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ON THE ESTIMATE OF THE ARITHMETIC GENUS FOR NORMAL TWO-DIMENSIONAL SINGULARITIES ON DOUBLE COVERINGS

By

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Abstract. In this paper, we deal with normal two-dimensional singularities with multiplicity two. We call such a singularity double point. The purpose of this paper is to give an estimate of the arithmetic genus for double points in terms of Horikawa's canonical resolution and a p_a-formula for some class of double points. It is known that, by using the data obtained from the canonical resolution, the geometric genus for double points is formulated, and rational double points and elliptic double points are characterized. We give a characterization of double points with the arithmetic genus two.

§ 1. Introduction

Let (S,p) be a germ of normal two-dimensional variety over $\boldsymbol{C}$ at a point p. We call it a normal two-dimensional singularity. Let $\psi : (\tilde{S}, A) \to (S,p)$ be a resolution with the exceptional set A. In his famous paper, M. Artin [2] introduced an important divisor, which is called fundamental cycle: $Z = \min\{C \mid \text{positive cycle on } \tilde{S}, C \cdot A_i \leq 0 \text{ for any } i\}$, where $A = \bigcup A_i$ is the decomposition to irreducible components of the exceptional set A, and showed that $p_a(Z) = 0$ if and only if $R^1\psi_*\mathcal{O}_{\tilde{S}} = 0$. Next, Wagreich [10] introduced two numerical invariants for two-dimensional singularity, which are called geometric genus: $p_g = \dim R^1\psi_*\mathcal{O}_{\tilde{S}}$, and arithmetic genus: $p_a = \max\{p_a(C) \mid C \text{ is positive cycle on } \tilde{S}\}$, where $p_a(C)$ is the virtual genus of C. He found a relation of the three invariants: $0 \leq p_a(Z) \leq p_a \leq p_g$. We note that these invariants are independent of resolution. He also studied the class of elliptic singularity, which is defined by $p_a = 1$, the condition of which is equivalent to the condition that

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$p_a(Z) = 1$. Since it is known that fundamental cycle is obtained by using the computation sequence, we can easily compute $p_a(Z)$ from the weighted dual graph of the exceptional set of a resolution. However, in the case of $p_a \geq 2$, in general, we have $p_a(Z) < p_a$, and since it is very difficult to compute arithmetic genus, on the class of singularity with more large arithmetic genus, useful theory is not developed yet.

In this paper, we discuss normal two-dimensional singularities with multiplicity two, which is called double points. We resolve a double point by Horikawa's method [4, 5, 6]. The resolution is called canonical resolution. E. Horikawa (see [4], Lemma 6) proved that $p_g = (1/2)\sum_{i=0}^{r} \gamma_i(\gamma_i + 1)$, where the invariants γ_i are easily obtained by applying Horikawa's canonical resolution. From this, we have the characterization of the rational double points as the class satisfying $\gamma_i = 0$ for all i. In [9], M. Tomari showed certain lower estimate of p_a in terms of γ_i. He used this to characterize the class of $p_a = 1$ by γ_i. However, in general, p_a is not a function of γ_i. In this paper, we can give an upper estimate of the difference of the arithmetic genus and the virtual genus of a divisor which is easily computed, by the multiplicity of the branch locus of double point (Corollary C in Section 4). This estimate is the best as the linear bound. Further we give a p_a-formula for some class of double points (Theorem D in Section 4), and a characterization of double points with $p_a = 2$ in terms of γ_i (Corollary E in Section 4). We can find general studies on double points in [7, 9].

We explain the content of each section.

Section 2 is devoted to a review of Horikawa's canonical resolution of double points. We remark that a double point is written as $(\{z^2 = g(x, y)\}, o)$. First we regard $\{z^2 = g(x, y)\}$ as a surface contained in a trivial line bundle over (x, y)-plane, which is the double covering with branch locus $\{g(x, y) = 0\}$ in the base space. We obtain a base space by blowing-up at an appropriate singularity of the branch locus, and according to a rule, we define a line bundle over the base space. We can define a normal surface, which is double covering over the base space, in the line bundle, and define a birational proper mapping of the surface onto the original surface. If there exists a singularity of the branch locus of the double covering, we repeat this process. After finite processes, we obtain a non-singular surface in a line bundle. Then the composition of these mappings gives a resolution.

In Section 3, we give several representations of the arithmetic genus of double points by using the canonical resolution. Roughly speaking, the arithmetic genus is determined by the information of the branch locus. Let σ be the associated covering transformation of the resolution space of order two. We show that the arithmetic genus equals the maximum of the virtual genus of σ-invariant effective cycles, and we represent $p_a - p_a(Y_{S,p})$ by the invariants Γ^{odd} computed from γ_i and $V^{\min}$ combinatorially determined from cycles on the base space (Theorem A). Here $Y_{S,p}$ is the cycle on the resolution space, which is determined from the canonical divisor, and $p_a(Y_{S,p})$ is computed from γ_i.

In Sections 4 and 5, using the representation obtained in Section 3, we will give a detailed estimate of $p_a - p_a(Y_{S,p})$ (Theorem B). From Theorem B, we obtain an estimate $p_a - p_a(Y_{S,p}) \leq (1/8)m_1$, where m_1 is the multiplicity of the branch locus at p (Corollary C). Furthermore we give an example such that the equality holds in the above inequality. In such a sense, we give the best estimate. A lower estimate of the arithmetic genus due to M. Tomari is written as $p_a(Y_{S,p}) \leq p_a$ in our terminology (see [9], Lemma 1). From our upper estimate and Tomari's lower estimate, under the assumption that $m_1 \leq 8$, we give a p_a-formula: $p_a = p_a(Y_{S,p})$ (Theorem D). Since we can easily compute $p_a(Y_{S,p})$ from the relation: $p_a(Y_{S,p}) = \sum_{\gamma_i:\mathrm{even}} \gamma_i^2/4 + \sum_{\gamma_i:\mathrm{odd}} (\gamma_i^2 - 1)/4 + 1$, it seems to be useful. As a corollary of Theorem D, we obtain the characterization of double points with $p_a = 2$ (Corollary E). Using this characterization, we can classify the weighted dual graph of the minimal resolution of the double points with $p_a = 2$ [8]. The essential part of the proof of Theorem B is given in Section 5.

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§ 2. Horikawa's Canonical Resolution

In this section, we shall review Horikawa's canonical resolution (see [4], Section 2, [5], Section 3 and [6], § 7.4). The resolution is applied for double points. Let (S_1, p_1) be a double point. We may assume that (S_1, p_1) is represented as follows:

$$S_1 = \{(x, y, z) \in F_1 \mid z^2 = g(x, y)\} \quad \text{and} \quad p_1 = (0, 0, 0)$$

(See [1]). Further we may assume that there exists no singularity on S_1 except p_1. Here F_1 is a trivial line bundle over an open neighborhood U_1 of the origin p_1 in $\boldsymbol{C}^2((x, y)$-plane), and $g : U_1 \to \boldsymbol{C}$ is a reduced holomorphic function with $g(0,0) = 0$, and z is the fiber coordinate of F_1. Let ψ_1 be the natural projection of S_1 onto U_1 and B_1 be the branch locus $\{g(x, y) = 0\}$ of the double covering S_1. Then the singularities of the surface S_1 corresponds to those of the branch locus B_1.

From now on, we use the following notation. For a real number a, $[a] = \max\{n \in \boldsymbol{Z} \mid a \geq n\}$, and for an integral divisor D, we denote by $[D]$ a line bundle associated to the divisor D.

We construct the first step of the canonical resolution. Let $\pi_1 : U_2 \to U_1$ be the blowing-up of U_1 at p_1. We define a line bundle F_2 with the base space U_2 and a reduced divisor B_2 on U_2 as follows:

$$F_2 = \pi_1^*(F_1) - \left[\frac{m_1}{2}\right][E],$$

$$B_2 = \pi_1^*(B_1) - 2\left[\frac{m_1}{2}\right]E.$$

Here m_1 is the multiplicity of B_1 at p_1 and E the exceptional curve of π_1. Then, from the isomorphism $2F_1 \simeq [B_1]$, we obtain the isomorphism $2F_2 \simeq [B_2]$.

We represent the divisor B_2 by local equations g_{2k}, where g_{2k} is a holomorphic function on U_{2k} and $\{U_{2k}\}$ is an open covering of U_2. Moreover we may assume that F_2 is isomorphic to a trivial line bundle F_{2k} over U_{2k}. Then we can define the double covering S_{2k} over U_{2k} as follows:

$$S_{2k} = \{(x_{2k}, y_{2k}, z_{2k}) \in F_{2k} \mid z_{2k}^2 = g_{2k}(x_{2k}, y_{2k})\}.$$

Since $2F_2 \simeq [B_2]$, gluing S_{2k}, we obtain a surface S_2 in F_2, which is a double covering over U_2. Since B_1 has no multiple component, B_2 has no multiple component. It follows that S_2 is a normal surface.

We denote by ψ_2 the projection of S_2 onto U_2. Then we can define the natural birational holomorphic mapping $\tilde{\pi}_1$ of S_2 onto S_1 such that $\psi_1 \circ \tilde{\pi}_1 = \pi_1 \circ \psi_2$. In this way, we obtain the quadruplet (B_2, U_2, S_2, F_2) from the quadruplet (B_1, U_1, S_1, F_1).

If B_2 has a singularity, we choose one of the singularities of B_2 as the center of the blowing-up for the second step. Since the singularity p_2 is a double point, by the same way as above, the quadruplet (B_3, U_3, S_3, F_3) is obtained from (B_2, U_2, S_2, F_2).

After we applied the above operation of finite times (r times), the branch locus B_{r+1} becomes to non-singular and we obtain the following commutative diagram.

DIAGRAM 2.1.

$$\begin{array}{ccccccc}
F_1 & & F_2 & & & & F_{r+1} \\
\cup & & \cup & & & & \cup \\
S_1 & \xleftarrow{\tilde{\pi}_1} & S_2 & \xleftarrow{\tilde{\pi}_2} & \cdots & \xleftarrow{\tilde{\pi}_r} & S_{r+1} \\
\downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle \psi_2} & & & & \downarrow{\scriptstyle \psi_{r+1}} \\
U_1 & \xleftarrow{\pi_1} & U_2 & \xleftarrow{\pi_2} & \cdots & \xleftarrow{\pi_r} & U_{r+1} \\
\cup & & \cup & & & & \cup \\
B_1 & & B_2 & & & & B_{r+1}
\end{array}$$

We denote by $\tilde{\pi}$ the composition mapping of the birational mappings π_i, $1 \leq i \leq r$, and by p_i the singularity of the branch locus B_i which is the center of the blowing-up of π_i.

Since B_{r+1} is a non-singular curve, S_{r+1} is a non-singular surface. The birational morphism $\tilde{\pi} : S_{r+1} \to S_1$ is a resolution of the singularity p_1. We have an ambiguity for the way of choosing the center of the blowing-up which induces the each step in the process of the resolution. However the resolution space S_{r+1} is independent of the way of choosing the center. The resolution is called the canonical resolution.

When we resolve a double point (S, p) in the above way, we say that *we desingularize* (S, p) *by using the canonical method.*

§3. Representation of the Arithmetic Genus of Double Points

In this section, we give several representations of the arithmetic genus for the double points (Theorem A). Our results show that the arithmetic genus for double points is determined by the divisors on the exceptional set of the base space of the canonical resolution.

Let (S, p) be a double point. We desingularize (S, p) by using the canonical method. We obtain Diagram 2.1. We introduce several notations by using Diagram 2.1. We define divisors $E_j^{(i)}$ and $Y_j^{(i)}$ on U_i as follows:

$$E_j^{(i)} = \begin{cases} \pi_j^{-1}(p_j) & (i = j+1) \\ \text{the proper transformation of } E_j^{(j+1)} & \\ \text{into } U_i \text{ through } \pi_{j+1} \circ \cdots \circ \pi_{i-1} & (i > j+1) \end{cases}$$

and

$$Y_j^{(i)} = (\pi_j \circ \cdots \circ \pi_{i-1})^{-1}(p_j)$$

for $1 \leq j < i \leq r+1$. We denote by $E^{(i)}$ the exceptional set $\bigcup_{1 \leq j \leq i-1} E_j^{(i)}$ of U_i, and define a divisor $Y_i = \psi_{r+1}^*(Y_i^{(r+1)})$ for $1 \leq i \leq r$. Then the divisor Y_i coincides with *maximal ideal cycle* for (S_i, p_i) in this resolution [9, 11]. We can easily see that the relations

$$Y_i \cdot Y_j = -2\delta_{i,j} \quad \text{and} \quad Y_i^{(r+1)} \cdot Y_j^{(r+1)} = -\delta_{i,j} \quad 1 \leq i, j \leq r \tag{3.1}$$

hold, where $\delta_{i,j}$ is Kronecker's delta. We denote by m_i the multiplicity at p_i of the branch locus B_i for $1 \leq i \leq r$. We define the invariants γ_i as follows:

$$\gamma_i = \left[\frac{m_i}{2}\right] - 1 \quad 1 \leq i \leq r. \tag{3.2}$$

In the following proposition, when D is a cycle on U_{r+1}, we represent the divisor $\psi_{r+1}^*(D)$ by using Y_i, $1 \leq i \leq r$.

PROPOSITION 3.1. *Let D be a $\boldsymbol{Q}$-divisor with support in the exceptional set of U_{r+1}. Then the relation*

$$\psi_{r+1}^*(D) = -\sum_{1 \leq i \leq r} (D \cdot Y_i^{(r+1)}) Y_i \tag{3.3}$$

holds.

