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ON THE MEAN, GAUSS, THE SECOND GAUSSIAN AND THE SECOND MEAN CURVATURE OF THE HELICOIDAL SURFACES WITH LIGHT-LIKE AXIS IN $\mathbf{R}_1^3$

By

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Abstract. In this paper, the second Gaussian and the second mean curvature of the helicoidal surfaces with light-like axis of type IV^+ is obtained in Minkowski 3-space. In addition, some relations between the mean, Gauss, the second Gaussian and the second mean curvature of the helicoidal surfaces with light-like axis of type IV^+ are given in Minkowski 3-space.

Introduction

Helicoidal surfaces are naturel generalization of rotation surfaces, of which many nice works have been done such as [1, 2, 3, 5, 7, 10, 11].

About helicoidal surfaces in Euclidean 3-space, M. P. do Carmo and M. Dajczer [5] proved that, by using a result of E. Bour [4], there exists a two-parameter family of helicoidal surfaces isometric to a given helicoidal surface. By making use of this parametrization, they found a representation formula for helicoidal surfaces with constant mean curvature.

In 2000, T. Ikawa [10] showed that a generalized helicoid and a rotation surface have an isometric relation by Bour's theorem in Euclidean 3-space. He determined pairs of surfaces with the additional condition that they have the same Gauss map using Bour's theorem. Ikawa [11] classified the spacelike and timelike surfaces as (axis, profile curve)-type in Minkowski 3-space in 2001. He proved an isometric relation between a spacelike (timelike) generalized helicoid and a spacelike (timelike) rotation surface with spacelike (timelike) axis by Bour's theorem. Beneki, Kaimakamis and Papantoniou [2] classified four kinds of helicoidal surface with spacelike, timelike and lightlike axes in 2002. In 2004,

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Kim and Yoon [12] studied some properties about the second Gaussian curvature of ruled surfaces in $\mathbf{R}_1^3$. They classified ruled surfaces in $\mathbf{R}_1^3$ in terms of the second Gaussian curvature, the mean curvature and the Gaussian curvature.

In 2006, the present authors [7] showed that a generalized helicoid and a rotation surface with light-like axis have an isometric relation by Bour's theorem in Minkowski 3-space. We determined pairs of surfaces with light-like axis with an additional condition that they have the same Gauss map by Bour's theorem. A surface $\mathbf{M}$ in a Euclidean 3-space with positive Gaussian curvature K possesses a positive definite second fundamental form II if appropriately orientated. Therefore, the second fundamental form defines a new Riemannian metric on $\mathbf{M}$. In turn, we can consider the Gaussian curvature K_{II} of the second fundamental form which is regarded as a Riemannian metric. If a surface has non-zero Gaussian curvature everywhere, K_{II} can be defined formally and it is the curvature of the Riemannian or pseudo-Riemannian manifold $(\mathbf{M}, II)$.

Since Brioschi's formulas in Euclidean ([9], p. 504) and Minkowski 3-spaces are the same, we are able to compute K_{II} of $\mathbf{M}$ by replacing the components of the first fundamental form E, F, G by the components of the second fundamental form L, M, N respectively in Brioschi's formula. Consequently, the second Gaussian curvature K_{II} of $\mathbf{M}$ is given by

$$(1) \qquad K_{II} = \frac{1}{\Delta^2}\left[\det\begin{pmatrix} -\frac{1}{2}L_{uu} + M_{uv} - \frac{1}{2}N_{vv} & \frac{1}{2}L_u & M_u - \frac{1}{2}L_v \\ M_v - \frac{1}{2}N_u & L & M \\ \frac{1}{2}N_v & M & N \end{pmatrix} - \det\begin{pmatrix} 0 & \frac{1}{2}L_v & \frac{1}{2}N_u \\ \frac{1}{2}L_v & L & M \\ \frac{1}{2}N_u & M & N \end{pmatrix}\right]$$

where $\Delta = |\det II|$. It is well known that a minimal surface has vanishing second Gaussian curvature but that a surface with vanishing second Gaussian curvature need not be minimal [12].

For study of the second Gaussian curvature, Koutroufiotis [14] has shown that a closed ovaloid is a sphere if $K_{II} = cK$ for some constant c or if $K_{II} = \sqrt{K}$ in 1974. Koufogiorgos and Hasanis [13] proved that the sphere is the only closed ovaloid satisfying $K_{II} = H$ in 1977. Also Kühnel [15] studied surfaces of revolution satisfying $K_{II} = H$. Baikoussis and Koufogiorgos [1] proved that the helicoidal surfaces satisfying $K_{II} = H$ are locally characterized by the constancy of the ratio of the principal curvatures. Blair and Koufogiorgos [3] investigated a non-developable ruled surface in $\mathbf{E}^3$ such that $aK_{II} + bH$, $2a + b \neq 0$, is a constant along each ruling in 1992. Also, they proved that a ruled surface with vanishing second Gaussian curvature is a helicoid.

On the other hand, in 2005, Dillen and Sodsiri [6] studied ruled linear Weingarten surfaces in Minkowski 3-space such that the linear combination $aK_{II} + bH + cH_{II} + dK$ is constant along each ruling for some constants a, b, c, d with $a^2 + b^2 + c^2 \neq 0$. They used the second mean curvature formula as follow

$$H_{II} = H - \frac{1}{2\sqrt{|\det II|}} \sum_{i,j} \frac{\partial}{\partial u^i}\left(\sqrt{|\det II|}\hbar_{ij}\frac{\partial}{\partial u^j}(\ln \sqrt{|K|})\right) \tag{2}$$

where $i, j \in \{1, 2\}$, $\hbar_{ij}$ are the coefficients of the second fundamental form II, u^1 and u^2 stand for u, v and H, K are the mean and Gauss curvatures respectively. Observe that the formula of H_{II} is similar to the one in Euclidean case, cf. [8] and [16].

