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RING OF THE WEIGHT ENUMERATORS OF d_n^+

By

Makoto Fujii and Manabu Oura

Abstract. We show that the ring of the weight enumerators of a self-dual doubly even code d_n^+ in arbitrary genus is finitely generated. Indeed enough elements to generate it are given. The latter result is applied to obtain a minimal set of generators of the ring in genus two.

1 Introduction

The weight enumerator plays an important role in coding theory and has connections with other branches in mathematics. We recall some of them to see the background of this paper.

Let C be a self-dual doubly even (Type II, for short) code of length n. The weight enumerator

$$W_C(x, y) = \sum_{v \in C} x^{n-wt(v)} y^{wt(v)}$$

has invariant properties. The so-called MacWilliams identity is described as

$$W_C(x, y) = W_C\left(\frac{x+y}{\sqrt{2}}, \frac{x-y}{\sqrt{2}}\right)$$

and the doubly evenness gives

$$W_C(x, y) = W_C(x, \sqrt{-1}y).$$

These being said, the weight enumerator of a Type II code is an element of the invariant ring

$$\mathbf{C}[x, y]^G = \{f(x, y) \in \mathbf{C}[x, y] : \sigma f = f \ \ \forall \sigma \in G\}$$

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of the finite group G where G is of order 192 generated by

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{-1} \end{pmatrix}$$

and

$$\sigma f(x, y) = f(ax + by, cx + dy), \quad \sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Gleason [5] showed that the invariant ring $\mathbf{C}[x, y]^G$ is generated by the weight enumerators of Type II codes. Indeed we have

$$\mathbf{C}[x, y]^G = \mathbf{C}[W_{d_8^+}(x, y), W_{d_{24}^+}(x, y)].$$

We shall mention two consequences of this equality. Since the degrees of the generators are 8 and 24, the length of a Type II code is a multiple of 8. Non-existence of an extremal Type II code for sufficiently large n also follows from the above equality.

We observe that $W_{d_8^+}(x, y)$ and $W_{d_{24}^+}(x, y)$ are algebraically independent over $\mathbf{C}$. A finite group having such a property (i.e., whose invariant ring is generated by the algebraically independent elements over $\mathbf{C}$) is called a finite unitary reflection group. See [20].

The generalization of the above correspondence is initiated in [1, 10] and inherited in [4, 18]. *cf.* [12].

The invariance property of the weight enumerator gives rise to the relation with the modular forms, (*cf.* [3, 4, 18]). In fact, the weight enumerator of a Type II code of length n is mapped under the theta map to the Siegel modular form of weight $n/2$ in genus g. The modular form of weight 8 which is obtained from the difference $\psi^{(g)}$ of the weight enumerators of $d_8^+ \oplus d_8^+$ and d_{16}^+ is of great importance. We just mention two points in genus three. Witt [21] asked if the modular form obtained from $\psi^{(3)}$ vanishes, and it was affirmatively answered in [8, 9]. Runge [16, 17] showed that the ring of Siegel modular forms for Γ_3 is isomorphic to the quotient ring of the invariant ring of some finite group divided by an ideal $(\psi^{(3)})$.

Let $\mathfrak{D}^{(g)}$ be the ring of the weight enumerator of d_n^+ in genus g. This is a subring of the ring of the weight enumerators of Type II codes. As indicated above, $\mathfrak{D}^{(1)}$ coincides with the ring of the weight enumerators. In this paper, we show that $\mathfrak{D}^{(g)}$ *is generated by the elements of*

$$8 \leq n \leq 2^{2g+3}.$$

Using this result, we show that $\mathfrak{D}^{(2)}$ *is minimally generated by nine weight enumerators of lengths*

$$8, 24, 32, 40, 48, 56, 64, 72, 80.$$

The computations were done with Magma [2] and SageMath [19].

