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HOMOTOPY TYPE OF THE BOX COMPLEXES OF GRAPHS WITHOUT 4-CYCLES

By

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Abstract. In this paper, we show that a graph G contains no 4-cycles if and only if $\|\bar{G}\|$ is a strong $\mathbf{Z}_2$-deformation retract of the box complex $\|\mathsf{B}(G)\|$ of G, where $\bar{G}$ is the 1-dimensional free simplicial $\mathbf{Z}_2$-complex introduced in [2].

1 Introduction

We assume that all graphs are finite, simple, undirected and have no isolated vertices. For a graph G, an abstract free simplicial $\mathbf{Z}_2$-complex $\mathsf{B}(G)$, called the box complex of G, is defined in [3]. The $\mathbf{Z}_2$-index of $\|\mathsf{B}(G)\|$ gives us a lower bound for the chromatic number $\chi(G)$; for any graph G, we have

$$\chi(G) \geq \mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G)\|) + 2.$$

In [4] p. 81, J. Matoušek and G. M. Ziegler pointed out that, for every graph G which contains no 4-cycles, we have $\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G)\|) \leq 1$. This indicates that the difference between $\mathrm{ind}_{\mathbf{Z}_2}(\|\mathsf{B}(G)\|) + 2$ and $\chi(G)$ can be arbitrarily large in general.

We are interested in the relation between topology of $\|\mathsf{B}(G)\|$ and combinatorics of G. In [2], the author showed that the box complex $\mathsf{B}(G)$ contains a natural double covering $\bar{G}$ of G which is a 1-dimensional free simplicial $\mathbf{Z}_2$-subcomplex of $\mathsf{B}(G)$. Also it is shown that the homotopy type of $\|\bar{G}\|$ is determined by the homotopy type of $\|G\|$ and combinatorics of G (see section 3). In this paper, we study the relation between $\mathsf{B}(G)$ and $\bar{G}$ when G contains no 4-cycles.

In [4] p. 81, J. Matoušek and G. M. Ziegler showed that, if G contains no 4-cycles, there is a $\mathbf{Z}_2$-retraction $r : \|\mathrm{sd}\,\mathsf{B}(G)\| \to \|\mathsf{L}\|$, where L is a 1-dimensional subcomplex of the first barycentric subdivision $\mathrm{sd}\,\mathsf{B}(G)$. It turns out that

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$\mathsf{L} = \mathrm{sd}\, \bar{G}$. In section 4, we show that the $\mathbf{Z}_2$-retract $\|\mathsf{L}\|$ is actually a strong $\mathbf{Z}_2$-deformation retract of $\|\mathrm{sd}\, \mathsf{B}(G)\|$. Thus, $\|\mathsf{B}(G)\|$ and $\|\bar{G}\|$ have the same homotopy type. Conversely, if $\|\mathsf{B}(G)\|$ admits a retraction onto $\|\bar{G}\|$, then G contains no 4-cycles (see Theorem 4.3).

2 Preliminaries

In this section, we recall some basic concepts on graphs, abstract simplicial complexes and the $\mathbf{Z}_2$-index of a $\mathbf{Z}_2$-space. We follow [1] with respect to the standard notation in graph theory.

A *graph* is a pair $G = (V(G), E(G))$ which consists of a nonempty finite set $V(G)$ and a family $E(G)$ of 2-elements subsets of $V(G)$. Elements of $V(G)$ (resp. $E(G)$) are called vertices (resp. edges) of G. By this definition, all graphs are simple, that is, they have no loops and multiple edges. Also all graphs are undirected and an edge $\{u, v\}$ of a graph is simply denoted by uv or vu. A vertex of G which is not contained in any edge of G is called an *isolated vertex* of G.

An *abstract simplicial complex* is a pair (V, K), where V is a finite set and K is a family of subsets of V such that if $\sigma \in \mathsf{K}$ and $\tau \subset \sigma$, then $\tau \in \mathsf{K}$. The *polyhedron* of K is denoted by $\|\mathsf{K}\|$.