In the present paper, we classified for the helicoidal surfaces with light-like axis of *type* IV^+ ((L, S)-*type* in [7]) in Minkowski 3-space satisfying the general conditions of curvatures in section 2 and we summarized as follow

Table 1. General relations between H, K, H_{II} and K_{II}

relation	function	condition
$H = \Phi(u)K$	$\Phi(u) = \frac{(u^3\varphi'' - 2u^2\varphi' + a^2)D^{1/2}}{2(2u^3\varphi'' + a^2)}$	$a^2 \neq -2u^3\varphi''$ $D \neq 0$
$K_{II} = \Psi(u)H$	$\Psi(u) = \frac{4D\mu(u)}{(2u^3\varphi'' - a^2)(-u^3\varphi'' + 2u^2\varphi' - a^2)}$	$a^2 \neq 2u^3\varphi''$ $a^2 \neq 2u^2\varphi' - u^3\varphi''$ $D \neq 0$
$K_{II} = \Upsilon(u)K$	$\Upsilon(u) = \frac{-2D^{3/2}\mu(u)}{4u^6\varphi''^2 - a^4}$	$a^2 \neq \lvert 2u^3\varphi''\rvert$ $D \neq 0$
$H_{II} = \Omega(u) + H$	$\Omega(u) = \frac{-2^4\xi(u)}{D^{7/2}\sqrt{\lvert -2u^3\varphi'' - a^2\rvert}}$	$a^2 \neq \lvert 2u^3\varphi''\rvert$ $D \neq 0$
$H_{II} = \Omega(u) + \Phi(u)K$	$\Omega(u) = \frac{-2^4\xi(u)}{D^{7/2}\sqrt{\lvert -2u^3\varphi'' - a^2\rvert}}$, $\Phi(u) = \frac{(u^3\varphi'' - 2u^2\varphi' + a^2)D^{1/2}}{2(2u^3\varphi'' + a^2)}$	$a^2 \neq \lvert 2u^3\varphi''\rvert$ $D \neq 0$
$H_{II} = \Omega(u) + \Lambda(u)K_{II}$	$\Omega(u) = \frac{-2^4\xi(u)}{D^{7/2}\sqrt{\lvert -2u^3\varphi'' - a^2\rvert}}$, $\Lambda(u) = \frac{-(u^3\varphi'' - 2u^2\varphi' + a^2)^2(2u^3\varphi'' - a^2)}{2^3D^{1/2}(2u^3\varphi'' + a^2)\mu(u)}$	$a^2 \neq \lvert 2u^3\varphi''\rvert$ $D \neq 0$

where $\varphi = \varphi(u)$ is a function on the profile curve, $D = 4(a^2 - 4u^2\varphi')$, $a \in \mathbf{R}\backslash\{0\}$ which is the pitch of the helicoidal surface ot type IV^+,

$$\mu(u) = \{a^6u^4\varphi''\varphi'''' + 2^{-1}a^8u\varphi'''' + (-2u^3-1)4a^4u^3\varphi'\varphi''\varphi'''' + (-2u^3-1)2a^6\varphi'\varphi''''$$
$$+ (u^3+2)2^4a^2u^5\varphi'^2\varphi''\varphi'''' + (u^3+2)2^3a^4u^2\varphi'^2\varphi'''' - 2^6u^7\varphi'^3\varphi''\varphi''''$$
$$- 2^5a^2u^4\varphi'^3\varphi'''' + (5-4u^2)a^6u^3\varphi''\varphi''' + a^8\varphi''' + 6a^4u^6\varphi''^2\varphi'''$$
$$- 2^5a^2u^8\varphi'^3\varphi''' + (-2^{-1}-2a^4u^2)a^2u^2\varphi'\varphi''' + (2-u^2)2^6a^2u^7\varphi'^2\varphi''\varphi'''$$
$$+ (5+2u^2)2^3a^4u^4\varphi'^2\varphi''' + (-2-3a^4+2^2a^4u^2)2^3u^5\varphi'\varphi''\varphi'''$$
$$- 3\cdot 2^4a^2u^8\varphi'\varphi''^2\varphi''' + 3\cdot 2^5u^{10}\varphi'^2\varphi''^2\varphi''' + (6a^2+3-2^3u^2)2a^2u^5\varphi''^3$$
$$+ (2-2^{-3}u^2)3a^6u^2\varphi''^2 - 3\cdot 2^4u^{10}\varphi'\varphi''^4 + (-6a^2u^7-3a^2+2^3u^9)2^3\varphi'\varphi''^3$$
$$+ (-12-13\cdot 2^{-1}u^2)a^4u^4\varphi'\varphi''^2 + 12u^8\varphi''^4 + (3-4u^2)2a^6u\varphi'\varphi''$$
$$+ (-3a^2+2u^2)2^4u^6\varphi'^2\varphi''^2 + (-3+4u^2)2^3a^4u^3\varphi'^2\varphi''$$
$$- 2^5a^6u^2\varphi'^2 - 4a^4u^4\varphi'^3\}$$

and

$$\xi(u) = \{-2^2u^6\varphi'\varphi''\varphi'''' + a^2u^4\varphi''\varphi'''' - 2^3a^2u^3\varphi''\varphi''' + a^2u^4\varphi'''^2 - 7\cdot 2^2u^6\varphi''^2\varphi'''$$
$$- 2^2u^6\varphi'\varphi'''^2 - (3\cdot 2^7a^2+5)2^4u^5\varphi'\varphi''\varphi''' - 2^3a^2u^2\varphi'\varphi''' + 3\cdot 2^{13}u^7\varphi'^2\varphi''\varphi'''$$
$$+ 3\cdot 2^{10}a^2u^6\varphi''^2\varphi''' + 3\cdot 2^{12}u^8\varphi'\varphi''^2\varphi''' - 11a^2u^2\varphi''^2 - 33\cdot 2^2u^4\varphi'\varphi''^2$$
$$- 19\cdot 2^2u^5\varphi''^3 - 2^4a^2u\varphi'\varphi'' + 15\cdot 2^{11}a^2u^4\varphi'\varphi''^2 + 9\cdot 2^{13}\varphi'^2\varphi''^2$$
$$+ 3\cdot 2^{14}u^6\varphi'^2\varphi''^2 + 3\cdot 2^{14}a^2u^3\varphi'^2\varphi'' + 3\cdot 2^{14}u^7\varphi'^2\varphi''^2 + 3\cdot 2^{10}a^2u^5\varphi''^3$$
$$+ 9\cdot 2^{12}u^9\varphi'\varphi''^3 + 3\cdot 2^{14}u^7\varphi'\varphi''^3 - 3\cdot 2^{13}u^8\varphi'\varphi''^3\}.$$

In section 3, we show some results of curvatures for the helicoidal surfaces with light-like axis of *type* IV^+ in Minkowski 3-space. In addition, we study helicoidal surfaces with light-like axis of *type* IV^+ in Minkowski 3-space such that the linear combination $pH + qK + rH_{II} + sK_{II}$ is constant along each ruling for some non-zero constants p, q, r, s.