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2 Preliminaries

We recall coding theory (*cf.* [6, 11, 15]). Let $\mathbf{F}_2 = \{0, 1\}$ be the field of two elements. Two vector spaces $\mathbf{F}_2^n$ and $\mathbf{F}_2^g$ appear in the following. For technical reason, an element of $\mathbf{F}_2^n$ is regarded as a row vector, while that of $\mathbf{F}_2^g$ as a column vector. The space $\mathbf{F}_2^n$ is equipped with the inner product

$$u \cdot v = u_1 v_1 + \cdots + u_n v_n, \quad u = (u_1, \ldots, u_n), \quad v = (v_1, \ldots, v_n)$$

and so is $\mathbf{F}_2^g$

$$\alpha \cdot \beta = \alpha_1 \beta_1 + \cdots + \alpha_g \beta_g, \quad \alpha = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_g \end{pmatrix}, \quad \beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_g \end{pmatrix}.$$

Since we deal with only binary linear codes in this paper, we call a subspace of $\mathbf{F}_2^n$ a code of length n. The weight $wt(u)$ of an element $u \in \mathbf{F}_2^n$ is the number of non-zero coordinates of u. A code C is said to be self-dual if it coincides with its dual

$$C^\perp = \{v \in \mathbf{F}_2^n : u \cdot v = 0, \ \forall u \in C\},$$

doubly even if

$$wt(u) \equiv 0 \pmod 4, \quad \forall u \in C.$$

Codes with those two properties (self-duality and doubly evenness) are particularly interesting. We use the term Type II instead of self-dual and doubly even. It is known that a Type II code of length n exists if and only if $n \equiv 0 \pmod 8$.

We proceed to generalize the weight enumerator that appeared in Introduction. In order to do so, we refresh the weight enumerator. We introduce new variables x_α $(\alpha \in \mathbf{F}_2)$ and the number $\omega_\alpha(u)$ of occurrences of α in a row vector u. Then the weight enumerator is defined as

$$W_C(x_0, x_1) = \sum_{u \in C} \prod_{\alpha \in \mathbf{F}_2} x_\alpha^{\omega_\alpha(u)}.$$

We also interpret this as

$$W_C(x_0, x_1) = \sum_{u=(u_1, u_2, \ldots, u_n) \in C} x_{u_1} x_{u_2} \cdots x_{u_n}.$$

This weight enumerator is essentially the same as that in Introduction. We shall define the weight enumerator in genus g. For any binary linear code C of length n, the weight enumerator of C in genus g is, by definition,

$$W_C^{(g)}(x_\alpha : \alpha \in \mathbf{F}_2^g) = \sum_{u_1, \ldots, u_g \in C} \prod_{\alpha \in \mathbf{F}_2^g} x_\alpha^{\omega_\alpha(u_1, \ldots, u_g)} \tag{1}$$

where $\omega_\alpha(u_1, \ldots, u_g)$ is the number of occurrences of α as a column in the matrix of row vectors $\begin{pmatrix} u_1 \\ \vdots \\ u_g \end{pmatrix}$. As before, we rewrite this as

$$W_C^{(g)}(x_\alpha : \alpha \in \mathbf{F}_2^g) = \sum_{\substack{u_1=(u_{11}, \ldots, u_{1n}) \in C \\ \vdots \\ u_g=(u_{g1}, \ldots, u_{gn}) \in C}} x_{\begin{pmatrix} u_{11} \\ \vdots \\ u_{g1} \end{pmatrix}} x_{\begin{pmatrix} u_{12} \\ \vdots \\ u_{g2} \end{pmatrix}} \cdots x_{\begin{pmatrix} u_{1n} \\ \vdots \\ u_{gn} \end{pmatrix}}.$$

Though this formulation might not be well known, it works well in the proof of the formula for d_n^+ in this section. Since there does not occur any confusion, we shall use an abridged notation $W_C^{(g)}$. It is clear that $W_C^{(g)}$ is a homogeneous polynomial of total degree n in $\mathbf{C}[x_\alpha : \alpha \in \mathbf{F}_2^g]$. Let $\mathfrak{W}^{(g)}$ be the ring over $\mathbf{C}$ generated by the weight enumerators of all Type II codes in genus g. It is known that $\mathfrak{W}^{(g)}$ is the invariant ring of the specified finite group (*cf.* [5, 18, 12]). In particular $\mathfrak{W}^{(g)}$ is finitely generated. In Introduction we discussed this topic for $g = 1$.