A $\mathbf{Z}_2$-*space* (X, ν) is a topological space X with a homeomorphism $\nu : X \to X$ such that $\nu \circ \nu = \mathrm{id}$, called a $\mathbf{Z}_2$-*action* on X. A $\mathbf{Z}_2$-action which has no fixed points is said to be *free*. A topological space X with a free $\mathbf{Z}_2$-action is called a *free* $\mathbf{Z}_2$-space. For two $\mathbf{Z}_2$-spaces (X, ν_X), (Y, ν_Y), a continuous map $f : X \to Y$ which satisfies $\nu_Y \circ f = f \circ \nu_X$ is called a $\mathbf{Z}_2$-*map* from X to Y. The $\mathbf{Z}_2$-*index* of a $\mathbf{Z}_2$-space (X, ν) is defined as

$$\mathrm{ind}_{\mathbf{Z}_2}(X, \nu) := \min\{n \mid \text{there exists a } \mathbf{Z}_2\text{-map } X \to S^n\},$$

where $S^n = \{x \in \mathbf{R}^{n+1} \mid \|x\| = 1\}$ with the free $\mathbf{Z}_2$-action given by $x \mapsto -x$.

3 The Box Complex of a Graph and Some Results

In this section, we define the box complex of a graph following [3] and present some results in [2].

Let G be a graph and U a subset of $V(G)$. A vertex $v \in V(G)$ is called a *common neighbor* of U in G if $uv \in E(G)$ for all $u \in U$. The set of all common neighbors of U in G is denoted by $\mathrm{CN}_G(U)$. For a vertex u of $V(G)$, $\mathrm{CN}_G(\{u\})$, the set of all neighbors of u in G, is simply denoted by $\mathrm{CN}_G(u)$. For convenience, we define $\mathrm{CN}_G(\phi) = V(G)$. For $U_1, U_2 \subseteq V(G)$ such that $U_1 \cap U_2 = \phi$, we define $G[U_1, U_2]$ as the bipartite subgraph of G with

$$V(G[U_1, U_2]) = U_1 \cup U_2 \quad \text{and}$$
$$E(G[U_1, U_2]) = \{u_1u_2 \in E(G) \mid u_1 \in U_1, u_2 \in U_2\}.$$

The graph $G[U_1, U_2]$ is said to be *complete* if $u_1u_2 \in E(G)$ for all $u_1 \in U_1$ and $u_2 \in U_2$. For convenience, $G[\phi, U_2]$ and $G[U_1, \phi]$ are also said to be complete.

Let U_1, U_2 be subsets of $V(G)$. The subset $U_1 \uplus U_2$ of $V(G) \times \{1,2\}$ is defined as

$$U_1 \uplus U_2 := (U_1 \times \{1\}) \cup (U_2 \times \{2\}).$$

For vertices $u_1, u_2 \in V(G)$, $\{u_1\} \uplus \phi$, $\phi \uplus \{u_2\}$, and $\{u_1\} \uplus \{u_2\}$ are simply denoted by $u_1 \uplus \phi$, $\phi \uplus u_2$ and $u_1 \uplus u_2$ respectively.

In this paper, we assume that all graphs contain no isolated vertices. The *box complex* of a graph G is an abstract simplicial complex with the vertex set $V(G) \times \{1,2\}$ and the family of simplices

$$\mathsf{B}(G) = \{U_1 \uplus U_2 \mid U_1, U_2 \subseteq V(G), U_1 \cap U_2 = \phi,$$
$$G[U_1, U_2] \text{ is complete}, \mathrm{CN}_G(U_1) \neq \phi \neq \mathrm{CN}_G(U_2)\}.$$

An abstract simplex $U_1 \uplus U_2$ and its geometric simplex are denoted by the same symbol $U_1 \uplus U_2$. The simplicial map $\nu : V(\mathsf{B}(G)) \to V(\mathsf{B}(G))$ given by

$$u \uplus \phi \mapsto \phi \uplus u \quad \text{and} \quad \phi \uplus u \mapsto u \uplus \phi \quad \text{for all } u \in V(G)$$

induces a free $\mathbf{Z}_2$-action on $\|\mathsf{B}(G)\|$. We always think of $\|\mathsf{B}(G)\|$ as a free $\mathbf{Z}_2$-space with this action. It is easy to see that the box complex $\|\mathsf{B}(G)\|$ is the disjoint union $\coprod_{i=1}^{k} \|\mathsf{B}(G_i)\|$, where $\{G_1, \dots, G_k\}$ is the set of all components of G. In what follows, we always assume that graphs under consideration are connected.