PROOF. We can easily see the equality $\sum \boldsymbol{Q}E_j^{(r+1)} = \sum \boldsymbol{Q}Y_j^{(r+1)}$. By the relation (3.1), the equality $D = -\sum_{1 \leq i \leq r}(D \cdot Y_i^{(r+1)}) Y_i^{(r+1)}$ holds. By taking ψ_{r+1}^* of this relation, we obtain the equality (3.3). Q.E.D.

We shall represent the canonical divisor $K_{S_{r+1}}$ on S_{r+1} by using Y_i, $1 \leq i \leq r$. By Hurwitz's formula, $K_{S_{r+1}} = \psi_{r+1}^*(K_{U_{r+1}} + F_{r+1})$ holds. The equality (3.3) implies the relation

$$K_{S_{r+1}} = -\sum_{1 \leq j \leq r} ((K_{U_{r+1}} + F_{r+1}) \cdot Y_j^{(r+1)}) Y_j. \tag{3.4}$$

Combing the relations $K_{U_{i+1}} = \pi_i^*(K_{U_i}) + E_i^{(i+1)}$ and $F_{i+1} = \pi_i^*(F_i) - [m_i/2]E_i^{(i+1)}$, we have

$$K_{U_{i+1}} + F_{i+1} = \pi_i^*(K_{U_i} + F_i) - \gamma_i E_i^{(i+1)} \tag{3.5}$$

for $1 \leq i \leq r$. Since the relation $E_i^{(i+1)} \cdot Y_j^{(i+1)} = 0$ holds for $j < i$, from (3.5), we obtain

$$(K_{U_{r+1}} + F_{r+1}) \cdot Y_j^{(r+1)} = -\gamma_j E_j^{(j+1)} \cdot Y_j^{(j+1)} = \gamma_j. \tag{3.6}$$

From (3.4) and (3.6), we obtain

$$K_{S_{r+1}} = -\sum_{1 \leq j \leq r} \gamma_j Y_j. \tag{3.7}$$

In the following proposition, we compute the virtual genus of $\psi_{r+1}^*(D)$, where D is the same divisor as in Proposition 3.1.

We define the virtual genus of a $\boldsymbol{Q}$-divisor E as follows:

$$p_a(E) = \frac{1}{2}(E \cdot E + E \cdot K_{S_{r+1}}) + 1.$$

Proposition 3.2. *Let D be a $\boldsymbol{Q}$-divisor with support in the exceptional set of U_{r+1}. Then the equality*

$$p_a(\psi_{r+1}^*(D)) = -\frac{1}{4}\sum_{1\le i\le r}(2D\cdot Y_i^{(r+1)}+\gamma_i)^2 - \frac{1}{8}K_{S_{r+1}}^2 + 1 \tag{3.8}$$

holds.

Proof. By the definition of the virtual genus, we have

$$\begin{aligned} p_a(\psi_{r+1}^*(D)) &= \frac{1}{2}(\psi_{r+1}^*(D)^2 + \psi_{r+1}^*(D)\cdot K_{S_{r+1}}) + 1 \\ &= \frac{1}{2}\left(\psi_{r+1}^*(D) + \frac{1}{2}K_{S_{r+1}}\right)^2 - \frac{1}{8}K_{S_{r+1}}^2 + 1. \end{aligned} \tag{3.9}$$

By putting (3.3) and (3.7) into (3.9), we obtain

$$p_a(\psi_{r+1}^*(D)) = \frac{1}{2}\left(-\sum_{1\le i\le r}(D\cdot Y_i^{(r+1)})Y_i - \frac{1}{2}\sum_{1\le i\le r}\gamma_i Y_i\right)^2 - \frac{1}{8}K_{S_{r+1}}^2 + 1. \tag{3.10}$$

By (3.1) and (3.10), the assertion follows. Q.E.D.

We denote by B_i^{exc} the union of the components of the branch locus B_i contained in the exceptional set of U_i for $2\le i\le r+1$.

Definition 3.3. We denote by $\mathfrak{P}(\{1,\dots,r\})$ the power set of the set $\{1,\dots,r\}$. We define $\Gamma_{S,p}^{\text{odd}}:\mathfrak{P}(\{1,\dots,r\})\to\{n\in\boldsymbol{Z}\mid n\ge 0\}$ and $V_{S,p}^{\min}:\mathfrak{P}(\{1,\dots,r\})\to\{n\in\boldsymbol{Z}\mid n\ge 0\}$ as follows:

$$\Gamma_{S,p}^{\text{odd}}(W) = \#\{j\in W\mid \gamma_j \text{ is an odd number}\}$$

and

$$V_{S,p}^{\min}(W) = \min\left\{\sum_{j\in W}\delta(D\cdot Y_j^{(r+1)}+\gamma_j)\mid D\in\operatorname{Div}(U_{r+1}), |D|\subset B_{r+1}^{\text{exc}}, D\ge 0\right\}$$

for $W\subset\{1,\dots,r\}$. Here, for $m\in\boldsymbol{Z}$, $\delta(m)=0$ if m is an even number, and $\delta(m)=1$ if m is an odd number. We regard the support of the zero-divisor in $\operatorname{Div}(U_{r+1})$ as the empty set.

Remark. The values $\Gamma^{\text{odd}}(\{1\le j\le r\})$ and $V^{\min}(\{1\le j\le r\})$ are independent of the way of choosing the center of the blowing-up in each step of the process of the canonical resolution. For $W_1, W_2 \subset \{1, \dots, r\}$ with $W_1 \cap W_2 = \phi$, we can easily see the relations

$$(1)\ \Gamma^{\mathrm{odd}}_{S,p}(W_1 \cup W_2) = \Gamma^{\mathrm{odd}}_{S,p}(W_1) + \Gamma^{\mathrm{odd}}_{S,p}(W_2), \tag{3.11}$$

$$(2)\ V^{\min}_{S,p}(W_1 \cup W_2) \geq V^{\min}_{S,p}(W_1) + V^{\min}_{S,p}(W_2). \tag{3.12}$$

DEFINITION 3.4. We define an effective divisor $Y_{S,p}$ on S_{r+1} as follows:

$$Y_{S,p} = \left[-\frac{K_{S_{r+1}}}{2}\right] = \sum_{1\leq i\leq r}\left[\frac{\gamma_i}{2}\right] Y_i. \tag{3.13}$$

From the relations (3.1) and (3.7), we can easily check the equality

$$p_a(Y_{S,p}) = \sum_{\gamma_j:\mathrm{even}} \frac{\gamma_j^2}{4} + \sum_{\gamma_j:\mathrm{odd}} \frac{\gamma_j^2 - 1}{4} + 1. \tag{3.14}$$

For the sake of our inductive arguments, we shall use an equality

$$p_a(S,p) = \max\{p_a(C) \mid C \text{ is } \textit{effective} \text{ cycle on } S_{r+1}\}$$

when $p_a(S,p) \geq 1$. We remark that the equality does not hold for a rational singularity, because $p_a(0) = 1$.

Now we state the main theorem of this section.

THEOREM A. *We assume that (S,p) is non-rational. Let σ be the associated covering transformation of the resolution space of order two. Let $\alpha : \mathrm{Div}(U_{r+1}) \to \mathrm{Div}(U_{r+1})$ be the mapping defined by $\alpha(D) = \sum_{1\leq j\leq r}\lceil (D \cdot Y_j^{(r+1)} + \gamma_j)/2 \rceil Y_j^{(r+1)}$. Then we have the equalities*

(1) $p_a(S,p) = \max\{p_a(D) \mid \sigma^*(D) = D, D$ *is a cycle on* $S_{r+1}, D \geq 0\}$,

(2) $p_a(S,p) = \max\{p_a(\psi^*_{r+1}(\frac{1}{2}D + \alpha(D))) \mid \mathrm{supp}(D) \subset B^{\mathrm{exc}}_{r+1}, D \geq 0\}$,

(3) $p_a(S,p) - p_a(Y_{S,p}) = \frac{1}{4}\Gamma^{\mathrm{odd}}(\{1 \leq j \leq r\}) - \frac{1}{4}V^{\min}(\{1 \leq j \leq r\})$.

PROOF. First we will show the assertion (1). Let D be a positive cycle such that $p_a(D) = p_a(S,p)$. We can take D to be $\mathrm{supp}(D) = \tilde{\pi}^{-1}(p)$. Then there exists a decomposition $D = D_0 + D_1$ to effective two cycles such that $\sigma^*(D_0) = D_0$ and $\sigma^*(D_1) \cdot D_1 \geq 0$. In fact, we can take the maximal cycle of the set of all effective σ-invariant cycles $\leq D$ as D_0. Then there exists no common irreducible component of $\mathrm{supp}(D_1)$ and $\mathrm{supp}(\sigma^*(D_1))$. Hence it follows that $D_1 \cdot \sigma^*(D_1) \geq 0$. To show the assertion (1), it is enough to show $p_a(D_0 + D_1 + \sigma^*(D_1)) = p_a(S,p)$. Since the inequality

$$0 \leq p_a(D_0 + D_1) - p_a(D_0) = p_a(D_1) + D_1 \cdot D_0 - 1$$

holds, the inequality

$$p_a(D_1) + D_1 \cdot D_0 > 0$$

follows. Hence we have

$$\begin{aligned} p_a(D_0 + D_1 + \sigma^*(D_1)) &= p_a(D_0 + D_1) + p_a(\sigma^*(D_1)) + (D_0 + D_1) \cdot \sigma^*(D_1) - 1 \\ &= p_a(D_0 + D_1) + (p_a(D_1) + D_0 \cdot D_1) + D_1 \cdot \sigma^*(D_1) - 1 \\ &\geq p_a(D_0 + D_1) \\ &= p_a(D). \end{aligned}$$

By the definition of $p_a(S, p)$ and the choice of D, we have

$$p_a(D_0 + D_1 + \sigma^*(D_1)) = p_a(D) = p_a(S, p).$$

Next we will show the assertions (2) and (3). From the assertion (1), there exists an effective divisor D on S_{r+1} such that $D = \sigma^*(D)$ and $p_a(D) = p_a(S, p)$. We can write as $D = \psi_{r+1}^*((1/2)D_1 + D_2)$, where D_1 and D_2 are effective divisors on U_{r+1} with support in B_{r+1}^{exc} and $\overline{E^{(r+1)} \setminus B_{r+1}^{\mathrm{exc}}}$ respectively. By (3.8), we obtain

$$p_a(D) = -\frac{1}{4} \sum_{1 \leq i \leq r} (D_1 \cdot Y_i^{(r+1)} + 2D_2 \cdot Y_i^{(r+1)} + \gamma_i)^2 - \frac{1}{8} K_{S_{r+1}}^2 + 1 \tag{3.15}$$

and

$$\begin{aligned} &p_a\left(\psi_{r+1}^*\left(\frac{1}{2}D_1 + \alpha(D_1)\right)\right) \\ &\quad = -\frac{1}{4} \sum_{1 \leq i \leq r} (D_1 \cdot Y_i^{(r+1)} + 2\alpha(D_1) \cdot Y_i^{(r+1)} + \gamma_i)^2 - \frac{1}{8} K_{S_{r+1}}^2 + 1. \end{aligned} \tag{3.16}$$

By the definition of α, the equality

$$D_1 \cdot Y_i^{(r+1)} + 2\alpha(D_1) \cdot Y_i^{(r+1)} + \gamma_i = -\delta(D_1 \cdot Y_i^{(r+1)} + \gamma_i) \tag{3.17}$$

holds for $1 \leq i \leq r$. By the definition of δ, the inequality

$$(D_1 \cdot Y_i^{(r+1)} + 2D_2 \cdot Y_i^{(r+1)} + \gamma_i)^2 \geq \delta(D_1 \cdot Y_i^{(r+1)} + \gamma_i) \tag{3.18}$$

holds for $1 \leq i \leq r$. Putting (3.17) into (3.16), from the relations (3.15) and (3.18), we obtain

$$p_a(D) \leq p_a\left(\psi_{r+1}^*\left(\frac{1}{2}D_1 + \alpha(D_1)\right)\right). \tag{3.19}$$

Now, by the way of the choice of $D_1, (1/2)D_1 + \alpha(D_1)$ is an effective divisor, and $\psi_{r+1}^*((1/2)D_1 + \alpha(D_1))$ is an integral divisor. Hence we have the inequality

$$p_a\left(\psi_{r+1}^*\left(\frac{1}{2}D_1 + \alpha(D_1)\right)\right) \leq p_a(S, p). \tag{3.20}$$

From (3.19) and (3.20), the assertion (2) is obtained. From the assertion (2), (3.16), (3.17) and the definition of $V^{\min}(*)$, we obtain

$$p_a(S, p) = -\frac{1}{4}V^{\min}(\{1 \leq j \leq r\}) - \frac{1}{8}K_{S_{r+1}}^2 + 1. \tag{3.21}$$

We can easily see that

$$p_a(Y_{S,p}) + \frac{1}{4}\Gamma^{\mathrm{odd}}(\{1 \leq j \leq r\}) = -\frac{1}{8}K_{S_{r+1}}^2 + 1. \tag{3.22}$$

By combing (3.21) and (3.22), the assertion (3) is obtained. Q.E.D.

§ 4. Estimate of the Arithmetic Genus of Double Points

The purpose of this section is to give estimates and a formula of the arithmetic genus for double points. We give a detailed estimate of $p_a(S, p) - p_a(Y_{S,p})$ by the data obtained from the process of the canonical resolution of (S, p) (Theorem B). As the corollary, we show that the difference is bounded by one-eighth of the multiplicity m_1 of the branch locus of S at p (Corollary C). Furthermore, we give a p_a-formula in case that (S, p) is a non-rational double point with $m_1 \leq 8$ (Theorem D).

To state Proposition 4.2, we need the following definition. We will introduce a relation between two double points.