Next we recall a Type II code d_n^+ of length n for $n \equiv 0 \pmod 8$ and its weight enumerator. It is nice to start with a repetition code R_n of length n. The dual code of R_n can be described as

$$R_n^\perp = \{(u_1, \dots, u_n) \in \mathbf{F}_2^n : u_1 + \cdots + u_n = 0\}$$

which has a generator matrix

$$\begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ & & & \ddots & \\ 1 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

The following $n/2 \times n$ matrix is a generator matrix of d_n^+, that is, the $n/2$ row vectors form a basis of d_n^+:

$$\begin{pmatrix} 11 & 11 & 00 & 00 & \cdots & 00 & 00 \\ 11 & 00 & 11 & 00 & \cdots & 00 & 00 \\ 11 & 00 & 00 & 11 & \cdots & 00 & 00 \\ & & & & \ddots & & \\ 11 & 00 & 00 & 00 & \cdots & 00 & 11 \\ 10 & 10 & 10 & 10 & \cdots & 10 & 10 \end{pmatrix}.$$

The code d_n^+ is then characterized as

$$\begin{aligned} d_n^+ = \{&(\alpha_1 + \gamma, \alpha_1, \alpha_2 + \gamma, \alpha_2, \dots, \alpha_{n/2} + \gamma, \alpha_{n/2}) : \alpha_1, \dots, \alpha_{n/2}, \gamma \in \mathbf{F}_2, \\ &\alpha_1 + \cdots + \alpha_{n/2} = 0\}. \end{aligned}$$

It is known to be Type II. The weight enumerator of d_n^+ in genus g is expressed as

$$W_{d_n^+}^{(g)} = \frac{1}{2^g} \sum_{\beta, \gamma \in \mathbf{F}_2^g} \left(\sum_{\alpha \in \mathbf{F}_2^g} (-1)^{\alpha \cdot \beta} x_{\alpha+\gamma} x_\alpha \right)^{n/2}.$$

We can find this formula of genus two in [4]. For the completeness of this paper, we add a proof. We have

$$\begin{aligned} \mathrm{RHS} &= \frac{1}{2^g} \sum_{\gamma \in \mathbf{F}_2^g} \left[\sum_{\beta \in \mathbf{F}_2^g} \left\{ \prod_{i=1}^{n/2} \left(\sum_{\alpha^{(i)} \in \mathbf{F}_2^g} (-1)^{\alpha^{(i)} \cdot \beta} x_{\alpha^{(i)}+\gamma} x_{\alpha^{(i)}} \right) \right\} \right] \\ &= \frac{1}{2^g} \sum_{\gamma \in \mathbf{F}_2^g} \left(\sum_{\substack{\alpha^{(1)}, \dots, \alpha^{(n/2)} \in \mathbf{F}_2^g \\ \beta \in \mathbf{F}_2^g}} (-1)^{(\alpha^{(1)} + \cdots + \alpha^{(n/2)}) \cdot \beta} x_{\alpha^{(1)}+\gamma} x_{\alpha^{(1)}} \cdots x_{\alpha^{(n/2)}+\gamma} x_{\alpha^{(n/2)}} \right). \end{aligned}$$

For a fixed γ, we divide the summation as

$$\sum_{\substack{\alpha^{(1)},\dots,\alpha^{(n/2)}\in\mathbf{F}_2^g\\ \beta\in\mathbf{F}_2^g}} = \sum_{\substack{\alpha^{(1)}+\cdots+\alpha^{(n/2)}=0\\ \beta}} + \sum_{\substack{\alpha^{(1)}+\cdots+\alpha^{(n/2)}\neq 0\\ \beta}}.$$