Let $\bar{G}$ be the following 1-dimensional simplicial subcomplex of $\mathsf{B}(G)$:

$$\bar{G} := \{u \uplus \phi, v \uplus \phi, \phi \uplus u, \phi \uplus v, u \uplus v, v \uplus u \mid uv \in E(G)\}.$$

Then, $\|\bar{G}\|$ is a free $\mathbf{Z}_2$-space with the restriction of the free $\mathbf{Z}_2$-action on $\|\mathsf{B}(G)\|$. Moreover, following [2], $\bar{G}$ is a natural double covering of G constructed from $\bar{T}$, where T is any spanning tree of G.

Let X be a $\mathbf{Z}_2$-space and A a $\mathbf{Z}_2$-subspace of X. A strong deformation retraction $\{f_t\}_{t \in [0,1]}$ of X onto A such that each $f_t : X \to X$ is a $\mathbf{Z}_2$-map is called a *strong $\mathbf{Z}_2$-deformation retraction* of X onto A. For two spaces X and Y, the symbol $X \simeq Y$ means that they have the same homotopy type. The following two theorems are useful when we investigate topological information of $\|\mathsf{B}(G)\|$.

Theorem 3.1 ([2], Theorem 4.1). *Let G be a connected graph with k induced cycles of G.*

(1) *If G contains no cycles of odd length, we have $\|\bar{G}\| \simeq \bigvee_k S^1 \amalg \bigvee_k S^1$.*

(2) *If G contains at least one cycle of odd length, we have $\|\bar{G}\| \simeq \bigvee_{2k-1} S^1$.* □

Theorem 3.2 ([2], Theorem 4.2). *Let G be a connected graph. Then, $\mathsf{B}(G)$ is connected if and only if $\bar{G}$ is connected.* □

Theorem 3.1 shows that a connected graph G contains at least one cycle of odd length if and only if $\bar{G}$ is connected. Thus, by Theorem 3.2, we see that a connected graph G contains a cycle of odd length if and only if $\mathsf{B}(G)$ is connected.

4 The Box Complexes of a Graph without 4-Cycles

First, if a graph G contains no 4-cycles, we characterize simplices of $\mathsf{B}(G)$.

Lemma 4.1 (cf. [4] p. 81, (H1)). *A graph G contains no 4-cycles if and only if for any simplices $U_1 \uplus U_2 \in \mathsf{B}(G)$, we have $|U_1| \le 1$ or $|U_2| \le 1$. For such a graph G, every maximal simplex $U_1 \uplus U_2 \in \mathsf{B}(G)$ satisfies $|U_1| = 1$ or $|U_2| = 1$.*

Proof. We assume that a graph G contains no 4-cycles. Suppose that $|U_1| \ge 2$ and $|U_2| \ge 2$ for some simplex $U_1 \uplus U_2 \in \mathsf{B}(G)$. Since $G[U_1, U_2]$ is complete, for any vertices $u_1, u_1' \in U_1$ and $u_2, u_2' \in U_2$, the four edges u_1u_2, u_2u_1', $u_1'u_2'$ and $u_2'u_1$ of G yield a 4-cycle of G, a contradiction. Hence, we have $|U_1| \le 1$ or $|U_2| \le 1$ for $U_1 \uplus U_2 \in \mathsf{B}(G)$.

Let $U_1 \uplus U_2$ be a maximal simplex of $\mathsf{B}(G)$ with $|U_1| \le 1$. Suppose that $|U_1| = 0$. Since $\phi \uplus U_2 = U_1 \uplus U_2 \in \mathsf{B}(G)$, there exists a common neighbor x of U_2. Then, we notice that $x \uplus U_2$ is a simplex of $\mathsf{B}(G)$. This contradicts the maximality of $\phi \uplus U_2$. Hence, we see $|U_1| = 1$.