1 Preliminaries

Let $\mathbf{R}^3_1$ be a 3-dimensional Minkowski space with natural Lorentzian metric $\langle\,,\rangle = dx^2 + dy^2 - dz^2$. A vector w in $\mathbf{R}^3_1$ is called spacelike (resp. timelike) if $\langle w, w\rangle > 0$ or $w = 0$ (resp. $\langle w, w\rangle < 0$). If $w \neq 0$ satisfies

$\langle w, w \rangle = 0$, then w is called lightlike. A surface in Minkowski 3-space $\mathbf{R}_1^3$ is called a *spacelike* (resp. *timelike*, *degenerate* (lightlike)) if the induced metric on the surface is a positive definite Riemannian (resp. Lorentzian, degenerate) metric.

In the rest of this paper we shall identify a vector (a, b, c) with its transpose $(a, b, c)^t$. Now we define a non degenerate rotation surface and generalized helicoid in $\mathbf{R}_1^3$. For an open interval $I \subset \mathbf{R}$, let $\gamma : I \to \Pi$ be a curve in a plane Π in $\mathbf{R}_1^3$, and let ℓ be a straight line in Π which does not intersect the curve $\gamma = \gamma(u)$. A *rotation surface* in $\mathbf{R}_1^3$ is defined as a non degenerate surface formed by rotating a curve γ around a line ℓ (these are called the *profile curve* and the *axis*, respectively). Suppose that when a profile curve γ rotates around the axis ℓ, it simultaneously displaces parallel to ℓ so that the speed of displacement is proportional to the speed of rotation. Then the resulting surface is called the *generalized helicoid* with axis ℓ and *pitch* a.

We say that a helicoidal surface in $\mathbf{R}_1^3$ is of *type* IV^+ or IV^- if the discriminant $D = EG - F^2$ of the first fundamental form is positive or negative, where E, F, G are the coefficients of the line element of helicoidal surface.

Since $D = 16u^2\varphi' - 4a^2 \neq 0$, if $\varphi > -\frac{a^2}{4u}$ then $H(u, v)$ is spacelike, if $\varphi < -\frac{a^2}{4u}$ then $H(u, v)$ is timelike.

In this work, we assume $D > 0$ and so helicoidal surfaces are spacelike of *type* IV^+. Suppose that the axis of rotation is a lightlike line, or equvalently the line of the plane x_2x_3 spanned by the vector $(0, 1, 1)$. Since the surface is non degenerate, we may assume that the profile curve γ lies in the x_2x_3-plane without loss of generality and its parametrization is given by $\gamma(u) = (0, \varphi(u) + u, \varphi(u) - u)$, where $\varphi(u) + u$ and $\varphi(u) - u$ are differentiable functions on I such that $\varphi(u) + u \neq \varphi(u) - u$ for all $u \in \mathbf{R} \backslash \{0\}$.

If the axis l is lightlike in Minkowski 3-space $\mathbf{R}_1^3$, then we may suppose that l is the line spanned by the vector $(0, 1, 1)$. The semi-orthogonal matrix given as follows is the subgroup of the Lorentzian group that fixes the above vector as invariant

$$A(v) = \begin{pmatrix} 1 & -v & v \\ v & 1 - \frac{v^2}{2} & \frac{v^2}{2} \\ v & -\frac{v^2}{2} & 1 + \frac{v^2}{2} \end{pmatrix} \tag{3}$$

where $\varepsilon = diag(1, 1, -1)$, $A^t \varepsilon A = \varepsilon$ and $\det A = +1$ for $v \in \mathbf{R}$. A helicoidal surface of *type* IV^+ in Minkowski 3-space with the lightlike axis which is spanned by $(0, 1, 1)$, and which has pitch $a \in \mathbf{R} \backslash \{0\}$ is as follows

$$(4)\quad H(u,v)=\begin{pmatrix}1 & -v & v\\ v & 1-\frac{v^2}{2} & \frac{v^2}{2}\\ v & -\frac{v^2}{2} & 1+\frac{v^2}{2}\end{pmatrix}\begin{pmatrix}0\\ \varphi+u\\ \varphi-u\end{pmatrix}+a\begin{pmatrix}0\\ v\\ v\end{pmatrix}=\begin{pmatrix}-2uv\\ \varphi+u-uv^2+av\\ \varphi-u-uv^2+av\end{pmatrix}.$$

$H(u,v)$ reduces to a rotation surface when $a=0$. Therefore, the rotation surface can be parametrized as (in [7])

$$(5)\qquad R(u,v)=(-2uv,\varphi+u-uv^2,\varphi-u-uv^2).$$

In section 2, we give some relations between the H, K, K_{II} and H_{II} to the helicoidal surfaces of *type* IV^+ and then in section 3 we obtain special conditions and relations of these curvatures of the helicoidal surfaces of *type* IV^+ for some differentiable functions $\varphi=\varphi(u)$ on the profile curves in Minkowski 3-space.

2 General Cases of the H, K, H_{II} and K_{II} to the Helicoidal Surfaces of Type IV^+

In this section, we give some theorems between the mean, Gauss, the second mean and the second Gaussian curvature of the helicoidal surfaces with $(0,1,1)$ light-like axis of *type* IV^+ in Minkowski 3-space. We classified these curvatures to general cases. We have following theorems.

THEOREM 2.1. *The mean curvature and the Gauss curvature of the helicoidal surfaces of type* IV^+ *are related as follow equation*

$$(6)\qquad H=\Phi(u)K$$

in Minkowski 3*-space, where* $\Phi(u)=\frac{(u^3\varphi''-2u^2\varphi'+a^2)D^{1/2}}{2(2u^3\varphi''+a^2)}$ *and* $a^2\neq-2u^3\varphi''$, $a\in\mathbf{R}\backslash\{0\}$, $D=4(a^2-4u^2\varphi')\neq0$.