From the $\sum_{\substack{\alpha^{(1)}+\cdots+\alpha^{(n/2)}=0\\ \beta}}$-part, we get

$$2^g \sum_{\substack{\alpha^{(1)},\dots,\alpha^{(n/2)}\in\mathbf{F}_2^g\\ \alpha^{(1)}+\cdots+\alpha^{(n/2)}=0}} x_{\alpha^{(1)}+\gamma}x_{\alpha^{(1)}}\cdots x_{\alpha^{(n/2)}+\gamma}x_{\alpha^{(n/2)}}$$

because of $(-1)^{(\alpha^{(1)}+\cdots+\alpha^{(n/2)})\cdot\beta}=1$ for any $\beta\in\mathbf{F}_2^g$. Next fix $\alpha^{(1)},\dots,\alpha^{(n/2)}$ such that $\alpha^{(1)}+\cdots+\alpha^{(n/2)}\neq 0$. Then the number of $\beta\in\mathbf{F}_2^g$ which is orthogonal to $\alpha^{(1)}+\cdots+\alpha^{(n/2)}(\neq 0)$ is 2^{g-1}. This could be easily understood if you consider the dual code of a code generated by $\alpha^{(1)}+\cdots+\alpha^{(n/2)}$. At any rate, from the $\sum_{\substack{\alpha^{(1)}+\cdots+\alpha^{(n/2)}\neq 0\\ \beta}}$-part, we get 0. Finally we have that

$$\sum_{\gamma\in\mathbf{F}_2^g}\sum_{\substack{\alpha^{(1)},\dots,\alpha^{(n/2)}\in\mathbf{F}_2^g\\ \alpha^{(1)}+\cdots+\alpha^{(n/2)}=0}} x_{\alpha^{(1)}+\gamma}x_{\alpha^{(1)}}\cdots x_{\alpha^{(n/2)}+\gamma}x_{\alpha^{(n/2)}}.$$

In view of the characterization of d_n^+ this is nothing else but the definition of the weight enumerator of d_n^+ in genus g. This proves the formula.

We denote by $\mathfrak{D}^{(g)}$ the ring generated over $\mathbf{C}$ by the weight enumerators of d_n^+ $(n=8,16,24,\dots)$ in genus g. The ring $\mathfrak{D}^{(g)}$ is a subring of $\mathfrak{W}^{(g)}$. These rings are graded as

$$\mathfrak{W}^{(g)} = \bigoplus_{n\equiv 0 \pmod 8}\mathfrak{W}_n^{(g)},$$

$$\mathfrak{D}^{(g)} = \bigoplus_{n\equiv 0 \pmod 8}\mathfrak{D}_n^{(g)}.$$

In [4], $\mathfrak{W}^{(2)}$ is determined. Let g_{24} be the Golay code of length 24. It is then

$$\mathfrak{W}^{(2)} = \mathbf{C}[W_{d_8^+}^{(2)}, W_{d_{24}^+}^{(2)}, W_{g_{24}}^{(2)}, W_{d_{40}^+}^{(2)}] \oplus \mathbf{C}[W_{d_8^+}^{(2)}, W_{d_{24}^+}^{(2)}, W_{g_{24}}^{(2)}, W_{d_{40}^+}^{(2)}]W_{d_{32}^+}^{(2)}$$

and the dimension formula is given as follows:

$$\sum_n \dim\mathfrak{W}_n^{(2)} = \frac{1+t^{32}}{(1-t^8)(1-t^{24})^2(1-t^{40})}.$$

n	8	16	24	32	40	48	56	64	72	80	88	96	104	112	120	128
$\dim \mathfrak{W}_n^{(2)}$	1	1	3	4	5	8	10	12	17	21	24	31	37	42	52	60
$\dim \mathfrak{D}_n^{(2)}$	1	1	2	3	4	6	8	11	15	20	24	30	36	42	51	59

Table: Dimensions of $\mathfrak{W}_n^{(2)}$ and $\mathfrak{D}_n^{(2)}$

The dimensions of small n in genus 2 are given in Table.