Definition 4.1. Let (S, p) be a double point and $\Pi(S, p)$ be the set of all canonical resolutions with the Diagram 2.1. Remark that the difference of each member of $\Pi(S, p)$ is just the numbering of the blowing-ups in the canonical resolutions. For each element of $\Pi(S, p)$, the number of the steps are same, we simply denote it by r. Let $\pi \in \Pi(S, p)$ with Diagram 2.1, and $J \subset \{1, \ldots, r\}$. We denote by $\tilde{B}_i$ the proper transformation of the branch locus $B_1 = \tilde{B}_1$ through $\pi_1 \circ \cdots \circ \pi_{i-1}$ for $2 \leq i \leq r$. We put $\tilde{m}_i = \mathrm{mult}_{p_i} \tilde{B}_i$ for $1 \leq i \leq r$. Then we attach the vector $\mathrm{mult}(\pi, J) = (m_i, \tilde{m}_i)_{i \in J}$, and the matrix $EY(\pi, J) = (E_i^{(s+1)} \cdot Y_j^{(s+1)})_{i,j \in J}$. Here $s = \max J$.

Let (S, p) and (S', p') be double points, and let $\pi \in \Pi(S, p)$ and $\pi' \in \Pi(S', p')$, and $J \subset \{1, \ldots, r\}$ and $J' \subset \{1, \ldots, r'\}$ be as above. Then we say that

(S, p) *with* π *and* J *and* (S', p') *with* π' *and* J' *are numerically equivalent* if there exists a one to one correspondence of the ordered sets $\mu : J \to J'$ such that the equalities $\mu_*(\mathrm{mult}(\pi, J)) = \mathrm{mult}(\pi', J')$, $\mu_*(EY(\pi, J)) = EY(\pi', J')$ hold. Here μ_* means the natural transformation of vectors induced from the map μ for the index. We will write by $(S, p)_{\pi, J} \sim (S', p')_{\pi', J'}$ in this situation.

We can construct a double point by using the process of the canonical resolution of double point.

PROPOSITION 4.2. *Let* $\tilde{\pi} : S_{r+1} \to S$ *be the canonical resolution of a double point* (S, p) *with Diagram* 2.1. *Suppose* Sing $B_{r'+1} \cap \tilde{B}^{\mathrm{exc}}_{r'+1} = \phi$ *for some* $1 \le r' \le r$, *where* Sing $B_{r'+1}$ *is the singular locus of* $B_{r'+1}$ *in* $U_{r'+1}$. *Then there exists a double point* (S', p') *such that the following conditions are satisfied.*

(1) *The number of the steps of the canonical resolution of* (S', p') *equals* r'.

(2) *There exists* $\tilde{\pi}' \in \Pi(S', p')$ *such that* $(S, p)_{\tilde{\pi}, 1 \le j \le r'} \sim (S', p')_{\tilde{\pi}', 1 \le j \le r'}$

(3) $(E'^{(r'+1)} \cdot \tilde{B}'_{r'+1})_q \le 1$ *for any* $q \in U'_{r'+1}$.

Here we attach prime to the notations appeared in the process of the canonical resolution of (S', p').

PROOF. There exists a curve $\tilde{C}_{r'+1}$ on a sufficiently small neighbourhood of $E^{(r'+1)} \subset U_{r'+1}$ such that the following conditions hold.

(a) $\tilde{C}_{r'+1}$ is a non-singular curve.

(b) $\tilde{B}_{r'+1} \cdot E_j^{(r'+1)} = \tilde{C}_{r'+1} \cdot E_j^{(r'+1)}$ for $1 \le j \le r'$.

(c) $(E^{(r'+1)} \cdot \tilde{C}_{r'+1})_q \le 1$ for any $q \in U_{r'+1}$.

In fact, for any non-singular point $p \in E^{(r'+1)}$, there exists a non-singular curve C on a sufficiently small neighbourhood of $E^{(r'+1)}$ passing through p with $C \cdot E^{(r'+1)} = 1$; we can construct $\tilde{C}_{r'+1}$ by taking a union of such curves. We set a reduced curve $\tilde{C}_i$ on a sufficiently small neighbourhood of $E^{(i)}$ as follows:

$$\tilde{C}_i = (\pi_i \circ \cdots \circ \pi_{r'})_* (\tilde{C}_{r'+1}) \tag{4.1}$$

for $1 \le i \le r'$. Then there exists a double covering S' with $\tilde{C}_1$ as its branch locus over an open neighbourhood of p. We put $p' = p$. We will show that (S', p') satisfies the conditions (1), (2) and (3) of the proposition. For $1 \le i \le r'$, the equality

$$\mathrm{mult}_{p_i} \tilde{B}_i = \mathrm{mult}_{p_i} \tilde{C}_i \tag{4.2}$$

holds from the condition (b) and the relation

$$\mathrm{mult}_{p_i} \tilde{C}_i = \tilde{C}_{i+1} \cdot E_i^{(i+1)} = \tilde{C}_{r'+1} \cdot (\pi_{i+1} \circ \cdots \circ \pi_{r'})^* (E_i^{(i+1)}), \tag{4.3}$$

and the same relation for $\tilde{B}$. We set $B_i' = \tilde{C}_i \cup E_i^{\mathrm{exc}}$ for $1 \leq i \leq r'$. Then, from the equality (4.2), we have $m_i = \mathrm{mult}_{p_i} B_i'$ for $1 \leq i \leq r'$. From the assumption of the proposition and the property of the curve $\tilde{C}_i$, we can easily see that $B_{r'+1}'$ is a non-singular curve. Hence we can desingularize (S', p') by partially using the same resolution process as one of (S, p), which has B_i' as the branch locus on the base space U_i at each step. We denote it by $\tilde{\pi}'$. Then the condition (1) is clearly satisfied. We have the relation $m_i = m_i'$ and $\tilde{m}_i = \tilde{m}_i'$, and we can easily see that the condition (2) is satisfied. The condition (3) follows from the condition (c).

Q.E.D.

We divide the singularities of the branch locus into the four types according to the situation of the exceptional set.

DEFINITION 4.3. Let $\tilde{\pi} : S_{r+1} \to S$ be the canonical resolution of a double point (S, p) with Diagram 2.1. Let $q \in U_i$ be a singularity of the branch locus B_i.

(1) If $\mathrm{mult}_q(B_i) = \mathrm{mult}_q(\tilde{B}_i)$, then we say that *$q$ is a singularity of type A.*

(2) If $\mathrm{mult}_q(B_i) = \mathrm{mult}_q(\tilde{B}_i) + 1$ and $\#\{k \mid E_k^{(i)} \ni q\} = 1$, then we say that *$q$ is a singularity of type B.*

(3) If $\mathrm{mult}_q(B_i) = \mathrm{mult}_q(\tilde{B}_i) + 1$ and $\#\{k \mid E_k^{(i)} \ni q\} = 2$, then we say that *$q$ is a singularity of type C.*

(4) If $\mathrm{mult}_q(B_i) = \mathrm{mult}_q(\tilde{B}_i) + 2$, then we say that *$q$ is a singularity of type D.*

REMARK. The number of the elements of $\{j \mid p_j$ is a singularity of type A (resp. B, C, D)$\}$ is independent of the way of choosing the center of the blowing-up in each step, we denote it by $n_{\mathrm{A}}(S, p)$ (resp. $n_{\mathrm{B}}(S, p), n_{\mathrm{C}}(S, p), n_{\mathrm{D}}(S, p)$).

The following proposition shows that $p_a(S, p) - p_a(Y_{S,p})$ is represented by the sum of the differences of the double points with good properties ((1), (2) and (3) of Proposition 4.2).

PROPOSITION 4.4. *Let $\tilde{\pi} : S_{r+1} \to S$ be the canonical resolution of a double point (S, p) with Diagram 2.1. We assume that $r \geq 2$. Suppose that there exists a number $2 \leq n \leq r$ such that the following conditions hold.*

(a) *p_n is the singularity of type A in sense of Definition* 4.3.

(b) *There exists no singularity of B_n except p_n.*

Let (S', p') be a double point which satisfies the conditions (1)–(3) *with $r' = n - 1$ of Proposition* 4.2.

(1) *If p_n is a rational singularity of S_n, then we have the equality*

$$p_a(S, p) - p_a(Y_{S,p}) = p_a(S', p') - p_a(Y_{S',p'}). \tag{4.4}$$

(2) *If p_n is a non-rational singularity of S_n, then we have the equality*

$$p_a(S,p) - p_a(Y_{S,p}) = p_a(S',p') - p_a(Y_{S',p'}) + p_a(S_n,p_n) - p_a(Y_{S_n,p_n}). \quad (4.5)$$

REMARK. From the above assumption (a) and (b), we have $\operatorname{Sing} B_n \cap B_n^{\mathrm{exc}} = \phi$. Hence we can construct (S',p').

PROOF. We set $N_1 = \{j \mid 1 \le j \le n-1\}$ and $N_2 = \{j \mid n \le j \le r\}$. We remark that, by the assumption (b) of the proposition, p_j is an infinitely near singularity of p_n if and only if $j \in N_2$. Let $\tilde{\pi}' \in \Pi(S',p')$ be a canonical resolution such that (S,p) with $\tilde{\pi}$ and N_1 and (S',p') with $\tilde{\pi}'$ and N_1 are numerically equivalent. We put sets N_1^{odd} and N_2^{odd} as follows:

$$N_1^{\mathrm{odd}} = \{j \in N_1 \mid m_j \text{ is an odd number}\}$$

and

$$N_2^{\mathrm{odd}} = \{j \in N_2 \mid m_j \text{ is an odd number}\}.$$

We remark that $j \in N_1^{\mathrm{odd}} \cup N_2^{\mathrm{odd}}$ if and only if $E_j^{(i)} \subset B_i$ for any $i > j$. Before we prove the proposition, we are going to show several claims.

CLAIM 4.5. *If (i,j) satisfies one of the cases*
(1) *$i \in N_1^{\mathrm{odd}}$ and $n \le j \le r$,*
(2) *$i \in N_2^{\mathrm{odd}}$ and $1 \le j \le n-1$,*
then the equality

$$E_i^{(r+1)} \cdot Y_j^{(r+1)} = 0 \quad (4.6)$$

holds.

PROOF. Obviously, the relation $E_i^{(r+1)} \cdot Y_j^{(r+1)} = 0$ holds for $j < i$. Therefore, in case of (2), the equality (4.6) holds. Next, we consider the case of (1). Because of the assumption (a) of the proposition and $E_i^{(n)} \subset B_n$, we have $p_n \notin E_i^{(n)}$. Since $E_j^{(j+1)}$ is collapsed on p_n through $\pi_n \circ \cdots \circ \pi_j$, from the relation $p_n \notin E_i^{(n)}$, we have

$$E_i^{(j+1)} \cap E_j^{(j+1)} = \phi. \quad (4.7)$$

Therefore, we obtain $E_i^{(r+1)} \cdot Y_j^{(r+1)} = E_i^{(j+1)} \cdot E_j^{(j+1)} = 0$.

Next we show the following claim, which plays an essential role in the proof of the proposition.

CLAIM 4.6.

$$V_{S,p}^{\min}(N_1 \cup N_2) = V_{S',p'}^{\min}(N_1) + V_{S_n,p_n}^{\min}(N_2). \tag{4.8}$$

PROOF. First we will prove the inequality

$$V_{S,p}^{\min}(N_1) \geq V_{S',p'}^{\min}(N_1). \tag{4.9}$$

Let D be a divisor on U_{r+1} with support in B_{r+1}. We set divisors D' on U_n and $\bar{D}'$ on U_{r+1} as follows:

$$D' = (\pi_n \circ \cdots \circ \pi_r)_*(D) \quad \text{and} \quad \bar{D}' = (\pi_n \circ \cdots \circ \pi_r)^*(D').$$

Then D' is supported in B_n^{exc}. By the relation $p_n \notin B_n^{\mathrm{exc}}$, the divisor $D - \bar{D}'$ is supported in $\bigcup_{i \in N_2} E_i^{(r+1)} \cap B_{r+1}$. Therefore, from Claim 4.5, we have

$$(D - \bar{D}') \cdot Y_i^{(r+1)} = 0 \quad \text{for } i \in N_1. \tag{4.10}$$

From the equality (4.10), we have

$$D \cdot Y_i^{(r+1)} = \bar{D}' \cdot Y_i^{(r+1)} = D' \cdot Y_i^{(n)} \quad \text{for } i \in N_1.$$

Hence we obtain

$$\sum_{i \in N_1} \delta(D \cdot Y_i^{(r+1)} + \gamma_i) = \sum_{i \in N_1} \delta(D' \cdot Y_i^{(n)} + \gamma_i). \tag{4.11}$$

Since (S, p) with $\tilde{\pi}$ and N_1 is numerically equivalent to (S', p') with $\tilde{\pi}'$ and N_1, we have

$$\sum_{i \in N_1} \delta(D' \cdot Y_i^{(n)} + \gamma_i) \geq V_{S',p'}^{\min}(N_1). \tag{4.12}$$

From (4.11) and (4.12), we obtain (4.9).

We set $N_2 - n + 1 = \{j \mid 1 \leq j \leq r - n + 1\}$.

Next we will prove the inequality

$$V_{S,p}^{\min}(N_2) \geq V_{S_n,p_n}^{\min}(N_2 - n + 1). \tag{4.13}$$

Let F be a divisor on U_{r+1} with support in B_{r+1}^{exc}. We can decompose F into $F = F_1 + F_2$ such that F_1 and F_2 are supported in $\bigcup_{i \in N_1} E_i^{(r+1)}$ and $\bigcup_{i \in N_2} E_i^{(r+1)}$ respectively. From Claim 4.5, we have

$$F_1 \cdot Y_i^{(r+1)} = 0 \quad \text{for } i \in N_2.$$

Hence we obtain the relation

$$F \cdot Y_i^{(r+1)} = F_2 \cdot Y_i^{(r+1)} \quad \text{for } i \in N_2. \tag{4.14}$$

From (4.14), we have

$$\sum_{i \in N_2} \delta(F \cdot Y_i^{(r+1)} + \gamma_i) = \sum_{i \in N_2} \delta(F_2 \cdot Y_i^{(r+1)} + \gamma_i). \tag{4.15}$$

Since (S, p) with $\tilde{\pi}$ and N_2 is numerically equivalent to (S_n, p_n) with $\tilde{\pi}_n \circ \cdots \circ \tilde{\pi}_r$ and $N_2 - n + 1$, we have

$$\sum_{i \in N_2} \delta(F_2 \cdot Y_i^{(r+1)} + \gamma_i) \geq V_{S_n, p_n}^{\min}(N_2 - n + 1). \tag{4.16}$$

From (4.15) and (4.16), we obtain (4.13).