Conversely, we assume that a graph G contains a 4-cycle $u_1u_2u_3u_4u_1$. Let $U_1 = \{u_1, u_3\}$ and $U_2 = \{u_2, u_4\}$. Then, we see $U_1 \uplus U_2 \in \mathsf{B}(G)$. □

Next, we notice the relation between any two distinct maximal simplices of $\mathsf{B}(G)$.

Lemma 4.2. *Let G be a graph without 4-cycles. For any two distinct maximal simplices of $\mathsf{B}(G)$, the intersection is a simplex of $\bar{G}$.*

Proof. Let $U_1 \uplus U_2$ and $V_1 \uplus V_2$ be distinct maximal simplices of $\mathsf{B}(G)$. By the definition, we see $(U_1 \uplus U_2) \cap (V_1 \uplus V_2) = (U_1 \cap V_1) \uplus (U_2 \cap V_2)$. It suffices to prove that $|U_1 \cap V_1| \leq 1$ and $|U_2 \cap V_2| \leq 1$ by the definition of $\bar{G}$.

Suppose that $|U_1 \cap V_1| \geq 2$. Then, we have $|U_2| = 1$ and $|V_2| = 1$ by the maximality of simplices and Lemma 4.1. We divide our consideration into the following two cases:

$$(1) \quad U_2 \cap V_2 \neq \phi \qquad \text{and} \qquad (2) \quad U_2 \cap V_2 = \phi.$$

(1) $U_2 \cap V_2 \neq \phi$. We have $U_2 = V_2$, and so $U_1 \neq V_1$ since $U_1 \uplus U_2$ and $V_1 \uplus V_2$ are distinct. By the maximality of simplices, we see $U_1 \setminus V_1 \neq \phi \neq V_1 \setminus U_1$, so we have $U_1, V_1 \subsetneq U_1 \cup V_1$. On the other hand, since $G[U_1, V_2]$ is complete, we see $U_1 \uplus U_2, V_1 \uplus V_2 \subsetneq (U_1 \cup V_1) \uplus V_2 \in \mathsf{B}(G)$. This contradicts the maximality of $U_1 \uplus U_2$ and $V_1 \uplus V_2$.

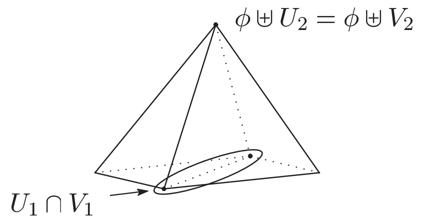


Figure 1. The simplices $U_1 \uplus U_2$ and $V_1 \uplus V_2$, if $U_2 = V_2$.

(2) $U_2 \cap V_2 = \phi$. Let $U_2 = \{u\}$ and $V_2 = \{v\}$. Recall $|U_1 \cap V_1| \geq 2$ and take two vertices $x_1, x_2 \in U_1 \cap V_1$. Then, ux_1, x_1v, vx_2 and x_2u are the edges of G since $U_1 \uplus u, V_1 \uplus v \in \mathsf{B}(G)$. We see that these edges yield a 4-cycle ux_1vx_2u of G, a contradiction.

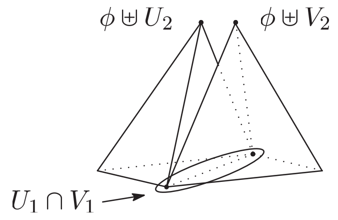


Figure 2. The simplices $U_1 \uplus U_2$ and $V_1 \uplus V_2$, if $U_2 \neq V_2$.