Finally we recall the following Φ-operator

$$\Phi(x_{\substack{\alpha\\0}}) = x_\alpha \quad \text{and} \quad \Phi(x_{\substack{\alpha\\1}}) = 0, \quad \alpha \in \mathbf{F}_2^{g-1}, \quad \begin{pmatrix}\alpha\\ *\end{pmatrix} \in \mathbf{F}_2^g.$$

It is known that the Φ-operator maps the weight enumerator of a code in genus g to that in genus $g-1$.

3 Results

Our first objective is to prove

Theorem 1. (1) *$\mathfrak{D}^{(g)}$ is finitely generated over $\mathbf{C}$.*
(2) *A set of generators of $\mathfrak{D}^{(g)}$ can be obtained from $W_{d_n^+}^{(g)}$ for $n \leq 2^{2g+3}$.*

Proof. Since (1) follows from (2), we shall show (2). If we ignore the coefficient $\dfrac{1}{2^g}$, the weight enumerator of d_n^+ has the form

$$X_1^{n/2} + X_2^{n/2} + \cdots + X_{2^{2g}}^{n/2}$$

for fixed $X_i \in \mathbf{C}[x_\alpha : \alpha \in \mathbf{F}_2^g]$. For a better understanding, we put $Z_i = X_i^4$. Here we remind that $n \equiv 0 \pmod 8$. Then we can say that $\mathfrak{D}^{(g)}$ is generated by the forms

$$Z_1 + Z_2 + \cdots + Z_{2^{2g}},$$

$$Z_1^2 + Z_2^2 + \cdots + Z_{2^{2g}}^2,$$

$$\vdots$$

If we apply the fundamental theorem of symmetric polynomials, we can conclude that $\mathfrak{D}^{(g)}$ can be generated by

$$Z_1^i + Z_2^i + \cdots + Z_{2^{2g}}^i, \quad 1 \leq i \leq 2^{2g}.$$

Translating this into the condition of the length n, we have

$$1 \leq \frac{n}{2} \cdot \frac{1}{4} \leq 2^{2g}.$$

Hence, in order to generate $\mathfrak{D}^{(2)}$, it is enough for n to range from 8 through 2^{2g+3} mod 8. This completes the proof. □

We shall examine the case $g = 1$. From (2) of Theorem 1, we possess four elements of lengths 8, 16, 24, 32 to generated $\mathfrak{D}^{(1)}$. Because of

$$(W_{d_8^+}^{(1)})^2 = W_{d_{16}^+}^{(1)} \quad \text{and} \quad W_{d_{32}^+}^{(1)} = -\frac{5}{3}(W_{d_8^+}^{(1)})^4 + \frac{8}{3} W_{d_8^+}^{(1)} W_{d_{24}^+}^{(1)},$$

we get

$$\mathfrak{D}^{(1)} = \mathbf{C}[W_{d_8^+}^{(1)}, W_{d_{24}^+}^{(1)}].$$

Notice that our argument in this section does not give guarantee as to the fact $\mathfrak{W}^{(1)} = \mathfrak{D}^{(1)}$.

We proceed to the higher genus. Table shows that $\mathfrak{D}^{(2)}$ is strictly smaller than $\mathfrak{W}^{(2)}$. In fact, we shall show

Proposition 2. *We have that*

$$\mathfrak{W}^{(g)} = \mathfrak{D}^{(g)} \quad \textit{if and only if} \quad g = 1.$$