From (3.12), (4.9) and (4.13), the left hand side of (4.8) is not less than the right hand side of (4.8). Next, we show the converse.

Let A' be an effective divisor on U_n with support in B_n^{exc}. Let A_n be an effective divisor on U_{r+1} with support in $\bigcup_{i \in N_2} E_i^{(r+1)} \cap B_{r+1}$. We set a divisor $\bar{A}'$ on U_{r+1} as follows:

$$\bar{A}' = (\pi_n \circ \cdots \circ \pi_r)^*(A').$$

Then we have

$$\bar{A}' \cdot Y_i^{(r+1)} = A' \cdot Y_i^{(n)} \quad \text{for } i \in N_1. \tag{4.17}$$

Since $\bar{A}'$ and A_n are supported in $\bigcup_{i \in N_1} E_i^{(r+1)}$ and $\bigcup_{i \in N_2} E_i^{(r+1)}$ respectively, from Claim 4.5, we have

$$\bar{A}' \cdot Y_i^{(r+1)} = 0 \text{ for } i \in N_2 \quad \text{and} \quad A_n \cdot Y_i^{(r+1)} = 0 \text{ for } i \in N_1. \tag{4.18}$$

We put $A = \bar{A}' + A_n$. Then A is an effective divisor and supported in B_{r+1}^{exc}. From (4.17) and (4.18), we have

$$\begin{aligned} \sum_{i \in N_1 \cup N_2} \delta(A \cdot Y_i^{(r+1)} + \gamma_i) &= \sum_{i \in N_1 \cup N_2} \delta(\bar{A}' \cdot Y_i^{(r+1)} + A_n \cdot Y_i^{(r+1)} + \gamma_i) \\ &= \sum_{i \in N_1} \delta(A' \cdot Y_i^{(n)} + \gamma_i) + \sum_{i \in N_2} \delta(A_n \cdot Y_i^{(r+1)} + \gamma_i). \end{aligned} \tag{4.19}$$

Since (S, p) with $\tilde{\pi}$ and N_1 is numerically equivalent to (S', p') with $\tilde{\pi}'$ and N_1, and (S, p) with $\tilde{\pi}$ and N_2 is numerically equivalent to (S_n, p_n) with $\tilde{\pi}_n \circ \cdots \circ \tilde{\pi}_r$ and $N_2 - n + 1$, from (4.19), the left hand side of (4.8) is not grater than the right hand side of (4.8). Therefore we obtain the equality (4.8).

Now we return to the proof of the proposition. If (S, p) is a rational singularity, then $\gamma_i = 0$ for $1 \le i \le r$ and $p_a(Y_{S,p}) = 1$. Therefore (S', p') is a rational singularity and $p_a(Y_{S',p'}) = 1$. Obviously, so is (S_n, p_n). Hence (4.4) holds. From now on, in this proof, we assume that (S, p) is a non-rational singularity. Then (S', p') is necessarily a non-rational singularity. In fact, if (S', p') is a rational singularity, then we have $\tilde{m}_j = \tilde{m}'_j \le 2$ for any $2 \le j \le n-1$ (see [4], Lemma 5). This follows $m_j \le 2$ for any $j \in N_2$ and that (S, p) is a rational singularity. This contradicts the assumption.

Now, obviously, we have

$$\Gamma^{\mathrm{odd}}_{S,p}(N_1) = \Gamma^{\mathrm{odd}}_{S',p'}(N_1) \tag{4.20}$$

and

$$\Gamma^{\mathrm{odd}}_{S,p}(N_2) = \Gamma^{\mathrm{odd}}_{S_n,p_n}(N_2 - n + 1). \tag{4.21}$$

By combing (4.20), (4.21) and (3.11), we obtain

$$\Gamma^{\mathrm{odd}}_{S,p}(N_1 \cup N_2) = \Gamma^{\mathrm{odd}}_{S',p'}(N_1) + \Gamma^{\mathrm{odd}}_{S_n,p_n}(N_2 - n + 1). \tag{4.22}$$

If (S_n, p_n) is a rational singularity, then we can easily see

$$V^{\min}_{S_n,p_n}(N_2 - n + 1) = \Gamma^{\mathrm{odd}}_{S_n,p_n}(N_2 - n + 1) = 0.$$

Therefore (4.4) and (4.5) follows from (3) of Theorem A, (4.8) and (4.22).
Q.E.D.

In general, when we desingularize a double point (S, p) by using the canonical method, it is possible that $(E^{(r+1)} \cdot \tilde{B}_{r+1})_p = 2$ for some $p \in U_{r+1}$. In the following proposition, we represent the difference of the arithmetic genus of the double point (S, p) and the virtual genus of $Y_{S,p}$ by the difference of that of a certain double point (S', p') with the property $(E'^{(r+1)} \cdot \tilde{B}'_{r+1})_p \le 1$ for any $p \in U'_{r+1}$.

PROPOSITION 4.7. *Let $\tilde{\pi} : S_{r+1} \to S$ be the canonical resolution of a double point (S, p) with Diagram* 2.1. *Then we have the equality*

$$p_a(S, p) - p_a(Y_{S,p}) = p_a(S', p') - p_a(Y_{S',p'}), \tag{4.23}$$

where (S', p') is a double point which satisfies the conditions (1)–(3) *with $r' = r$ of Proposition* 4.2.

PROOF. Obviously we have

$$\Gamma^{\mathrm{odd}}_{S,p}(\{1 \le j \le r\}) = \Gamma^{\mathrm{odd}}_{S',p'}(\{1 \le j \le r\})$$

and

$$V^{\min}_{S,p}(\{1 \le j \le r\}) = V^{\min}_{S',p'}(\{1 \le j \le r\}).$$

Therefore, by (3) of Theorem A, we obtain (4.23). Q.E.D.

DEFINITION 4.8. Let $\tilde{\pi} : S_{r+1} \to S$ be the canonical resolution of a double point (S, p) with Diagram 2.1. We define an integer J_i as follows:

$$J_i = \sum_{p \notin \operatorname{Sing} B_{i+1}} (E_i^{(i+1)} \cdot \tilde{B}_{i+1})_p \tag{4.24}$$

for $1 \le i \le r$, where $\operatorname{Sing} B_i$ is the singular locus of B_i in U_i.

REMARK. If the canonical resolution of (S, p) satisfies the condition (3) with $r' = r$ of Proposition 4.2, then we have $J_i = E_i^{(r+1)} \cdot \tilde{B}_{r+1}$.

We can construct canonical resolution with a priority as follows:

DEFINITION 4.9 (canonical resolution with priority). Let (S, p) be a double point. We denote by r the number of the steps of the process of the canonical resolution of (S, p). Let $H^r = (\{1, 2, 3, 4\} \times \{m \in \mathbf{Z} \mid 0 \le m\})^{\times r}$ be the set with the lexicographic order, and let $\tau : \{A, B, C, D\} \to \{1, 2, 3, 4\}$ be the one to one correspondence such that $\tau(A) = 1$, $\tau(B) = 2$, $\tau(C) = 3$ and $\tau(D) = 4$. We define a mapping $\Phi : \Pi(S, p) \to H^r$ by $\Phi(\pi) = (\tau(\operatorname{type}(U_i, p_i)), \mu_i)_{i=1,\dots,r}$, where $\mu_i = r - n_{\mathrm{D}}(S_i, p_i)$ if p_i is of type B and $\mu_i = \tilde{m}_i$ if not for $1 \le i \le r$. Then, there exists a element $\pi \in \Pi(S, p)$ corresponding to the unique maximal element of $\Phi(\Pi(S, p))$. When we construct canonical resolution by π, we say that *we desingularize* (S, p) *by using the canonical method with priority* $D > C > B > A$.

For example, for the singularity of D_5-type, the number of the elements of the set $\Phi(\Pi(S, p))$ equals 2, and the maximal element is

$$((\tau(A), 3), (\tau(B), 0), (\tau(C), 2), (\tau(B), 0)) = (1, 3, 2, 0, 3, 2, 2, 0),$$

and non-maximal element is

$$(1, 3, 2, 0, 2, 0, 3, 2)$$

By the following lemma and the above proposition, it is sufficient to consider the double point with $n_{\mathrm{A}}(S, p) = 1$.

LEMMA 4.10. *Let $q > 0$. We assume that*

$$p_a(S, p) - p_a(Y_{S,p}) \leq \left\lceil \frac{m_1 - 1}{q} \right\rceil - \sum_{\substack{1 \leq i \leq r \\ J_i > 0}} \left\lceil \frac{J_i - 1}{q} \right\rceil \tag{4.25}$$

for any double point (S, p) with $n_{\mathrm{A}}(S, p) = 1$. Then the above inequality holds for any double point (S, p).

PROOF. We are going to prove that, by the induction on l, the inequality (4.25) holds for any double point (S, p) with $n_{\mathrm{A}}(S, p) \leq l$. In case of $n_{\mathrm{A}}(S, p) = 1$, the inequality (4.25) holds by assumption. We assume that it is true for $l - 1$ ($l \geq 2$). Let (S, p) be a double point with $n_{\mathrm{A}}(S, p) = l$. We desingularize (S, p) by the canonical method with priority D > C > B > A. We obtain Diagram 2.1. We put $I_i^{(j)} = (E_i^{(j)} \cdot \tilde{B}_j)_{p_j}$ for $1 \leq i < j \leq r + 1$. By Proposition 4.7, we may assume without loss of generality that

$$(E^{(r+1)} \cdot \tilde{B}_{r+1})_p \leq 1 \quad \text{for any } p \in U_{r+1}. \tag{4.26}$$

We put

$$n = \max\{j \mid p_j \text{ is a singularity of type A}, 1 \leq j \leq r\}.$$

By $n_{\mathrm{A}}(S, p) \geq 2$, we obtain $2 \leq n$. Then, from the priority D > C > B > A of the construction of the canonical resolution of (S, p), the assumptions (a) and (b) of Proposition 4.4 are satisfied. Let (S', p') be as in Proposition 4.4. By the definition of n, we have $n_{\mathrm{A}}(S', p') = l - 1$. Hence, by the assumption of the induction, we have

$$p_a(S', p') - p_a(Y_{S',p'}) \leq \left\lceil \frac{m_1 - 1}{q} \right\rceil - \sum_{\substack{1 \leq j \leq n-1 \\ J_j' > 0}} \left\lceil \frac{J_j' - 1}{q} \right\rceil. \tag{4.27}$$

Now we set

$$T = \{k \mid p_n \in E_k^{(n)}, 1 \leq k \leq n - 1\}.$$

Since p_n is a singularity of type A, we have $E_k^{(n)} \not\subset B_n$ for any $k \in T$. By (4.26), $\tilde{B}_{r+1}$ does not pass through the intersection point of the two distinct irreducible exceptional curves of U_{r+1}. Hence we have

$$J_k' = I_k^{(n)} + J_k \quad \text{for } k \in T \tag{4.28}$$

and

$$J_i' = J_i \quad \text{for any } i \notin T,\ 1 \leq i \leq n - 1. \tag{4.29}$$

Then, for $k \in T$, we have

$$\left[\frac{J_k' - 1}{q}\right] \geq \left[\frac{I_k^{(n)} - 1}{q}\right] + \left[\frac{J_k - 1}{q}\right] \quad \text{from (4.28)}$$

$$\geq \left[\frac{m_n - 1}{q}\right] + \left[\frac{J_k - 1}{q}\right] \quad \text{by } m_n \leq I_k^{(n)}. \tag{4.30}$$

By combing (4.27), (4.29) and (4.30), we have

$$p_a(S', p') - p_a(Y_{S',p'}) \leq \left[\frac{m_1 - 1}{q}\right] - \sum_{\substack{1 \leq i \leq n-1 \\ J_i > 0}} \left[\frac{J_i - 1}{q}\right] - \left[\frac{m_n - 1}{q}\right]. \tag{4.31}$$

If (S_n, p_n) is a rational singularity, then we have $[(J_i - 1)/q] = 0$ or $J_i = 0$ for $n \leq i \leq r$. Hence, from (4.4) and (4.31), we obtain (4.25).

If (S_n, p_n) is a non-rational singularity, from the equality $n_{\mathrm{A}}(S_n, p_n) = 1$, we obtain

$$p_a(S_n, p_n) - p_a(Y_{S_n,p_n}) \leq \left[\frac{m_n - 1}{q}\right] - \sum_{\substack{n \leq i \leq r \\ J_i > 0}} \left[\frac{J_i - 1}{q}\right]. \tag{4.32}$$

By combing (4.5), (4.31) and (4.32), we obtain (4.25). Q.E.D.

We give a detailed estimate of the arithmetic genus of double point.

THEOREM B. *The inequality*

$$p_a(S, p) - p_a(Y_{S,p}) \leq \left[\frac{m_1 - 1}{8}\right] - \sum_{\substack{1 \leq j \leq r \\ J_j > 0}} \left[\frac{J_j - 1}{8}\right] \tag{4.33}$$

holds for any double point (S, p), where m_1 is the multiplicity at p of the branch locus B_1 of S and $Y_{S,p}$ is the cycle on S_{r+1} defined in Section 3.