PROOF. Firstly we compute the Eq. (1) for helicoidal surfaces of *type* IV^+. We obtain the following equation

$$(7)\quad K_{II}=\frac{1}{2(|LN|-M^2)^2}\left[-L_{uu}LN+L_{uu}M^2+\frac{1}{2}L_uN_uN-N_uMM_u+\frac{1}{2}L(N_u)^2\right].$$

Differentiating H_u and H_v,

$$H_{uu}=(0,\varphi'',\varphi''),\quad H_{uv}=(-2,-2v,-2v),\quad H_{vv}=(0,-2u,-2u).$$

The coefficients of the first and second fundamental forms of the generalized helicoid of *type* IV^+ are given by

$$E = 4\varphi', \quad F = 2a, \quad G_H = 4u^2,$$
$$L = \frac{4u\varphi''}{D^{1/2}}, \quad M = \frac{-4a}{D^{1/2}}, \quad N = \frac{-8u^2}{D^{1/2}}$$

where $D = 4(a^2 - 4u^2\varphi') \neq 0$. Hence, the Gauss and the mean curvatures are respectively

$$K = \frac{-16(2u^3\varphi'' + a^2)}{D^2} \tag{8}$$

and

$$H = \frac{8(-u^3\varphi'' + 2u^2\varphi' - a^2)}{D^{3/2}} \tag{9}$$

where $a^2 \neq 4u^2\varphi'$. Therefore the relation between H and K is

$$H = \frac{(u^3\varphi'' - 2u^2\varphi' + a^2)D^{1/2}}{2(2u^3\varphi'' + a^2)} K. \tag{10}$$

□

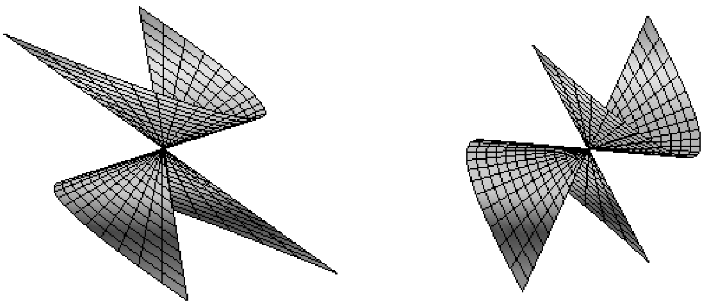

Figure 1.–2. Helicoidal surfaces of *type* IV^+ ($\varphi(u) = 1$ and $\varphi(u) = u$).

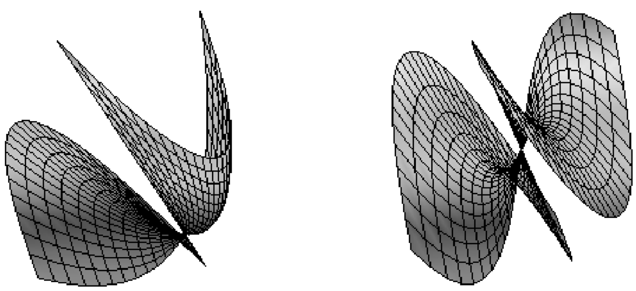

Figure 3.–4. Helicoidal surfaces of *type* IV^+ ($\varphi(u) = u^2 + u$ and $\varphi(u) = u^3$).

EXAMPLE 2.2. Spacelike helicoidal surfaces of *type* IV^+ with $(0,1,1)$ lightlike axis in Minkowski 3-space are in Figures 1–7, where the functions $\varphi(u) = \sum c_i u^i$, $0 \leq i \leq 4$, $c_i \in \mathbf{R}^+$ on profile curves $\gamma(u)$.

THEOREM 2.3. *The second Gaussian curvature and the mean curvature of the helicoidal surfaces of type* IV^+ *are related by the following equation*

$$K_{II} = \Psi(u)H \tag{11}$$

in Minkowski 3*-space, where* $\Psi(u) = \frac{4D\mu(u)}{(2u^3\varphi''-a^2)(-u^3\varphi''+2u^2\varphi'-a^2)}$ *and* $a^2 \neq 2u^2\varphi' - u^3\varphi''$, $a^2 \neq 2u^3\varphi''$, $D = 4(a^2 - 4u^2\varphi') \neq 0$, $\mu(u)$ *is in Table* 1, $a \in \mathbf{R}\backslash\{0\}$.

PROOF. We have the second Gaussian curvature as follow

$$K_{II} = \frac{32\mu(u)}{(2u^3\varphi'' - a^2)D^{1/2}} \tag{12}$$

where $D = 4(a^2 - 4u^2\varphi')$ and $\mu(u)$ is in Table 1. Hence the relation between the second Gaussian curvature and the mean curvature is

$$K_{II} = \frac{4D\mu(u)}{(2u^3\varphi'' - a^2)(-u^3\varphi'' + 2u^2\varphi' - a^2)}H \tag{13}$$

where $2u^3\varphi'' \neq a^2$, $u^3\varphi'' - 2u^2\varphi' + a^2 \neq 0$, $D \neq 0$, $a \in \mathbf{R}\backslash\{0\}$. □

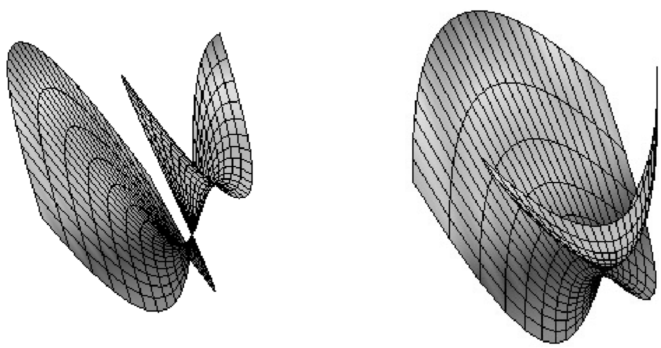

Figure 5.–6. Helicoidal surfaces of *type* IV^+ ($\varphi(u) = u^3 + u^2$ and $\varphi(u) = u^4 + u^3$).