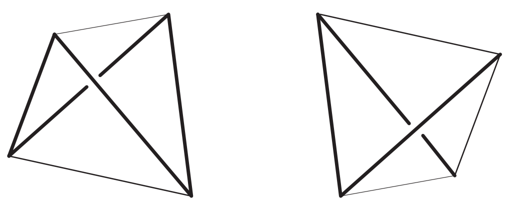
Thus, we conclude that $|U_1 \cup V_1| \leq 1$. By the same argument as above, we have $|U_2 \cap V_2| \leq 1$. Hence, the conclusion follows. $\square$

For each maximal simplex $u_1 \uplus U_2$ (resp. $U_1 \uplus u_2$) of $\mathsf{B}(G)$, we notice that $\phi \uplus U_2$ is a free face of $u_1 \uplus U_2$ (resp. $U_1 \uplus \phi$ is a free face of $U_1 \uplus u_2$). Thus, we can consider a collapsing from these free faces in $\|\mathsf{B}(G)\|$.

Theorem 4.3. *A graph G contains no 4-cycles if and only if $\|\bar{G}\|$ is a strong $\mathbf{Z}_2$-deformation retract of $\|\mathsf{B}(G)\|$.*

Proof. We assume that a graph G contains a 4-cycle C_4. By the definition of box complexes, we see that $\|\mathsf{B}(C_4)\|$ is the disjoint union of two 3-simplices. We notice that $\|\overline{C_4}\|$ is homeomorphic to the disjoint union of two circles, each of which is contractible in $\|\mathsf{B}(C_4)\|$ (see Figure 3). On the other hand, each component of $\|\overline{C_4}\|$ is not contractible in $\|\bar{G}\|$. Suppose that there exists a retraction $r : \|\mathsf{B}(G)\| \to \|\bar{G}\|$. We consider a loop l in $\|\mathsf{B}(G)\|$ which is one of two circles of $\|\overline{C_4}\|$. Then, we see that $r \circ l$ is the circle in $\|\bar{G}\|$ which must be nullhomotopic. This is impossible since $\|\bar{G}\|$ is the 1-dimensional polyhedron. Hence, $\|\bar{G}\|$ is not a retract of $\|\mathsf{B}(G)\|$.

$\|\mathsf{B}(C_4)\|$



(The polyhedron $\|\overline{C_4}\|$ is illustrated with ── .)

Figure 3. The box complex $\|\mathsf{B}(C_4)\|$.

Conversely, we assume that G is a graph without 4-cycles. First, we define a strong deformation retraction of each maximal simplex of $\|\mathsf{B}(G)\|$. By Lemma 4.1, we can divide all maximal simplices of $\|\mathsf{B}(G)\|$ into the two sets of simplices

$$B_1 = \{v \uplus U \mid v \uplus U \text{ is maximal}\} \quad \text{and} \quad B_2 = \{U \uplus v \mid U \uplus v \text{ is maximal}\}.$$

The $\mathbf{Z}_2$-action ν on $\|\mathsf{B}(G)\|$ induces a one-to-one correspondence between B_1 and B_2. For each simplex $v \uplus U \in B_1$, we define a strong deformation retraction $\{f_t^v\}_{t\in[0,1]}$ of $v \uplus U$ onto $K_v^- := \bigcup_{x\in U} v \uplus x$ starting with a collapsing from the free face $\phi \uplus U$ of $v \uplus U$ (see Figure 4).

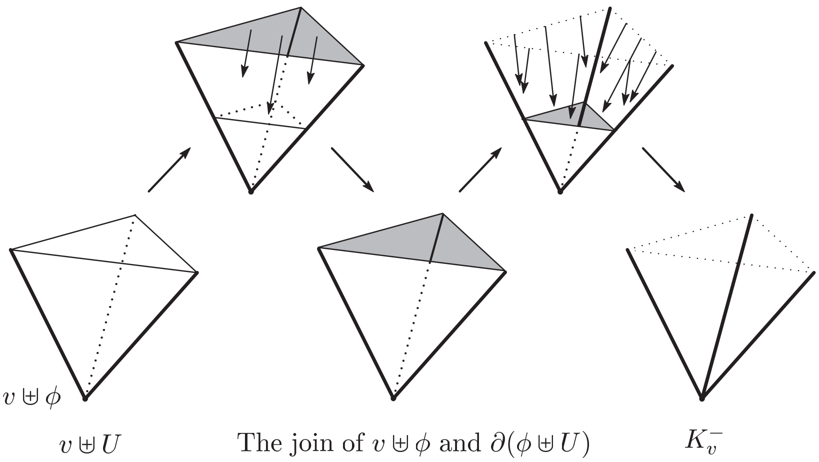


Figure 4. The strong deformation retraction $\{f_t^v\}_{t\in[0,1]}$ of $v \uplus U$ onto K_v^-.