REMARK. The left and right hand sides of the inequality (4.33) are independent of the way of choosing the center of the blowing-up which induces each step of the process of the canonical resolution.

PROOF. In the next section, we will prove the inequality

$$\Gamma_{S,p}^{\mathrm{odd}}(\{1 \leq j \leq r\}) - V_{S,p}^{\min}(\{1 \leq j \leq r\}) \leq 4\left[\frac{m_1 - 1}{8}\right] - \sum_{\substack{1 \leq j \leq r \\ J_j > 0}} 4\left[\frac{J_j - 1}{8}\right] \tag{4.34}$$

for any double point (S, p) with $n_{\mathrm{A}}(S, p) = 1$. If (S, p) is a rational double point, then, since $J_j = 0$ for any $1 \leq j \leq r$, $m_1 \leq 3$, $p_a(S, p) = 0$ and $p_a(Y_{S,p}) = 1$, the inequality (4.33) holds. If (S, p) is a non-rational double point, then, from (4.34), (3) of Theorem A and Lemma 4.10, the inequality (4.33) holds for any double point (S, p). Q.E.D.

COROLLARY C. In the above situation, we have $p_a(S, p) - p_a(Y_{S,p}) \leq (1/8)m_1$.

REMARK. From (3) of Theorem A, we have

$$p_a(S, p) - p_a(Y_{S,p}) \leq \frac{1}{4}\Gamma^{\mathrm{odd}}_{S,p}(\{1 \leq j \leq r\}) \tag{4.35}$$

for any double point (S, p). In general, we can not give estimate of the right hand side of (4.35) in the first order on m_1. We will see such an example.

EXAMPLE 4.11. Let $(S^{(n)}, p) = (\{z^2 = x^4 - y^{4n}\}, o)$. Then $\mathrm{mult}_p B_1^{(n)} = 4$ holds for any n, where $B_1^{(n)}$ is the branch locus of $S^{(n)}$, and the canonical resolution of $(S^{(n)}, p)$ is obtained in the n-th step. We can easily see that

$$\Gamma^{\mathrm{odd}}_{S^{(n)},p}(\{1 \leq j \leq n\}) = n.$$

We compute the arithmetic genus on several examples.

EXAMPLE 4.12. Let $(S^{(n)}, p) = (\{z^2 = x^{2n} - y^{2n}\}, o)$ for $n \geq 2$. We desingularize $(S^{(n)}, p)$ by using the canonical method. Then the number r of the steps of canonical method of $(S^{(n)}, p)$ equals one and we have $m_1 = J_1 = 2n$ and $\gamma_1 = n - 1$. Hence, from Theorem B, we have

$$p_a(S^{(n)}, p) = p_a(Y_{S^{(n)},p}) = \begin{cases} \dfrac{(n-1)^2}{4} + 1 & (n\text{: odd}) \\ \dfrac{(n-1)^2 - 1}{4} + 1 & (n\text{: even}). \end{cases} \tag{4.36}$$

Next, we see an example that the equality holds in (4.33). From the example, we know that Corollary C gives the best estimate of $p_a(S, p) - p_a(Y_{S,p})$ on the first order of m_1.

EXAMPLE 4.13. Let $\{g_k\}_{k \geq 1}$ be a sequence, which consists of polynomials, defined as follows:

$$g_k(x,y) = y\prod_{i=1}^{k}\{(x^{4i}-y^2)(x^{6i}-(y+x^{2i-1})^3)(x^{6i}-(y+2x^{2i-1})^3)\}.$$

Here, we have the relation

$$g_k(x,y) = x^{16(k-1)+2} g_{k-1}\left(x, \frac{y}{x^2}\right)(x^4-y^2)(x^6-(y+x)^3)(x^6-(y+2x)^3)$$

for $k \geq 2$. We put $(S^{(k)}, p) = (\{z^2 = g_k(x,y)\}, o)$ for $k \geq 1$. Then the equality holds in (4.33) for every $(S^{(k)}, p)$. We will see that. There exists $\tilde{\pi}^{(k)} \in \Pi(S^{(k)}, p)$ such that

$$(\gamma_i^{(k)}) = (4k-1, 1, 1, 1, 4k-3, 4k-5, 1, 1, 4k-7, \dots, 3, 1, 1, 1).$$

We have $m_1^{(k)} = \operatorname{mult}_p B_1^{(k)} = 8k+1$ and $r(S^{(k)}, p) = 4k$, where $B_1^{(k)}$ is the branch locus of $S^{(k)}$ and $r(S,p)$ is the number of the steps of the canonical resolution of (S,p). In the situation, the singularity p_5 satisfies the assumptions (a) and (b) with $n=5$ of Proposition 4.4. Therefore, we will construct a double point as in Proposition 4.2 for $(S^{(k)}, p)$. Let $\{f_k\}_{k\geq 1}$ be the sequence, which consists of polynomials, defined as follows:

$$f_k(x,y) = (x^{6+16(k-1)} - y^{3+8(k-1)})(x^6-(y+x)^3)(x^6-(y+2x)^3).$$

We put $(S'^{(k)}, p') = (\{z^2 = f_k(x,y)\}, o)$ for $k \geq 1$. Then, there exists $\tilde{\pi}'^{(k)} \in \Pi(S'^{(k)}, p')$ such that $(\gamma_i'^{(k)}) = (4k-1, 1, 1, 4k-3)$. We have $\operatorname{mult}_p B_1'^{(k)} = 8k+1$ and $r(S'^{(k)}, p') = 4$ where $B_1'^{(k)}$ is the branch locus of $S'^{(k)}$. In this situation, we can easily see that

$$(S^{(k)}, p)_{\tilde{\pi}^{(k)}, 1\leq j\leq 4} \sim (S'^{(k)}, p')_{\tilde{\pi}'^{(k)}, 1\leq j\leq 4}$$

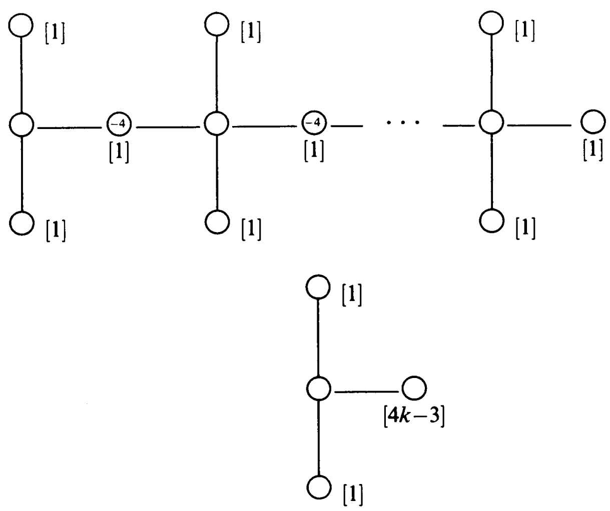
and

$$(S^{(k)}, p)_{\tilde{\pi}^{(k)}, 5\leq j\leq 4k} \sim (S^{(k-1)}, p)_{\tilde{\pi}^{(k-1)}, 1\leq j\leq 4(k-1)}.$$

Therefore, by using Proposition 4.4 successively, we have the equality

$$p_a(S^{(k)}, p) - p_a(Y_{S^{(k)}, p}) = \sum_{1\leq j\leq k}(p_a(S'^{(j)}, p') - p_a(Y_{S'^{(j)}, p'})). \tag{4.37}$$

We can write the weighted dual graph of the exceptional set of the minimal resolution of $(S^{(k)}, p)$ and $(S'^{(k)}, p')$, which have $4k$ vertexs and 4 vertexs respectively, as follows:



where we denote by $\textcircled{n}_{[g]}$ the curve with genus g and self-intersection number n, and, by $\bigcirc$ the rational curve with self-intersection number -2.

Next, we will compute $p_a(S'^{(j)}, p') - p_a(Y_{S'^{(j)},p'})$ for $1 \leq j \leq k$. We have

$$\Gamma^{\mathrm{odd}}_{S'^{(j)},p'}(\{1 \leq l \leq 4\}) = 4. \tag{4.38}$$

Let $D = \sum_{1 \leq j \leq 4} a_j E_j^{(5)}$ be an effective divisor on U_5 with support in B_5^{exc}, where U_5 is the base space of the canonical resolution of $(S'^{(j)}, p')$. Then we have

$$D \cdot Y_1^{(5)} + \gamma_1 = \gamma_1 - a_1,$$

$$D \cdot Y_2^{(5)} + \gamma_2 = a_1 + \gamma_2,$$

$$D \cdot Y_3^{(5)} + \gamma_3 = a_1 + \gamma_3,$$

$$D \cdot Y_4^{(5)} + \gamma_4 = a_1 + \gamma_4,$$

and γ_i is an odd number for $1 \leq i \leq 4$. Therefore we have

$$V^{\min}_{S'^{(j)},p'}(\{1 \leq l \leq 4\}) = 0. \tag{4.39}$$

From (4.38) and (4.39), using (3) of Theorem A, we have

$$p_a(S'^{(j)}, p') - p_a(Y_{S'^{(j)},p'}) = 1.$$

Therefore, from (4.37), we obtain

$$p_a(S^{(k)}, p) - p_a(Y_{S^{(k)},p}) = k.$$

Now we can easily see

$$\left[\frac{m_1^{(k)}-1}{8}\right]-\sum_{\substack{1\le j\le 4k\\ J_j^{(k)}>0}}\left[\frac{J_j^{(k)}-1}{8}\right]=k.$$

Under the assumption that the multiplicity at p of the branch locus of double point (S, p) is not greater than eight, we can give a p_a-formula.

THEOREM D. *We assume that* (S, p) *is a non-rational double point with the branch locus* B_1 *such that* $m_1 \le 8$, *where* $m_1 = \mathrm{mult}_p B_1$. *Then we have*

$$p_a(S, p) = p_a(Y_{S,p}), \tag{4.40}$$

where $Y_{S,p}$ *is the cycle on* S_{r+1} *defined in Section* 3.

PROOF. Under our assumption, we have $p_a(Y_{S,p}) \le p_a(S, p)$ (see [9], Lemma 1). The assertion now follows from Corollary C. Q.E.D.

The following corollary characterizes double point (S, p) with $p_a(S, p) = 2$.

COROLLARY E. *Let* (S, p) *be a double point. We have the following characterization.* $p_a(S, p) = 2$ *if and only if there exists a number i such that* $\gamma_i = 2$ *and* $\gamma_j \le 1$ $(j \ne i)$.

PROOF. If the inequality $1 \le p_a(S, p) \le 2$ holds, then it follows from (4.40) and (3.14) that $\gamma_1 \le 2$. Hence we have $m_1 \le 8$. Therefore, from (3.14) and Theorem D, it is easy to show the assertion of this corollary.

§5. Proof of the Inequality (4.34) for Double Points (S, p) with $n_A(S, p) = 1$

To begin with, we prove the following proposition.

PROPOSITION 5.1. *Let* $\tilde{m}_j = \mathrm{mult}_{p_j} \tilde{B}_j$. *We have the relation*

$$m_1 = \sum_{p_j:\text{type } C \text{ or } D} \tilde{m}_j + E^{(r+1)} \cdot \tilde{B}_{r+1}. \tag{5.1}$$

PROOF. We put $I^{(i)} = E^{(i)} \cdot \tilde{B}_i$ for any $2 \le i \le r+1$. We will prove the equality

$$I^{(i)} = \begin{cases} I^{(i+1)} & (p_i : \text{type A or B}) \\ I^{(i+1)} + \tilde{m}_i & (p_i : \text{type C or D}) \end{cases} \tag{5.2}$$

for $2 \leq i \leq r$. In case that p_i is a singularity of type A or B, then $\#\{j \mid E_j^{(i)} \ni p_i\} = 1$ holds. Then, by $E^{(i+1)} = \pi_i^*(E^{(i)})$, we have

$$\begin{aligned} I^{(i+1)} &= \pi_i^*(E^{(i)}) \cdot \tilde{B}_{i+1} \\ &= E^{(i)} \cdot \tilde{B}_i \\ &= I^{(i)}. \end{aligned} \tag{5.3}$$

In case that p_i is a singularity of type C or D, then $\#\{j \mid E_j^{(i)} \ni p_i\} = 2$ holds. Then, by $E^{(i+1)} = \pi_i^*(E^{(i)}) - E_i^{(i+1)}$, we have

$$\begin{aligned} I^{(i+1)} &= (\pi_i^*(E^{(i)}) - E_i^{(i+1)}) \cdot \tilde{B}_{i+1} \\ &= E^{(i)} \cdot \tilde{B}_i - \tilde{m}_i \quad \text{by } \tilde{m}_i = E_i^{(i+1)} \cdot \tilde{B}_{i+1} \\ &= I^{(i)} - \tilde{m}_i. \end{aligned} \tag{5.4}$$

From (5.3) and (5.4), we obtain (5.2) for $2 \leq i \leq r$. By $m_1 = I^{(2)}$, from (5.2), we can easily show (5.1). Q.E.D.

In what follows, we assume that double point (S, p) satisfies $n_{\mathrm{A}}(S, p) = 1$. We desingularize (S, p) by using the canonical method with priority D > C > B > A. We obtain Diagram 2.1. From Proposition 4.7, we may assume without loss of generality that $(E^{(r+1)} \cdot \tilde{B}_{r+1})_p \leq 1$ for any $p \in U_{r+1}$ and $(E^{(i)} \cdot \tilde{B}_i)_p \leq 1$ for any $p \in U_i \backslash \operatorname{Sing} B_i$.

First we show the inequality (4.34) for the case that m_1 is an even number. Suppose that m_1 is an even number. By the assumption, we have $r = 1$ and $B_2^{\mathrm{exc}} = \phi$. It is easy to see that $\Gamma^{\mathrm{odd}}(\{1\}) = V^{\min}(\{1\}) = \delta(\gamma_1)$ and $m_1 = J_1$. Now, we obtain (4.34).