THEOREM 2.4. *The second Gaussian curvature and the Gauss curvature of the helicoidal surfaces of type* IV^+ *are related by the following equation*

$$K_{II} = \Upsilon(u)K \tag{14}$$

in Minkowski 3*-space*, *where* $\Upsilon(u) = \frac{-2D^{3/2}\mu(u)}{4u^6\varphi''^2 - a^4}$ *and* $a^2 \neq |2u^3\varphi''|$, $D = 4(a^2 - 4u^2\varphi') \neq 0$, $\mu(u)$ *is in Table* 1, $a \in \mathbf{R}\backslash\{0\}$.

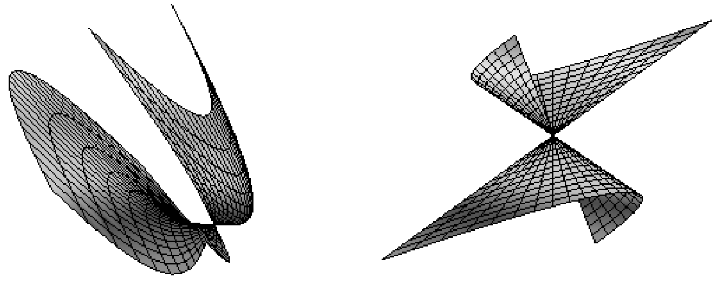

Figure 7.–8. Helicoidal surfaces of *type* IV^+ ($\varphi(u) = u^4 + u$ and $\varphi(u) = -1$).

PROOF. We can easily computate following equation

$$K_{II} = \frac{-2D^{3/2}\mu(u)}{4u^6\varphi''^2 - a^4} K, \tag{15}$$

where $a^2 \neq |2u^3\varphi''|$, $a \in \mathbf{R}\backslash\{0\}$, $\mu(u)$ is in Table 1 and $D \neq 0$. □

THEOREM 2.5. *The second mean curvature and the mean curvature of the helicoidal surfaces of type* IV^+ *are related by the following equation*

$$H_{II} = \Omega(u) + H \tag{16}$$

in Minkowski 3*-space*, *where* $\Omega(u) = \frac{-2^4\xi(u)}{D^{7/2}\sqrt{|-2u^3\varphi'' - a^2|}}$ *and* $a^2 \neq |2u^3\varphi''|$, $D = 4(a^2 - 4u^2\varphi') \neq 0$, $\xi(u)$ *is in Table* 1, $a \in \mathbf{R}\backslash\{0\}$.

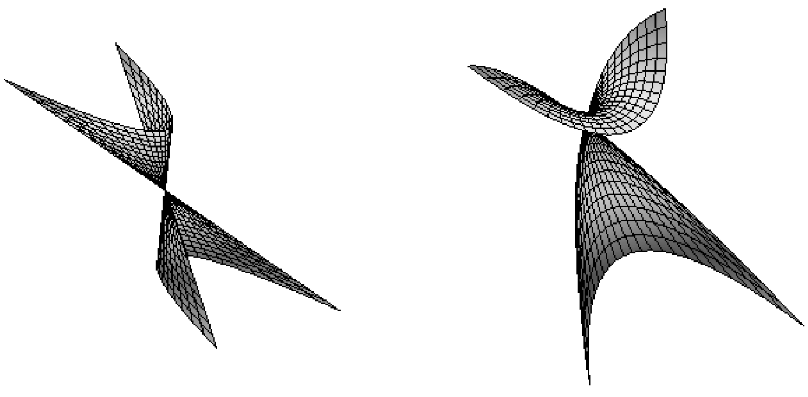

Figure 9.–10. Helicoidal surfaces of *type* IV^+ ($\varphi(u) = -u$ and $\varphi(u) = -u^2 - u$).

PROOF. We can easily computate H_{II} using (2) and have following equation

$$H_{II} = \frac{-2^4 \xi(u)}{D^{7/2}\sqrt{|-2u^3\varphi'' - a^2|}} + H, \tag{17}$$

where $a^2 \neq |2u^3\varphi''|$, $a \in \mathbf{R}\setminus\{0\}$, $\xi(u)$ is in Table 1 and $D \neq 0$. □

THEOREM 2.6. *The second mean curvature and Gauss curvature of the helicoidal surfaces of type IV^+ are related by the following equation*

$$H_{II} = \Omega(u) + \Phi(u)K \tag{18}$$

in Minkowski 3-space, where $\Omega(u) = \frac{-2^4\xi(u)}{D^{7/2}\sqrt{|-2u^3\varphi''-a^2|}}$, $\Phi(u) = \frac{(u^3\varphi''-2u^2\varphi'+a^2)D^{1/2}}{2(2u^3\varphi''+a^2)}$ *and* $a^2 \neq |2u^3\varphi''|$, $D = 4(a^2 - 4u^2\varphi') \neq 0$, $\xi(u)$ *is in Table* 1, $a \in \mathbf{R}\setminus\{0\}$.

PROOF. Using (2) then we have

$$H_{II} = \frac{-2^4 \xi(u)}{D^{7/2}\sqrt{|-2u^3\varphi'' - a^2|}} + \frac{(u^3\varphi'' - 2u^2\varphi' + a^2)D^{1/2}}{2(2u^3\varphi'' + a^2)}K, \tag{19}$$

where $a^2 \neq |2u^3\varphi''|$, $a \in \mathbf{R}\setminus\{0\}$, $\xi(u)$ is in Table 1 and $D \neq 0$. □

THEOREM 2.7. *The second mean curvature and the second Gaussian curvature of the helicoidal surfaces of type IV^+ are related by the following equation*

$$H_{II} = \Omega(u) + \Lambda(u)K_{II} \tag{20}$$

in Minkowski 3-space, where $\Omega(u) = \frac{-2^4\xi(u)}{D^{7/2}\sqrt{|-2u^3\varphi''-a^2|}}$, $\Lambda(u) = \frac{-(u^3\varphi''-2u^2\varphi'+a^2)^2(2u^3\varphi''-a^2)}{2^3D^{1/2}(2u^3\varphi''+a^2)\mu(u)}$ *and* $a^2 \neq |2u^3\varphi''|$, $D = 4(a^2 - 4u^2\varphi') \neq 0$, $\xi(u)$ *is in Table* 1, $a \in \mathbf{R}\setminus\{0\}$.

