e'_{S,T})$ is bijective. □

3. Proof of Chen's alternative Kneser coloring theorem

In this section, we prove Chen's alternative Kneser coloring theorem. Our proof follows the line given in [3], replacing a combinatorial argument with Proposition 2.4 obtained by making use of topological method.

Let $c : \binom{[n]}{k} \to [n-2k+2]$ be a proper coloring of $KG_{n,k}$. For a subset S of $[n]$, we define

$$c'(S) = \begin{cases} \max\{c(A) \mid A \subset S,\ |A| = k\} & (|S| \geqq k) \\ 0 & (|S| < k) \end{cases}.$$

Let (S,T) be an element of Λ_n such that $|S| + |T| \geqq 2k-1$. Then $|S| \geqq k$ or $|T| \geqq k$. If (S,T) satisfies both $|S| \geqq k$ and $|T| \geqq k$, then there exists a subset A of S and a subset B of T such that $c'(S) = c(A)$ and $c'(T) = c(B)$. Since $A \cap B \subset S \cap T = \emptyset$, $\{A, B\}$ is an edge of $KG_{n,k}$ and hence $c(A) \neq c(B)$. Therefore if $(S,T) \in \Lambda_n$ satisfies $|S| + |T| \geqq 2k-1$, then $c'(S) \neq c'(T)$.

We define a map λ from the vertex set of $J^{n,n-2k+1}$ to $\{\pm 1, \pm 2, \ldots, \pm(n-2k+2)\}$ by

$$\lambda(v) = \begin{cases} c'(v_+) & c'(v_+) > c'(v_-) \\ -c'(v_-) & c'(v_+) < c'(v_-) \end{cases}.$$

Because $g(v_+, v_-) = (v_-, v_+)$, λ satisfies $\lambda(gv) = -\lambda(v)$.

For an edge $\{u, v\}$ of $J^{n,n-2k+1}$, we easily see that $u_+ \cap v_- = \emptyset$ and $u_- \cap v_+ = \emptyset$. Therefore we see that $\lambda(u) \neq -\lambda(v)$ from the definition of λ. By Proposition 2.4, there exists an (S,T) in $\Lambda_{n,n-2k+2}$ such that $\lambda | V(\partial e'_{S,T}) : V(\partial e'_{S,T}) \to \{\pm 1, \pm 2, \ldots, \pm(n-2k+2)\}$ is bijective. Note that $|S| + |T| = 2k-2$ and that every vertex $v = (v_+, v_-)$ of $\partial e'_{S,T}$ satisfies $|v| = 2k-1$, $S \subset v_+$ and $T \subset v_-$. Therefore we have $|S| \leqq |v_+| \leqq |S| + 1$ and $|T| \leqq |v_-| \leqq |T| + 1$. We show that $|S| = |T| = k-1$. Suppose $|S| > |T|$. Then $|v_-| \leqq |T| + 1 \leqq k-1$ and $c'(v_-) = 0$ for $v \in V(\partial e_{S,T})$. Hence the map λ takes a positive value on every vertex of $\partial e'_{S,T}$, which contradicts that $\lambda | V(\partial e'_{S,T})$ is bijective. Anologously, the strict inequality $|S| < |T|$ is impossible. Therefore $|S| = |T| = k-1$.

Since $\lambda | V(\partial e'_{S,T})$ is bijective, there exist vertices $v_1, \ldots, v_{n-2k+2}$ in $V(\partial e'_{S,T})$ such that $\lambda(v_1) = 1, \ldots, \lambda(v_{n-2k+2}) = n-2k+2$.

Next we observe $|(v_j)_+| = k$ as follows: since $\lambda(v_j) = j > 0$ we have $c'((v_j)_+) > c'((v_j)_-) \geqq 0$ and hence $|(v_j)_+| \geqq k$. On the other hand, $v_j \in V(\partial e'_{S,T})$ and $|S| = k-1$ implies $|(v_j)_+| \leqq |S| + 1 \leqq k$ and we obtain the desired equality.

It follows from this that the elements of $[n] \backslash (S \cup T)$ are enumerated as $i_1, i_2, \ldots, i_{n-2k+2}$ such that $v_{1+} = S \cup \{i_1\}, \ldots, (v_{n-2k+2})_+ = S \cup \{i_{n-2k+2}\}$. Then $c(S \cup \{i_1\}) = 1, \ldots, c(S \cup \{i_{n-2k+2}\}) = n-2k+2$. If $a \neq b$, then $(S \cup \{i_a\}) \cap (T \cup \{i_b\}) = \emptyset$. Hence $S \cup \{i_a\}$ and $T \cup \{i_b\}$ are adjacent in the graph $KG_{n,k}$, we have $c(S \cup \{i_a\}) \neq c(T \cup \{i_b\})$ for $a \neq b$. Therefore we have $c(S \cup \{i_j\}) = c(T \cup \{i_j\}) = j$ for $j = 1, 2, \ldots, n-2k+2$. $\square$

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