From now on, we consider the case that m_1 is an odd number. Then, we have $r \geq 2$. To prove the case, we introduce several notations.

We define $N_8 : \{n \in \boldsymbol{Z} \mid n \geq 0\} \to \{n \in \boldsymbol{Z} \mid n \geq 0\}$ as follows:

$$N_8(n) = \begin{cases} 0 & \text{if } n = 0 \\ n - 8\left[\dfrac{n-1}{8}\right] & \text{if } n > 0. \end{cases} \tag{5.5}$$

REMARK. We can easily see that $N_8(*)$ has the following properties.

(1) $N_8(n_1 + n_2) \leq N_8(n_1) + N_8(n_2)$ for any $n_1, n_2 \geq 0$.

(2) $N_8(n) \geq 1$ for any $n \geq 1$.

(3) If $n \leq 8$, then we have the relation $N_8(n) = n$.

From the definition of $N_8(*)$, Proposition 5.1, and the remark of Definition 4.8, we have

$$\frac{1}{8}\left(\sum_{p_j:\text{type C or D}} \tilde{m}_j + \sum_{1\le j\le r} N_8(J_j)\right) = \frac{m_1}{8} - \sum_{\substack{1\le j\le r\\ J_j>0}} \left[\frac{J_j-1}{8}\right] \tag{5.6}$$

From (3) of Theorem A, 4 divides $\Gamma^{\text{odd}}(\{1\le j\le r\}) - V^{\min}(\{1\le j\le r\})$. Therefore, since m_1 is an odd number, to show the inequality (4.34), from (5.6), it is sufficient to show

$$\Gamma^{\text{odd}}(\{1\le j\le r\}) - V^{\min}(\{1\le j\le r\}) \le \frac{1}{2}\left(\sum_{p_j:\text{type C or D}} \tilde{m}_j + \sum_{1\le j\le r} N_8(J_j)\right). \tag{5.7}$$

We will prove it by dividing the process of the canonical method into blocks. We define $\mu = n_{\text{B}}(S,p)$. Let $b:\{1,\dots,\mu\}\to\{j\,|\,p_j$ is a singularity of type B$\}$ be the one to one correspondence of the order sets. Let $b_{\mu+1}=\infty$. We remark that, since m_1 is an odd number, there exists a number $1\le j\le r$ such that p_j is a singularity of type B. The singularities of the branch locus in U_{b_α} are of type B, and p_j, $b_\alpha<j<b_{\alpha+1}$ are of type C or D. We define sets $\mathscr{C}_\alpha$ and $\mathscr{D}_\alpha$ as follows:

$$\mathscr{C}_\alpha(\text{resp. } \mathscr{D}_\alpha) = \{j\,|\,p_j \text{ is type C (resp. D)}, b_\alpha<j<b_{\alpha+1}\}$$

for any $1\le\alpha\le\mu$. From the definition of priority $\text{D}>\text{C}>\text{B}>\text{A}$, then p_j is an infinitely near singularity of p_{b_α} for $b_\alpha\le j<b_{\alpha+1}$. We define several types for singularities p_{b_α}, $1\le\alpha\le\mu$ of type B appeared in the process as follows:

(1) If $\mathscr{C}_\alpha=\phi$ and $\mathscr{D}_\alpha=\phi$, then we say that p_{b_α} *is a singularity of type* B^I.

(2) If $\mathscr{C}_\alpha\ne\phi$ and $\mathscr{D}_\alpha=\phi$, then we say that p_{b_α} *is a singularity of type* B^{II}.

(3) If $\mathscr{D}_\alpha\ne\phi$ and $p_{b_\alpha+1}$ is type C, then we say that p_{b_α} *is a singularity of type* B^{III}.

(4) If $\mathscr{D}_\alpha\ne\phi$ and $p_{b_\alpha+1}$ is type D, then we say that p_{b_α} *is a singularity of type* B^{IV}.

We remark that one of the above cases occurs. Then, we have the relation

$$\{j\,|\,2\le j\le r\} = \textstyle\bigcup_{1\le\alpha\le\mu}(\{b_\alpha\}\cup\mathscr{C}_\alpha\cup\mathscr{D}_\alpha). \tag{5.8}$$

We will prove (5.7). First, we consider the case that, for any $1\le\alpha\le\mu$, p_{b_α} is a singularity of type B^{I} or B^{II}.

If $\mu=1$ and p_{b_1} is a singularity of type B^{I}, then, the relations $r=b_1=2$, $J_1=0$, and $J_2=m_1$ hold. Therefore, we obtain (5.7) from the following proposition.

PROPOSITION 5.2. *If $\mu = 1$ and p_{b_1} is a singularity of type B^{I}, then we obtain the equality*

$$\Gamma^{\mathrm{odd}}(\{1 \le j \le r\}) = V^{\min}(\{1 \le j \le r\}). \tag{5.9}$$

Otherwise, one of the following cases occurs.

(a) $\mu \ge 2$.

(b) p_{b_1} is of type B^{II}.

In any case, we can show that

$$\Gamma^{\mathrm{odd}}(\{2 \le j \le r\}) = \sum_{1 \le \alpha \le \mu} \Gamma^{\mathrm{odd}}(\{b_\alpha\} \cup \mathscr{C}_\alpha)$$
$$\le \sum_{p_j:type\ \mathrm{C}\text{ or }\mathrm{D}} \frac{\tilde{m}_j}{2} + \sum_{2 \le j \le r} \frac{N_8(J_j)}{2} - 1 \tag{5.10}$$

by using the following proposition.

PROPOSITION 5.3. *Let $1 \le \alpha \le \mu$.*

(1) *If p_{b_α} is a singularity of type B^I, then we have*

$$\Gamma^{\mathrm{odd}}(\{b_\alpha\}) \le \frac{N_8(J_{b_\alpha})}{2} - \frac{1}{2}. \tag{5.11}$$

(2) *If p_{b_α} is a singularity of type B^{II}, then we have*

$$\Gamma^{\mathrm{odd}}(\{b_\alpha\} \cup \mathscr{C}_\alpha) \le \sum_{j \in \mathscr{C}_\alpha} \frac{\tilde{m}_j}{2} + \sum_{j \in \{b_\alpha\} \cup \mathscr{C}_\alpha} \frac{N_8(J_j)}{2} - 1. \tag{5.12}$$

REMARK. In case of (1) (resp. (2)), we have $\{j \mid b_\alpha \le j < b_{\alpha+1}\} = \{b_\alpha\}$ (resp. $\{b_\alpha\} \cup \mathscr{C}_\alpha$).

The proofs of the propositions are given in the latter. From the relation $\Gamma^{\mathrm{odd}}(\{1\}) = \delta(\gamma_1) \le 1$ and inequality (5.10), we obtain (5.7).

From now on, we assume that there exists $1 \le \alpha \le \mu$ such that p_{b_α} is a singularity of type $\mathrm{B}^{\mathrm{III}}$ or B^{IV}. We put

$$\tau = \max\{\alpha \mid p_{b_\alpha} \text{ is a singularity of type } \mathrm{B}^{\mathrm{III}} \text{ or } \mathrm{B}^{\mathrm{IV}}, 1 \le \alpha \le \mu\}.$$

From $n_{\mathrm{A}}(S, p) = 1$, p_{b_α} is a singularity of type B^{I} or of type B^{II} if and only if $n_{\mathrm{D}}(S_{b_\alpha}, p_{b_\alpha}) = 0$. Therefore, from the definition of priority $\mathrm{D} > \mathrm{C} > \mathrm{B} > \mathrm{A}$, we can easily see that p_{b_τ} is a unique singularity. The following propositions are proved in the latter.

PROPOSITION 5.4.

(1) *For* $1 \leq \alpha \leq \mu$, *if* p_{b_α} *is a singularity of type* B^{III}, *then we have the inequality*

$$\Gamma^{\text{odd}}(\{b_\alpha\} \cup \mathscr{C}_\alpha \cup \mathscr{D}_\alpha) \leq \sum_{j \in \mathscr{C}_\alpha \cup \mathscr{D}_\alpha} \frac{\tilde{m}_j}{2} + \sum_{j \in \{b_\alpha\} \cup \mathscr{C}_\alpha \cup \mathscr{D}_\alpha} \frac{N_8(J_j)}{2} - \frac{1}{2}. \tag{5.13}$$

(2) *If* p_{b_τ} *is a singularity of type* B^{III}, *then we have the inequality*

$$\Gamma^{\text{odd}}(\{j \mid b_\tau \leq j \leq r\}) \leq \sum_{\substack{b_\tau \leq j \leq r \\ p_j\text{:type } C \text{ or } D}} \frac{\tilde{m}_j}{2} + \sum_{b_\tau \leq j \leq r} \frac{N_8(J_j)}{2} - 1. \tag{5.14}$$

PROPOSITION 5.5. *For* $1 \leq \alpha \leq \mu$, *if* p_{b_α} *is a singularity of type* B^{IV}, *then we have the inequality*

$$\begin{aligned} &\Gamma^{\text{odd}}(\{b_\alpha\} \cup \mathscr{C}_\alpha \cup \mathscr{D}_\alpha) - V^{\min}(\{b_\alpha\} \cup \mathscr{C}_\alpha \cup \mathscr{D}_\alpha) \\ &\leq \sum_{j \in \mathscr{C}_\alpha \cup \mathscr{D}_\alpha} \frac{\tilde{m}_j}{2} + \sum_{j \in \{b_\alpha\} \cup \mathscr{C}_\alpha \cup \mathscr{D}_\alpha} \frac{N_8(J_j)}{2}. \end{aligned} \tag{5.15}$$

PROPOSITION 5.6. *If* $\mu = \tau$ *and* p_{b_τ} *is a singularity of type* B^{IV}, *or* $\mu = \tau + 1$ *and* p_{b_τ} *is a singularity of type* B^{IV} *and* p_{b_μ} *is of type* B^{I}, *then we have the inequality*

$$\Gamma^{\text{odd}}(\{j \mid b_\tau \leq j \leq r\}) \leq \sum_{\substack{b_\tau \leq j \leq r \\ p_j\text{:type } C \text{ or } D}} \frac{\tilde{m}_j}{2} + \sum_{b_\tau \leq j \leq r} \frac{N_8(J_j)}{2} - 1. \tag{5.16}$$

From the inequalities (5.11), (5.12), (5.13) and (5.15), we have the inequality

$$\begin{aligned} &\Gamma^{\text{odd}}(\{b_\alpha\} \cup \mathscr{C}_\alpha \cup \mathscr{D}_\alpha) - V^{\min}(\{b_\alpha\} \cup \mathscr{C}_\alpha \cup \mathscr{D}_\alpha) \\ &\leq \sum_{j \in \mathscr{C}_\alpha \cup \mathscr{D}_\alpha} \frac{\tilde{m}_j}{2} + \sum_{j \in \{b_\alpha\} \cup \mathscr{C}_\alpha \cup \mathscr{D}_\alpha} \frac{N_8(J_j)}{2} \end{aligned} \tag{5.17}$$

for any $1 \leq \alpha \leq \mu$. We continue the proof of the inequality (5.7). We consider the following cases.

(1) $\mu \geq \tau + 2$.

(2) $\mu = \tau + 1$ and p_{b_μ} is a singularity of type B^{II}.

(3) p_{b_τ} is a singularity of type $\mathrm{B}^{\mathrm{III}}$.

(4) None of (1), (2) and (3) are satisfied.

In case of (1) or (2), from (5.11), (5.12) and $\Gamma^{\mathrm{odd}}(\{1\}) = \delta(\gamma_1) \leq 1$, we have

$$\Gamma^{\mathrm{odd}}\left(\{1\} \cup \bigcup_{\tau+1 \leq \alpha \leq \mu} (\{b_\alpha\} \cup \mathscr{C}_\alpha \cup \mathscr{D}_\alpha)\right)$$

$$\leq \sum_{\tau+1 \leq \alpha \leq \mu} \left(\sum_{j \in \mathscr{C}_\alpha \cup \mathscr{D}_\alpha} \frac{\tilde{m}_j}{2} + \sum_{j \in \{b_\alpha\} \cup \mathscr{C}_\alpha \cup \mathscr{D}_\alpha} \frac{N_8(J_j)}{2} \right) \tag{5.18}$$

From (5.17) and (5.18), we obtain (5.7).

In case of (3), from (5.14) and the inequality $\delta(\gamma_1) \leq 1$, we obtain

$$\Gamma^{\mathrm{odd}}(\{1\} \cup \{j \mid b_\tau \leq j \leq r\}) \leq \sum_{\substack{b_\tau \leq j \leq r \\ p_j:\text{type C or D}}} \frac{\tilde{m}_j}{2} + \sum_{b_\tau \leq j \leq r} \frac{N_8(J_j)}{2}. \tag{5.19}$$

Therefore, from (5.17) and (5.19), we obtain (5.7).

In case of (4), the assumption of Proposition 5.6. From (5.16), we obtain (5.19). Hence we obtain (5.7).

We finish the proof of the inequality (4.34).

We give the proof of the above propositions. We define $I_i^{(j)} = (E_i^{(j)} \cdot \tilde{B}_j)_{p_j}$ for $1 \leq i < j \leq r+1$ and $I^{(j)} = (E^{(j)} \cdot \tilde{B}_j)_{p_j}$ for $2 \leq j \leq r+1$. If p_j is an infinitely near singularity of p_i, we write as $p_i \preceq p_j$.