PROOF. Using (1) and (2) then we have

$$H_{II} = \frac{-2^4 \xi(u)}{D^{7/2}\sqrt{|-2u^3\varphi'' - a^2|}} + \frac{-(u^3\varphi'' - 2u^2\varphi' + a^2)^2(2u^3\varphi'' - a^2)}{2^3D^{1/2}(2u^3\varphi'' + a^2)\mu(u)}K_{II}, \tag{21}$$

where $a^2 \neq |2u^3\varphi''|$, $a \in \mathbf{R}\setminus\{0\}$, $D \neq 0$, $\xi(u)$ and $\mu(u) \neq 0$ are in Table 1. □

THEOREM 2.8. *Let $H(u,v)$ be the helicoidal surfaces of type IV^+, and let p, q, r, s be constants with*

$$p = \frac{D}{-u^3\varphi'' + 2u^2\varphi' - a^2}, \quad q = \frac{D^{3/2}}{-2(2u^3\varphi'' + a^2)},$$

$$r = \frac{D^3}{D^2(-u^3\varphi'' + 2u^2\varphi' - a^2) - 2^4\xi(u)}, \quad s = \frac{2u^3\varphi'' - a^2}{4\mu(u)}.$$

Then

$$pH + qK + rH_{II} + sK_{II} = 2^5 D^{-1/2} \tag{22}$$

where $-u^3\varphi'' + 2u^2\varphi' \neq a^2$, $-2u^3\varphi'' \neq a^2$, $\mu(u) \neq 0$ *and* $\xi(u) \neq 0$ *in Table* 1, $D = 4(a^2 - 4u^2\varphi') \neq 0$.

Proof. Using (8), (9), (12) and (17) we obtain that $pH + qK + rH_{II} + sK_{II}$ is non-zero constant along each ruling. □

Theorem 2.9. *Let* $H(u, v)$ *be the helicoidal surfaces of type* IV^+, *and let* p, q, r, s *be constants and if* $\mu(u) = 0$ *in previous Theorem, then*

$$K_{II} = 0 \tag{23}$$

where $\varphi(u) = c_1 \in \mathbf{R}$, $\xi(u) \neq 0$ *in Table* 1, $D = 4a^2 \neq 0$ *and* $a \in \mathbf{R}\backslash\{0\}$.

Proof. Using (8), (9), (12) and (17) we have

$$\begin{aligned}
&\left(\frac{D\mu(u)}{-u^3\varphi'' + 2u^2\varphi' - a^2}\right)H + \left(\frac{D^{3/2}\mu(u)}{-2(2u^3\varphi'' + a^2)}\right)K \\
&\qquad + \left(\frac{D^3\mu(u)}{D^2(-u^3\varphi'' + 2u^2\varphi' - a^2) - 2^4\xi(u)}\right)H_{II} + \left(\frac{2u^3\varphi'' - a^2}{4}\right)K_{II} \\
&\quad = 2^5\mu(u)D^{-1/2}.
\end{aligned}$$

If $\varphi(u) = c_1 \in \mathbf{R}$, then $\mu(u) = 0$ and $s \neq 0$. Hence the relation $K_{II} = 0$ holds. □

3 Some Results of the H, K, H_{II} and K_{II} to the Helicoidal Surfaces of Type IV^+

In this section, we obtain some results between the mean, Gauss, the second mean and the second Gaussian curvature of the helicoidal surfaces with $(0, 1, 1)$ light-like axis of *type* IV^+ in Minkowski 3-space. Special conditions of curvatures as follow

Table 2. Special relations between H, K, H_{II} and K_{II}

case	function on profile curve	relation	condition
1	$\varphi(u) = c_1$, $c_1 \in \mathbf{R}$	$H = H_{II}$ $H^2 + K = 0$ $K_{II} = 0$	$D_1 \neq 0$ $a \neq 0$
2	$\varphi(u) = c_1 u + c_2$, $c_1 = 1,\ c_2 = 0$	$H = H_{II}$ $H - \frac{(2u^2 - a^2)D^{1/2}}{2a^2} K = 0$, $K_{II} = 0$	$D_2 \neq 0$ $a \neq 0$
3	$\varphi(u) = c_1 u^2 + c_2 u + c_3$, $c_1 = c_2 = 1,\ c_3 = 0$	$H = \phi(u)K$ $K_{II} = \psi(u)H$ $K_{II} = \eta(u)K$ $H_{II} = \theta(u) + H$ $H_{II} = \theta(u) + \phi(u)K$ $H_{II} = \theta(u) + \rho(u)K_{II}$	$D_3 \neq 0$ $a \neq 0$ $a^2 \neq -4u^3$ $a^2 \neq 2u^3 + 2u^2$

where $\phi(u) = \frac{(2u^3+2u^2-a^2)D^{1/2}}{2(4u^3+a^2)}$, $\psi(u) = \frac{\delta(u)}{2^{11}(4u^3+a^2)(2u^3+2u^2-a^2)D}$, $\eta(u) = \frac{\delta(u)}{2^{12}(4u^3+a^2)^2 D^{1/2}}$, $\theta(u) = \frac{2^4\zeta(u)}{D^{7/2}\sqrt{|-4u^3-a^2|}}$, $\rho(u) = \frac{-(-2u^3-2u^2+a^2)^2(4u^3-a^2)}{2^3(4u^3+a^2)\delta(u)D^{1/2}}$, $D_1 = 4a^2$, $D_2 = 4(a^2 - 4u^2)$, $D_3 = 4(-8u^3 - 4u^2 + a^2)$,