Proof. We have only to prove $\mathfrak{D}^{(g)} \subsetneq \mathfrak{W}^{(g)}$ for all $g \geq 2$. We know that $W_{g_{24}}^{(2)} \notin \mathfrak{D}^{(2)}$. Now suppose that

$$W_{g_{24}}^{(g)} = a(W_{d_8^+}^{(g)})^3 + bW_{d_8^+}^{(g)} W_{d_{16}^+}^{(g)} + cW_{d_{24}^+}^{(g)}.$$

for some $g \geq 3$. If we successively apply the Φ-operator to both sides, we get

$$W_{g_{24}}^{(2)} = a(W_{d_8^+}^{(2)})^3 + bW_{d_8^+}^{(2)} W_{d_{16}^+}^{(2)} + cW_{d_{24}^+}^{(2)}.$$

We have thus a contradiction to the fact $W_{g_{24}}^{(2)} \notin \mathfrak{D}^{(2)}$. This completes the proof. □

We turn our attention to the case $g = 2$.

Theorem 3. *The ring $\mathfrak{D}^{(2)}$ is minimally generated by nine elements $W_{d_n^+}^{(2)}$ of lengths*

$$8, 24, 32, 40, 48, 56, 64, 72, 80.$$

PROOF. By (2) of Theorem 1, $\mathfrak{D}^{(2)}$ is generated by the weight enumerators $W_{d_n^+}^{(2)}(x)$ of d_n^+ of lengths $8, 16, \ldots, 2^{2\cdot 2+3} = 128$. By calculating the dimension of the homogeneous part for each n, we get the result. This completes the proof. $\square$

THEOREM 4. *The ring* $\mathfrak{W}^{(2)}$ *is the normalization of* $\mathfrak{D}^{(2)}$ *in its field of fractions.*

PROOF. The ring $\mathfrak{W}^{(2)}$ is generated by the $W_{d_n^+}^{(2)}$'s and $W_{g_{24}}^{(2)}$. Because of $\dim \mathfrak{W}_{88}^{(2)} = \dim \mathfrak{D}_{88}^{(2)} (= 24)$, $(W_{d_8^+}^{(2)})^8 W_{g_{24}}^{(2)}$ should be written as a linear combination of the $W_{d_n^+}^{(2)}$'s. We can say more. The product $(W_{d_8^+}^{(2)})^7 W_{g_{24}}^{(2)}$ is indeed in $\mathfrak{D}_{80}^{(2)}$ by calculation. At any rate, we see that $\mathfrak{W}^{(2)}$ and $\mathfrak{D}^{(2)}$ have the same field of fractions. Since it can be shown that $W_{g_{24}}^{(2)}$ is a root of a monic quadratic equation over $\mathfrak{D}^{(2)}$ by explicit calculation, $\mathfrak{W}^{(2)}$ is integral over $\mathfrak{D}^{(2)}$. We give the mentioned forms above explicitly in Appendix. Since the invariant ring of a finite group is normal, so is $\mathfrak{W}^{(2)}$. This completes the proof. $\square$

We conclude this paper with some comments.

As a finite analogue of Eisenstein series, we studied E-polynomials (*cf.* [13, 14]). Since d_8^+ is a unique Type II code of length 8, we obtain the identity between an E-polynomial of weight 8 and $W_{d_8^+}^{(g)}$. The resulting identity seems to be non-trivial.

Let τ be an element of the Siegel upper-half space of genus g. For $\alpha, \beta \in \mathbf{F}_2^g$, we define a Thetanullwert

$$\theta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau) = \sum_{p \in \mathbf{Z}^g} \exp 2\pi\sqrt{-1}\left\{\frac{1}{2}{}^t\left(p + \frac{1}{2}\alpha\right)\tau\left(p + \frac{1}{2}\alpha\right) + {}^t\left(p + \frac{1}{2}\alpha\right)\frac{1}{2}\beta\right\}.$$

We put $f_\alpha(\tau) = \theta\begin{bmatrix}\alpha\\ 0\end{bmatrix}(2\tau)$. It is known (*cf.* [7]) that

$$\left(\theta\begin{bmatrix}\gamma\\ \beta\end{bmatrix}(\tau)\right)^2 = \sum_{\alpha \in \mathbf{F}_2^g} (-1)^{\alpha\cdot\beta} f_{\alpha+\gamma}(\tau) f_\alpha(\tau).$$