PROOF OF PROPOSITION 5.2. From the assumption of the proposition, we have $r = b_1 = 2$, and there exists no singularity of B_2 on $E_1^{(2)}$ except p_{b_1}. Hence, $\tilde{m}_1 = E_1^{(2)} \cdot \tilde{B}_2 = I_1^{(b_1)}$ holds. Since p_{b_1} is a singularity of type B^{I}, $I_1^{(b_1)} = \tilde{m}_{b_1}$ holds. Therefore, we have the inequality $\tilde{m}_{b_1} = \tilde{m}_1$. It follows the equality $\gamma_{b_1} = \gamma_1 + 1$ and the relation $E_1^{(3)} = B_3^{\mathrm{exc}}$. Hence we have the equality

$$\Gamma^{\mathrm{odd}}(\{1 \leq j \leq 2\}) = 1. \tag{5.20}$$

Now let D be an effective divisor on U_3 with support in B_3^{exc}. We can write as $D = a_1 E_1^{(3)}$, $a_i \in \boldsymbol{Z}$. We have

$$D \cdot Y_1^{(3)} = -a_1 \quad \text{and} \quad D \cdot Y_2^{(3)} = a_1.$$

It follows the inequality

$$\sum_{1 \leq i \leq 2} \delta(D \cdot Y_i^{(3)} + \gamma_i) = 1.$$

Therefore we have

$$V^{\min}(\{1 \le j \le 2\}) = 1. \tag{5.21}$$

From (5.20) and (5.21), we obtain (5.9). Q.E.D.

PROOF OF PROPOSITION 5.3. In case that p_{b_α} is a singularity of type B^{I} or of type B^{II}, $E_{b_\alpha}^{(b_\alpha+1)} \not\subset B_{b_\alpha+1}$ and $m_{b_\alpha} = \tilde{m}_{b_\alpha} + 1$ hold. Hence, $\tilde{m}_{b_\alpha}$ is an odd number.

CLAIM 5.7.

$$\tilde{m}_{b_\alpha} = \sum_{j \in \{b_\alpha\} \cup \mathscr{C}_\alpha} J_j. \tag{5.22}$$

PROOF. When p_{b_α} is a singularity of type B^{I}, $B_{b_\alpha+1}$ has no singularity on $E_{b_\alpha}^{(b_\alpha+1)}$ since there exists no singularity of type A on $E_{b_\alpha}^{(b_\alpha+1)}$. Therefore, from the relation $\tilde{m}_{b_\alpha} = \tilde{B}_{b_\alpha+1} \cdot E_{b_\alpha}^{(b_\alpha+1)}$, we have the inequality $J_{b_\alpha} = \tilde{m}_{b_\alpha}$. Hence, (5.22) follows.

Next, we consider the case that p_{b_α} is a singularity of type B^{II}. Then, $p_{b_\alpha+1} = E_{b_\alpha}^{(b_\alpha+1)} \cap \tilde{B}_{b_\alpha+1}$ is the singularity of type C and $B_{b_\alpha+1}$ has no singularity on $E_{b_\alpha}^{(b_\alpha+1)}$ except $p_{b_\alpha+1}$. Therefore, from the relation $\tilde{m}_{b_\alpha} = \tilde{B}_{b_\alpha+1} \cdot E_{b_\alpha}^{(b_\alpha+1)}$, we have the inequality $J_{b_\alpha} = \tilde{m}_{b_\alpha} - I_{b_\alpha}^{(b_\alpha+1)}$. Let $p = E_{b_\alpha}^{(b_\alpha+2)} \cap E_{b_\alpha+1}^{(b_\alpha+2)}$. From the way of choosing of canonical resolution, we have $(\tilde{B}_{b_\alpha+2} \cdot E_{b_\alpha}^{(b_\alpha+2)})_p = 0$. Hence, the equality $\tilde{m}_{b_\alpha+1} = I_{b_\alpha}^{(b_\alpha+1)}$ necessarily holds. Therefore we have

$$\tilde{m}_{b_\alpha} = J_{b_\alpha} + \tilde{m}_{b_\alpha+1}.$$

If $b_\alpha + 2 = b_{\alpha+1}$, then we have $\tilde{m}_{b_\alpha+1} = J_{b_\alpha+1}$ by the same way as the case of B^{I}, we obtain (5.22). Otherwise, repeating the same discussion as above, we obtain (5.22).

Now we return to the proof of the proposition. If $\tilde{m}_{b_\alpha} = 8p + 1$ $(p \in \boldsymbol{Z})$, then the relations $\delta(\gamma_{b_\alpha}) = 0$ and $N_8(\tilde{m}_{b_\alpha}) \ge 1$ hold. If $\tilde{m}_{b_\alpha} = 8p + q$ $(p \in \boldsymbol{Z}, q \in \{3, 5, 7\})$, the inequality $N_8(\tilde{m}_{b_\alpha}) \ge 3$ holds. In any case, we can easily see that

$$\Gamma^{\mathrm{odd}}(\{b_\alpha\}) \le \frac{N_8(\tilde{m}_{b_\alpha})}{2} - \frac{1}{2}. \tag{5.23}$$

Therefore, if p_{b_α} is a singularity of type B^{I}, then we obtain (5.11) by putting (5.22) into (5.23). If p_{b_α} is a singularity of type B^{II}, then we have

$$\Gamma^{\mathrm{odd}}(\mathscr{C}_\alpha) \le \sum_{j \in \mathscr{C}_\alpha} \frac{\tilde{m}_j}{2} - \frac{1}{2} \tag{5.24}$$

by $\delta(\gamma_j) \leq \tilde{m}_j/2 - 1/2$ for any $j \in \mathscr{C}_\alpha$. From the relation (5.22), we have

$$N_8(\tilde{m}_{b_\alpha}) \leq \sum_{j \in \{b_\alpha\} \cup \mathscr{C}_\alpha} N_8(J_j). \tag{5.25}$$

Hence, from (5.23), (5.24) and (5.25), we obtain (5.12). Q.E.D.

PROOF OF PROPOSITION 5.4. We set $d_\alpha = \min\{j \mid p_j$ is type D, $b_\alpha < j < b_{\alpha+1}\}$. Since $p_{b_\alpha+1}$ is a singularity of type C by the definition of type $\mathrm{B^{III}}$, we have $b_\alpha + 1 < d_\alpha$. We can easily see that

$$\delta(\gamma_k) \leq \frac{\tilde{m}_k}{2} \tag{5.26}$$

for any $k \in \mathscr{C}_\alpha \cup \mathscr{D}_\alpha$. Since $p_{d_\alpha} \in E^{(d_\alpha)}_{d_\alpha-1}$ holds, we have $E^{(d_\alpha)}_{d_\alpha-1} \subset B^{\mathrm{exc}}_{d_\alpha}$, and $\tilde{m}_{d_\alpha-1}$ is necessarily a positive even number. It follows the inequality

$$\delta(\gamma_{d_\alpha-1}) \leq \frac{\tilde{m}_{d_\alpha-1}}{2} - 1. \tag{5.27}$$

From (5.26) and (5.27), we have

$$\Gamma^{\mathrm{odd}}(\{b_\alpha\} \cup \mathscr{C}_\alpha \cup \mathscr{D}_\alpha) \leq \sum_{j \in \mathscr{C}_\alpha \cup \mathscr{D}_\alpha} \frac{\tilde{m}_j}{2}. \tag{5.28}$$

Now, since $B_{d_\alpha-1}$ has no singularity on $E^{(d_\alpha-1)}_{d_\alpha-2}$ except $p_{d_\alpha-1}$.

$$\begin{aligned}
\tilde{m}_{d_\alpha-2} &= E^{(d_\alpha-1)}_{d_\alpha-2} \cdot \tilde{B}_{d_\alpha-1} \\
&= \sum_{p \in \mathrm{Sing}\, B_{d_\alpha-1}} (E^{(d_\alpha-1)}_{d_\alpha-2} \cdot \tilde{B}_{d_\alpha-1})_p + J_{d_\alpha-2} \\
&= I^{(d_\alpha-1)}_{d_\alpha-2} + J_{d_\alpha-2}.
\end{aligned}$$

Hence, by $I^{(d_\alpha-1)}_{d_\alpha-2} \geq \tilde{m}_{d_\alpha-1}$, the inequality $\tilde{m}_{d_\alpha-2} \geq \tilde{m}_{d_\alpha-1} + J_{d_\alpha-2}$ holds. If $\tilde{m}_{d_\alpha-2} = \tilde{m}_{d_\alpha-1} + J_{d_\alpha-2}$, then the inequality $J_{d_\alpha-2} \geq 1$ holds since $\tilde{m}_{d_\alpha-2}$ is an odd number. Hence we obtain

$$0 \leq \frac{N_8(J_{d_\alpha-2})}{2} - \frac{1}{2}. \tag{5.29}$$

From (5.28) and (5.29), we obtain (5.13). If $\tilde{m}_{d_\alpha-2} > \tilde{m}_{d_\alpha-1} + J_{d_\alpha-2}$, then the inequality $I^{(d_\alpha-1)}_{d_\alpha-2} > \tilde{m}_{d_\alpha-1}$ holds. Then, the point $E^{(d_\alpha)}_{d_\alpha-1} \cap E^{(d_\alpha)}_{d_\alpha-2}$ is of type C.

Hence, there exists a number $d_\alpha - 1 < j < b_{\alpha+1}$ such that $p_j = E^{(j)}_{d_\alpha-1} \cap E^{(j)}_{d_\alpha-2}$. We have

$$\delta(\gamma_j) \le \frac{\tilde{m}_j}{2} - \frac{1}{2}. \tag{5.30}$$

From (5.26), (5.27) and (5.30), we have

$$\Gamma^{\mathrm{odd}}(\{b_\alpha\} \cup \mathscr{C}_\alpha \cup \mathscr{D}_\alpha) \le \sum_{j \in \mathscr{C}_\alpha \cup \mathscr{D}_\alpha} \frac{\tilde{m}_j}{2} - \frac{1}{2}. \tag{5.31}$$

Hence, we obtain (5.13).

Next we will discuss the case that p_{b_τ} is a singularity of type $\mathrm{B}^{\mathrm{III}}$. In case of $\tau < \mu$, for any $\tau < \beta \le \mu$, p_{b_β} is a singularity of type B^{I} or of type B^{II}. We have

$$\{j \mid b_\tau \le j\} = \bigcup\nolimits_{\tau \le \beta \le \mu} (\{b_\beta\} \cup \mathscr{C}_\beta \cup \mathscr{D}_\beta). \tag{5.32}$$

From (5.13), Proposition 5.3, we obtain (5.14). In case of $\tau = \mu$, then, by $\sum_{p_{d_\tau - 1} \preceq p_j} J_j \ge 1$, we have

$$\sum_{p_{d_\tau - 1} \preceq p_j} N_8(J_j) \ge 1. \tag{5.33}$$

Hence, from (5.28), (5.29) and (5.33) or from (5.31) and (5.33), we obtain (5.14).

Q.E.D.

LEMMA 5.8. *Let $1 \le i \le r$. We assume that p_i is a singularity with $\tilde{m}_i \le 8$. Then we have the equality*

$$I^{(i)} = \sum_{\substack{p_i \preceq p_j \\ p_j : type\ C\ or\ D}} \tilde{m}_j + \sum_{p_i \preceq p_j} N_8(J_j). \tag{5.34}$$

PROOF. By the same way as the proof of Proposition 5.1, we can easily see that

$$I^{(i)} = \sum_{\substack{p_i \preceq p_j \\ p_j : \text{type C } or \text{ D}}} \tilde{m}_j + \sum_{p_i \preceq p_j} J_j. \tag{5.35}$$

Now, since $\tilde{m}_i \le 8$, we have $J_j \le \tilde{m}_j \le \tilde{m}_i \le 8$ for any $p_j \succeq p_i$. Hence we have the equality $N_8(J_j) = J_j$ for $p_j \succeq p_i$. Therefore, from (5.35), we obtain (5.34).

Q.E.D.

PROOF OF PROPOSITION 5.5. We fix α such that p_{b_α} is a singularity of type B^{IV}. One of the following cases occurs.

(1) $b_\alpha < \exists k < b_{\alpha+1}$ such that $\tilde{m}_k \geq 4$.

(2) $\exists k \in \mathscr{C}_\alpha$ such that $\tilde{m}_k = 2$.

(3) $\exists k \in \mathscr{D}_\alpha$ such that $\tilde{m}_k = 2$, and $\tilde{m}_j \neq 2$ for any $j \in \mathscr{C}_\alpha$.

(4) $\#\{j \mid \tilde{m}_j \text{ is an odd number}, b_\alpha < j < b_{\alpha+1}\} \geq 2$.

(5) $\tilde{m}_{b_\alpha+1}$ is 1 or 3, and $\tilde{m}_j = 0$ for $b_\alpha + 1 < j < b_{\alpha+1}$.

(6) $\tilde{m}_{b_\alpha+1} = 0$.

In fact, if none of (1), (2), (3) and (4) are satisfied, then we have $\tilde{m}_j = 0, 1$ or 3 for any $b_\alpha < j < b_{\alpha+1}$ and $\#\{j \mid \tilde{m}_j \text{ is an odd number}, b_\alpha < j < b_{\alpha+1}\} \leq 1$. This is equivalent that the condition (5) or (6) holds.

We will give an estimate of the left hand side of (5.15) in each case above.

We remark that the inequality

$$\delta(\gamma_j) \leq \frac{\tilde{m}_j}{2} \tag{5.36}$$

holds for $j \in \mathscr{C}_\alpha \cup \mathscr{D}_\alpha$.

In case of (1), we can easily see

$$\delta(\gamma_k) \leq \frac{\tilde{m}_k}{2} - 1. \tag{5.37}$$

From the inequality $\delta(\gamma_{b_\alpha}) \leq 1$, (5.37) and (5.36), we have

$$\Gamma^{\mathrm{odd}}(\{b_\alpha\} \cup \mathscr{C}_\alpha \cup \mathscr{D}_\alpha) \leq \sum_{j \in \mathscr{C}_\alpha \cup \mathscr{D}_\alpha} \frac{\tilde{m}_j}{2}. \tag{5.38}$$

From the inequality $N_8(J_j) \geq 0$ and (5.38), we obtain (5.15).