$$\begin{aligned}\delta(u) = \{&-6u^{12} + 9u^{11} + (3\cdot 2^6 + 5)2^{-5}u^9 + (3\cdot 2^5 - 65)2^{-6}u^8 + (12 - 19a^2)2^{-1}u^7 \\ &+ (-7a^4 - 3\cdot 2^3)2^{-4}u^6 + [2^{-3} + (-2^8 - 3\cdot 2^5 - 1)2^{-8}a^2]3a^2u^5 - 2a^4u^4 \\ &+ (7\cdot 2^6 + 3a^6)2^{-10}u^3 + 13\cdot 2^{-5}a^6u^2 + 3\cdot 2^{-5}a^2u\}\end{aligned}$$

and

$$\begin{aligned}\zeta(u) = \{&9\cdot 2^{16}u^{10} + 3^3\cdot 2^{15}u^9 + 29\cdot 2^{16}u^8 + 9\cdot 2^{17}u^7 + 3\cdot 2^{16}u^6 \\ &+ (-13 + 3^4\cdot 2^6a^2)2^7u^5 + (-11 + 21\cdot 2^9a^2)3\cdot 2^4u^4 + 3\cdot 2^{15}a^2u^3 \\ &+ (-3a^2 + 2^{15})9\cdot 2^2u^2 + (-a^2 + 3^2\cdot 2^{12})2^5u + 9\cdot 2^{15}\}\end{aligned}$$

in case 3. We classified these curvatures for the special functions $\varphi = \varphi(u)$ on profile curves as following corollaries.

COROLLARY 3.1. *The mean, the second mean, Gauss and the second Gaussian curvatures of the helicoidal surfaces of type* IV^+ *are related as follow equations*

$$H = H_{II}, \quad H^2 + K = 0, \quad K_{II} = 0 \tag{24}$$

in Minkowski 3-*space, where* $\varphi(u) = c_1,\ c_1 \in \mathbf{R},\ a \in \mathbf{R}\backslash\{0\}$.

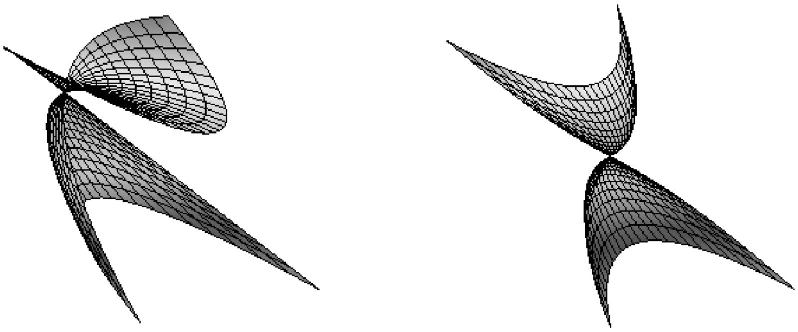

Figure 11.–12. Helicoidal surfaces of *type* IV^+ ($\varphi(u) = -u^2$ and $\varphi(u) = -u^3$).

Proof. If $\varphi(u) = c_1$, $c_1 \in \mathbf{R}$, (see Fig. 1.) then the mean, the second mean, Gauss and the second Gaussian curvatures of the helicoidal surfaces of *type* IV^+ are

$$H = H_{II} = -\frac{1}{a}, \quad K = -\frac{1}{a^2}, \quad K_{II} = 0$$

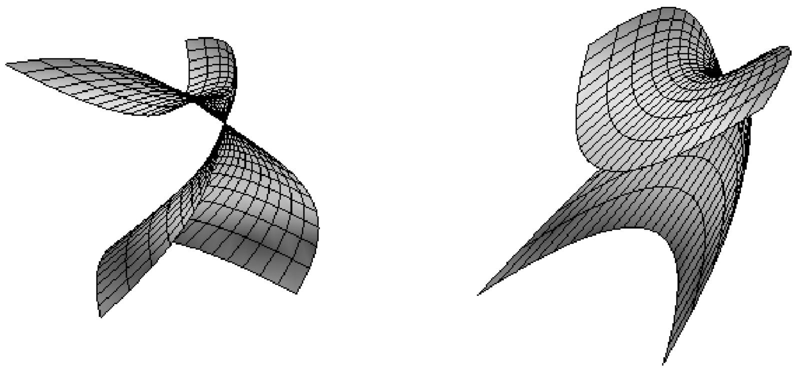

Figure 13.–14. Helicoidal surfaces of *type* IV^+ ($\varphi(u) = -u^4 - u$ and $\varphi(u) = -u^4 - 1$).

where $a \in \mathbf{R} \backslash \{0\}$, $c_1 \in \mathbf{R}$. Therefore we have the following equation

$$H = H_{II}, \quad H^2 + K = 0, \quad K_{II} = 0. \qquad \square$$

Corollary 3.2. *The mean, the second mean, Gauss and the second Gaussian curvatures of the helicoidal surfaces of type* IV^+ *are related as follow equations*

$$H = H_{II}, \quad H - \frac{(2u^2 - a^2)D^{1/2}}{2a^2} K = 0, \quad K_{II} = 0 \tag{25}$$

in Minkowski 3*-space, where* $\varphi(u) = c_1 u + c_2$, $c_1 = 1$, $c_2 = 0$ *and* $a \in \mathbf{R}\backslash\{0\}$, $a^2 \neq 4u^2$.

PROOF. If $\varphi(u) = c_1 u + c_2$, $c_1 = 1$, $c_2 = 0$, (see Fig. 2.) then the curvatures of the helicoidal surfaces of *type* IV^+ as follow

$$H = H_{II} = \frac{16u^2 - 8a^2}{D^{3/2}}, \quad K = \frac{16a^2}{D^2}, \quad K_{II} = 0$$

where $D = 4(a^2 - 4u^2) \neq 0$ and $a \in \mathbf{R}\backslash\{0\}$. Hence

$$H = H_{II}, \quad H - \frac{(2u^2 - a^2)D^{1/2}}{2a^2} K = 0, \quad K_{II} = 0$$

where $\varphi(u) = c_1 u + c_2$, $c_1 = 1$, $c_2 = 0$. □

COROLLARY 3.3. *The mean, the second mean, Gauss and the second Gaussian curvatures of the helicoidal surfaces of type* IV^+ *are related as follow equations*