For each simplex $U \uplus v \in B_2$, a strong deformation retraction of $U \uplus v$ onto $K_v^+ := \bigcup_{x\in U} x \uplus v$ is defined as $\{\nu \circ f_t^v \circ \nu\}_{t\in[0,1]}$. Let $X_v = (v \uplus U) \cup (U \uplus v)$, for any $v \in V(G)$. Then, a strong $\mathbf{Z}_2$-deformation retraction F_v of X_v onto $K_v^- \cup K_v^+$ is defined as

$$F_v(x,t) = \begin{cases} f_t^v(x) & \text{if } x \in v \uplus U, \\ \nu \circ f_t^v \circ \nu(x) & \text{if } x \in U \uplus v, \end{cases}$$

where $t \in [0,1]$. By Lemma 4.2, we can check

$$X_u \cap X_v = \|\bar{G}\| \cap X_u \cap X_v = (K_u^- \cup K_u^+) \cap (K_v^- \cup K_v^+)$$

for $u, v \in V(G)$ with $u \neq v$. Notice that $\bar{G} = \bigcup_{v\in V(G)} (K_v^- \cup K_v^+)$. Since the homotopy F_v is stationary on $K_v^- \cup K_v^+$ for each $v \in V(G)$, we see that the homotopies $\{F_v \mid v \in V(G)\}$ induce a strong $\mathbf{Z}_2$-deformation retraction of $\|\mathsf{B}(G)\|$ onto $\|\bar{G}\|$. $\square$

Let K be an abstract simplicial complex. The first barycentric subdivision of K, denoted by $\operatorname{sd}\mathsf{K}$, is the abstract simplicial complex with the vertex set

$V(\mathrm{sd}\,\mathsf{K}) = \mathsf{K}$ and the family of simplices consisting of all chains, where K is ordered by inclusion. In [4] p. 81, J. Matoušek and G. M. Ziegler pointed out that if a graph G contains no 4-cycles, there is a $\mathbf{Z}_2$-retraction from $\mathrm{sd}\,\mathsf{B}(G)$ to a 1-dimensional subcomplex L of $\mathrm{sd}\,\mathsf{B}(G)$, where L consists of the vertex set

$$V(\mathsf{L}) := \{A' \uplus A'' \mid A' \uplus A'' \in \mathsf{B}(G), |A'| \leq 1, |A''| \leq 1\}$$

and the family of simplices

$$V(\mathsf{L}) \cup \{(A' \uplus \phi, A' \uplus A''), (\phi \uplus A'', A' \uplus A'') \mid A' \uplus A'' \in \mathsf{B}(G), |A'| = 1, |A''| = 1\}.$$

We notice that $\mathrm{sd}\,\bar{G} = \mathsf{L}$, and hence, $\|\bar{G}\| = \|\mathrm{sd}\,\bar{G}\| = \|\mathsf{L}\|$. Theorem 4.3 shows that $\|\mathsf{L}\|$ is indeed a strong $\mathbf{Z}_2$-deformation retract of $\|\mathsf{B}(G)\|$ if G contains no 4-cycles. The theorem also shows that the converse of this also holds.

As a conclusion, we obtain the following corollary from Theorem 3.1 and 4.3.

COROLLARY 4.4. *Let G be a graph without 4-cycles and k the number of induced cycles of G.*

(1) *If G contains no cycles of odd length, we have* $\|\mathsf{B}(G)\| \simeq \bigvee_k S^1 \amalg \bigvee_k S^1$.

(2) *If G contains at least one cycle of odd length, we have* $\|\mathsf{B}(G)\| \simeq \bigvee_{2k-1} S^1$.

□

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