Under the theta map $x_\alpha \to f_\alpha(\tau)$, we derive the theta series $\vartheta_{D_n^+}^{(g)}(\tau)$ of an even unimodular lattice D_n^+ from the weight enumerator of d_n^+ in genus g. Therefore we have that

$$\begin{aligned}\vartheta_{D_n^+}^{(g)}(\tau) &= W_{d_n^+}^{(g)}(f_\alpha(\tau) : \alpha \in \mathbf{F}_2^g)\\
&= \frac{1}{2^g}\sum_{\beta,\gamma\in\mathbf{F}_2^g}\left(\sum_{\alpha\in\mathbf{F}_2^g}(-1)^{\alpha\cdot\beta} f_{\alpha+\gamma}(\tau)f_\alpha(\tau)\right)^{n/2}\\
&= \frac{1}{2^g}\sum_{\beta,\gamma\in\mathbf{F}_2^g}\left(\theta\begin{bmatrix}\gamma\\ \beta\end{bmatrix}(\tau)\right)^n\end{aligned}$$

which was given in [8] without coding theory.

Appendix: Expressions of $W_{g_{24}}^{(2)}$

We shall denote by d_n instead of d_n^+ and by C instead of $W_C^{(2)}$. For example, d_8^7 means $(W_{d_8^+}^{(2)})^7$. In the first formula, if we divide both sides by d_8^7, we get a rational expression of g_{24} by the d_n's. In the second formula, we can see that g_{24} is a root of a monic quadratic equation over $\mathfrak{D}^{(2)}$.

$$\begin{aligned}d_8^7 g_{24} = {} & 60068993523/2765440 \cdot d_{80} - 180183157847/10370400 \cdot d_{40}^2\\
& - 20022997841/553088 \cdot d_{32}d_{48} - 2860428263/69136 \cdot d_{24}d_{56}\\
& + 20022997841/414816 \cdot d_{24}^2 d_{32} - 20022997841/207408 \cdot d_8 d_{72}\\
& + 240240240009/1382720 \cdot d_8 d_{32} d_{40} + 20022997841/103704 \cdot d_8 d_{24} d_{48}\\
& - 20022997841/233334 \cdot d_8 d_{24}^3 + 361030987317/2212352 \cdot d_8^2 d_{64}\\
& - 721615331745/4424704 \cdot d_8^2 d_{32}^2 - 180492013471/518520 \cdot d_8^2 d_{24} d_{40}\\
& - 11605081037/138272 \cdot d_8^3 d_{56} + 162089538457/829632 \cdot d_8^3 d_{24} d_{32}\\
& - 6162271423/51852 \cdot d_8^4 d_{48} + 98965418167/622224 \cdot d_8^4 d_{24}^2\\
& + 1437603895651/6913600 \cdot d_8^5 d_{40} - 1819759052111/33185280 \cdot d_8^6 d_{32}\\
& - 1943814249461/12444480 \cdot d_8^7 d_{24} + 119236217012539/2986675200 \cdot d_8^{10}.\end{aligned}$$

$$\begin{aligned}g_{24}^2 = {} & -53361/9728 \cdot d_{48} + 41699/4864 \cdot d_{24}^2 + 2863707/124640 \cdot d_8 d_{40}\\
& - 55228635/1595392 \cdot d_8^2 d_{32} + 200123/199424 \cdot d_8^3 d_{24}\\
& + 61863307/7976960 \cdot d_8^6 + (161/152 \cdot d_{24} - 3289/12464 \cdot d_8^3) g_{24}.\end{aligned}$$

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Graduate School of Natural Science and Technology
Kanazawa University
Kakuma-machi, Kanazawa
Ishikawa 920-1192, Japan
E-mail: makotofkanazawa@gmail.com
oura@se.kanazawa-u.ac.jp