$$H = \phi(u)K, \quad K_{II} = \psi(u)H, \quad K_{II} = \eta(u)K, \tag{26}$$

and

$$H_{II} = \theta(u) + H, \quad H_{II} = \theta(u) + \phi(u)K, \quad H_{II} = \theta(u) + \rho(u)K_{II} \tag{27}$$

in Minkowski 3*-space, where* $\varphi(u) = c_1 u^2 + c_2 u + c_3$, $c_1 = c_2 = 1$, $c_3 = 0$, $\phi(u) = \frac{(2u^3+2u^2-a^2)D^{1/2}}{2(4u^3+a^2)}$, $\psi(u) = \frac{\delta(u)}{2^{11}(4u^3+a^2)(2u^3+2u^2-a^2)D}$, $\eta(u) = \frac{\delta(u)}{2^{12}(4u^3+a^2)^2 D^{1/2}}$, $\theta(u) = \frac{2^4\zeta(u)}{D^{7/2}\sqrt{|-4u^3-a^2|}}$, $\rho(u) = \frac{-(-2u^3-2u^2+a^2)^2(4u^3-a^2)}{2^3(4u^3+a^2)\delta(u)D^{1/2}}$, $\delta(u)$ *and* $\zeta(u)$ *are in Table* 2, $a \in \mathbf{R}\backslash\{0\}$, $a^2 \neq -4u^3$, $a^2 \neq 2u^3 + 2u^2$, $a^2 \neq 8u^3 + 4u^2$.

PROOF. If $\varphi(u) = c_1 u^2 + c_2 u + c_3$, $c_1 = c_2 = 1$, $c_3 = 0$ (see Fig. 3.) then we compute the mean, the second mean, Gauss and second Gaussian curvature respectively

$$H = \frac{8(2u^3 + 2u^2 - a^2)}{D^{3/2}}, \tag{28}$$

$$K = \frac{16(4u^3 + a^2)}{D^2}, \tag{29}$$

$$K_{II} = \frac{\delta(u)}{2^8(4u^3 + a^2)D^{5/2}} \tag{30}$$

and

$$(31) \qquad H_{II} = \frac{8(2u^3 + 2u^2 - a^2)}{D^{3/2}} - \frac{2^4\zeta(u)}{D^{7/2}\sqrt{|-4u^3 - a^2|}}$$

where $D = 4(-8u^3 - 4u^2 + a^2) \neq 0$, $a^2 \neq -4u^3$, $a \in \mathbf{R}\backslash\{0\}$, $\delta(u)$ and $\zeta(u)$ are in Table 2. Therefore the relations of these curvatures as follows

$$H = \frac{(2u^3 + 2u^2 - a^2)D^{1/2}}{2(4u^3 + a^2)}K,$$

$$K_{II} = \frac{\delta(u)}{2^{11}(4u^3 + a^2)(2u^3 + 2u^2 - a^2)D}H,$$

$$K_{II} = \frac{\delta(u)}{2^{12}(4u^3 + a^2)^2 D^{1/2}}K,$$

$$H_{II} = -\frac{2^4\zeta(u)}{D^{7/2}\sqrt{|-4u^3 - a^2|}} + H,$$

$$H_{II} = -\frac{2^4\zeta(u)}{D^{7/2}\sqrt{|-4u^3 - a^2|}} + \frac{(2u^3 + 2u^2 - a^2)D^{1/2}}{2(4u^3 + a^2)}K$$

and

$$H_{II} = -\frac{2^4\zeta(u)}{D^{7/2}\sqrt{|-4u^3 - a^2|}} + \frac{-(-2u^3 - 2u^2 + a^2)^2(4u^3 - a^2)}{2^3(4u^3 + a^2)\delta(u)D^{1/2}}K_{II}$$

where $\varphi(u) = c_1u^2 + c_2u + c_3$, $c_1 = c_2 = 1$, $c_3 = 0$. □

Example 3.4. Spacelike helicoidal surfaces of *type* IV^+ with $(0, 1, 1)$ lightlike axis in Minkowski 3-space are in Figures 8–14, where the functions $\varphi(u) = -\sum c_i u^i$, $0 \leq i \leq 4$, $c_i \in \mathbf{R}^+$ on profile curves $\gamma(u)$.

Corollary 3.5. *If* $H(u, v)$ *the helicoidal surfaces of type* IV^+ *such that* $H = K$, *where* $\varphi(u) = \mp\frac{-\sqrt{u^2+4}u^3 \mp 4\sqrt{u^2+4}u + u^4 \mp 6u^2 + 6a^2}{24u} + c$ *function on the profile curve* $\gamma(u)$, $c \in \mathbf{R}$, $u \in \mathbf{R}\backslash\{0\}$.

Proof. Using (8) and (9) we have (see Fig. 15.–16.) as follow equation

$$2(a^2 - 4u^2\varphi')(-u^3\varphi'' + 2u^2\varphi' - a^2) + (2u^3\varphi'' + a^2) = 0.$$

Then, it reduces to

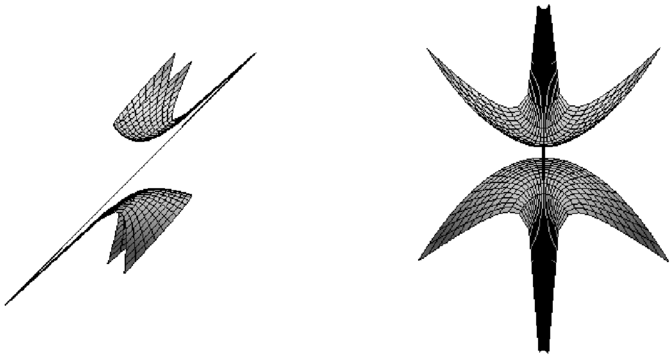

Figure 15.–16. Helicoidal surfaces of *type* IV^+ $(\varphi(u) = -\frac{-\sqrt{u^2+4}u^3-4\sqrt{u^2+4}u+u^4+6u^2+6a^2}{24u})$.

$$2(a^2-1)u^3\varphi'' + 8u^5\varphi'\varphi'' + 2u^4\varphi'^2 + 12a^2u^2\varphi' + a^2(1-2a^2) = 0.$$

Therefore, $\varphi(u)$ can be seen easily from the solution of this equation. □

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