In case of (2), we have $\delta(\gamma_k) = 0 = \tilde{m}_k/2 - 1$. Therefore, since (5.36) also holds in this case, the same inequality as (5.38) holds. Hence (5.15) follows.

In case of (3), $\gamma_j = 0$ for any $p_k \prec p_j$. Hence we have the equality

$$\Gamma^{\mathrm{odd}}(\{j \mid p_k \preceq p_j\}) = \delta(\gamma_k) = 1 = \tilde{m}_k - 1. \tag{5.39}$$

Now, by $\#\{j \mid E_j^{(k)} \ni p_k\} = 2$, the inequality $2\tilde{m}_k \leq I^{(k)}$ follows. Hence, from (5.39), we have

$$\Gamma^{\mathrm{odd}}(\{j \mid p_k \preceq p_j\}) \leq \frac{1}{2} I^{(k)} - 1. \tag{5.40}$$

Because of Lemma 5.8 and (5.40), we have

$$\Gamma^{\mathrm{odd}}(\{j \mid p_k \preceq p_j\}) \leq \sum_{\substack{p_k \preceq p_j \\ p_j:\text{type C or D}}} \frac{\tilde{m}_j}{2} + \sum_{p_k \preceq p_j} \frac{N_8(J_j)}{2} - 1. \tag{5.41}$$

By (5.36), we have the inequality

$$\Gamma^{\mathrm{odd}}(\{j \mid p_k \npreceq p_j, j \in \mathscr{C}_\alpha \cup \mathscr{D}_\alpha\}) \leq \sum_{\substack{p_k \npreceq p_j \\ j \in \mathscr{C}_\alpha \cup \mathscr{D}_\alpha}} \frac{\tilde{m}_j}{2}. \tag{5.42}$$

Since $p_j \succ p_k$ are singularities of type C and the process of the canonical method has priority D > C > B, we have $\{j \mid p_k \preceq p_j\} \subset \mathscr{C}_\alpha \cup \mathscr{D}_\alpha$. Hence, using the additivity of $\Gamma^{\mathrm{odd}}(*)$, by combing $\Gamma^{\mathrm{odd}}(\{b_\alpha\}) \leq 1$, (5.41) and (5.42), we obtain (5.15).

In case of (4), we can choose distinct numbers $b_\alpha < k_1, k_2 < b_{\alpha+1}$ such that $\tilde{m}_{k_i}$ is an odd number for $i = 1, 2$. We can easily check

$$\delta(\gamma_{k_i}) \leq \frac{\tilde{m}_{k_i}}{2} - \frac{1}{2}$$

for $i = 1, 2$. Hence, by $\delta(\gamma_{b_\alpha}) \leq 1$, we have

$$\Gamma^{\mathrm{odd}}(\{b_\alpha\} \cup \mathscr{C}_\alpha \cup \mathscr{D}_\alpha) \leq \sum_{j \in \mathscr{C}_\alpha \cup \mathscr{D}_\alpha} \frac{\tilde{m}_j}{2}. \tag{5.43}$$

Therefore (5.15) follows.

In case of (5), $m_{b_\alpha+1}$ is odd, and $E_{b_\alpha+1}^{(b_\alpha+2)} \subset B_{b_\alpha+2}^{\mathrm{exc}}$. Let $E_k^{(b_\alpha)}$ be the component of $B_{b_\alpha}^{\mathrm{exc}}$ such that $E_k^{(b_\alpha)} \ni p_{b_\alpha}$. Let $D = \sum_{1 \leq j \leq r} a_i E_i^{(r+1)}$, $a_i \in \boldsymbol{Z}$. Suppose $|D| \subset B_{r+1}^{\mathrm{exc}}$. We have

$$\begin{aligned} D \cdot Y_{b_\alpha}^{(r+1)} &= \sum_{1 \leq i \leq r} a_i (E_i^{(r+1)} \cdot Y_{b_\alpha}^{(r+1)}) \\ &= a_k - a_{b_\alpha}. \end{aligned} \tag{5.44}$$

Since $p_{b_\alpha+1}$ is of type D, $E_{b_\alpha}^{(b_\alpha+1)} \subset B_{b_\alpha+1}^{\mathrm{exc}}$ holds and the point $E_k^{(b_\alpha+1)} \cap E_{b_\alpha}^{(b_\alpha+1)}$ is $p_{b_\alpha+1}$. By the conditions $\tilde{m}_j = 0$ and $b_\alpha + 1 < j < b_{\alpha+1}$, we can see that the remaining singularities of type C or D on $E_{b_\alpha+1}^{(b_\alpha+2)}$ are exactly two points $E_k^{(b_\alpha+2)} \cap E_{b_\alpha+1}^{(b_\alpha+2)}$ and $E_{b_\alpha}^{(b_\alpha+2)} \cap E_{b_\alpha+1}^{(b_\alpha+2)}$. Both are rational singularities of A_1 type, so we may assume that $E_k^{(b_\alpha+2)} \cap E_{b_\alpha+1}^{(b_\alpha+2)} = p_{b_\alpha+2}$ and $E_{b_\alpha}^{(b_\alpha+3)} \cap E_{b_\alpha+1}^{(b_\alpha+3)} = p_{b_\alpha+3}$. Since $|D| \subset B_{r+1}^{\mathrm{exc}}$, we have $a_{b_\alpha+2} = 0$. Hence, we obtain

$$D \cdot Y_{b_\alpha+2}^{(r+1)} = a_k + a_{b_\alpha+1}. \tag{5.45}$$

By the same way as above, we have

$$D \cdot Y_{b_\alpha+3}^{(r+1)} = a_{b_\alpha} + a_{b_\alpha+1}. \tag{5.46}$$

Therefore, we have

$$\sum_{j=0,2,3} D\cdot Y^{(r+1)}_{b_\alpha+j}=2a_k+2a_{b_\alpha+1}. \tag{5.47}$$

Now we have $\gamma_{b_\alpha+2}=\gamma_{b_\alpha+3}=0$. Hence, from (5.47), we have

$$\begin{aligned}\sum_{j=0,2,3}\delta(D\cdot Y^{(r+1)}_{b_\alpha+j}+\gamma_{b_\alpha+j}) &\geq \delta\left(\sum_{j=0,2,3}(D\cdot Y^{(r+1)}_{b_\alpha+j}+\gamma_{b_\alpha+j})\right)\\ &=\delta(\gamma_{b_\alpha}).\end{aligned}$$

Therefore we obtain

$$V^{\min}(\{b_\alpha, b_\alpha+2, b_\alpha+3\})\geq\delta(\gamma_{b_\alpha}). \tag{5.48}$$

From (5.48), we obtain the inequality

$$V^{\min}(\{b_\alpha\}\cup\mathscr{C}_\alpha\cup\mathscr{D}_\alpha)\geq\delta(\gamma_{b_\alpha}). \tag{5.49}$$

Now, by (5.36), we have the inequality

$$\Gamma^{\mathrm{odd}}(\mathscr{C}_\alpha\cup\mathscr{D}_\alpha)\leq\sum_{j\in\mathscr{C}_\alpha\cup\mathscr{D}_\alpha}\frac{\tilde{m}_j}{2}. \tag{5.50}$$

From (5.49) and (5.50), we have

$$\begin{aligned}&\Gamma^{\mathrm{odd}}(\{b_\alpha\}\cup\mathscr{C}_\alpha\cup\mathscr{D}_\alpha)-V^{\min}(\{b_\alpha\}\cup\mathscr{C}_\alpha\cup\mathscr{D}_\alpha)\\ &=(\delta(\gamma_{b_\alpha})-V^{\min}(\{b_\alpha\}\cup\mathscr{C}_\alpha\cup\mathscr{D}_\alpha))+\Gamma^{\mathrm{odd}}(\mathscr{C}_\alpha\cup\mathscr{D}_\alpha)\\ &\leq\sum_{j\in\mathscr{C}_\alpha\cup\mathscr{D}_\alpha}\frac{\tilde{m}_j}{2}.\end{aligned}$$

Hence we obtain (5.15). Q.E.D.

In case of (6), we have $m_{b_\alpha+1}=2$. Let $E^{(b_\alpha)}_k\ni p_{b_\alpha}$. Let $D=\sum_{1\leq i\leq r}a_iE^{(r+1)}_i$ with support in B^{exc}_{r+1}. Then, we have

$$D\cdot Y^{(r+1)}_{b_\alpha}=a_k-a_{b_\alpha}\quad\text{and}\quad D\cdot Y^{(r+1)}_{b_\alpha+1}=a_k+a_{b_\alpha}. \tag{5.51}$$

Hence the equality

$$\sum_{j=0,1}D\cdot Y^{(r+1)}_{b_\alpha+j}=2a_k \tag{5.52}$$

follows. Therefore, by the same way as the case (5), we have the inequality (5.49) and (5.50), and we obtain (5.15). Q.E.D.

PROOF OF PROPOSITION 5.6. We consider the following conditions.

(a) $\exists k \in \mathscr{D}_\tau$ such that $\tilde{m}_k \geq 4$.

(b) $\exists k \in \mathscr{C}_\tau$ such that $\tilde{m}_k = 2$.

(c) $\exists k \in \mathscr{D}_\tau$ such that $\tilde{m}_k = 2$.

(d) $\exists k \in \mathscr{D}_\tau$ such that $\tilde{m}_k = 3$.

(e) $\exists k \in \mathscr{D}_\tau$ such that $\tilde{m}_k = 1$.

One of the following cases occurs.

(1) The condition (a) is satisfied.

(2) The condition (b) is satisfied, and (a) is not.

(3) The conditions (c) and (d) are satisfied, and neither (a) nor (b) are.

(4) The condition (c) is satisfied, and none of (a), (b) and (d) are.

(5) The conditions (d) and (e) are satisfied, and none of (a), (b) and (c) are.

(6) The condition (d) is satisfied, and none of (a), (b), (c) and (e) are.

(7) The condition (e) is satisfied, and none of (a), (b), (c) and (d) are.

(8) None of (a), (b), (c), (d) and (e) are satisfied.

We will give an estimate of the left hand side of (5.16) on each case above.

We will prove the following useful claim to prove the proposition.

CLAIM 5.9. *Let W be a subset of $\{j \mid b_\tau \leq j\}$. We assume that one of the following conditions is satisfied.*

(1) $\mu = \tau$.

(2) $\mu = \tau + 1$ *and* $W \supset \{j \mid b_\mu \leq j\}$.

(3) $\mu = \tau + 1$ *and* $W \subset \mathscr{C}_\tau \cup \mathscr{D}_\tau$.

Furthermore we assume that the inequality

$$\Gamma^{\mathrm{odd}}(W) \leq \sum_{\substack{j \in W \\ p_j : \text{type C or D}}} \frac{\tilde{m}_j}{2} + \sum_{j \in W} \frac{N_8(J_j)}{2} - 2 \tag{5.53}$$

holds. Then we obtain the inequality (5.16).

PROOF. In case that the condition (1) or (2) of the assumption is satisfied, we decompose $\{j \mid b_\tau \leq j\}$ into the direct sum as follows:

$$\{j \mid b_\tau \leq j\} = \{b_\tau\} \cup W \cup W_1,$$

where $W_1 = \{j \mid j \notin W, \mathscr{C}_\tau \cup \mathscr{D}_\tau\}$. We can easily see that

$$\delta(\gamma_j) \leq \frac{\tilde{m}_j}{2}$$

for $j \in \mathscr{C}_\tau \cup \mathscr{D}_\tau$. By the additivity of $\Gamma^{\mathrm{odd}}(*)$, we have

$$\Gamma^{\mathrm{odd}}(\{j \mid b_\tau \leq j\}) = \delta(\gamma_{b_\tau}) + \Gamma^{\mathrm{odd}}(W) + \Gamma^{\mathrm{odd}}(W_1)$$
$$\leq 1 + \Gamma^{\mathrm{odd}}(W) + \sum_{j \in W_1} \frac{\tilde{m}_j}{2}. \tag{5.54}$$

By combing (5.53) and (5.54), we obtain (5.16).

In case that the condition (3) is satisfied, we decompose $\{j \mid b_\tau \leq j\}$ into the direct sum as follows:

$$\{j \mid b_\tau \leq j\} = \{b_\tau\} \cup W \cup W_1 \cup \{j \mid b_\mu \leq j\}.$$

Therefore, since p_{b_μ} is a singularity of type B^{I} or of type B^{II}, from the estimate of Proposition 5.3 and (5.53), by the same way as above, we obtain (5.16).

We return to the proof of the proposition.

In case of (1), if $\tilde{m}_k = 4$ or 5, then the equality $\gamma_k = 2$ holds because p_k is a singularity of type D. Hence $\delta(\gamma_k) = 0$ follows. If $\tilde{m}_k \geq 6$, then the inequality $\delta(\gamma_k) \leq 1 \leq \tilde{m}_k/2 - 2$ holds. In any case, when we put $W = \{k\}$, the assumption of Claim 5.9 is satisfied. Therefore we obtain (5.16).

In case of (2), we define an integer $b_\tau < j_\tau < b_{\tau+1}$ as follows:

$$j_\tau = \max\{j \mid j \in \mathscr{C}_\tau \cup \mathscr{D}_\tau, p_k \preceq p_j, \tilde{m}_j = 2\}.$$

We see that

$$\tilde{m}_j \leq 1 \quad \text{for } p_{j_\tau} \prec p_j.$$

To show the above inequality, we assume that there exists a singularity p_j with $\tilde{m}_j = 2$ such that $p_{j_\tau} \prec p_j$. By the definition of j_τ, p_j is a singularity of type B. By the assumption of Proposition 5.6, $j = b_\mu$ and p_j is of type B^{I}. This contradicts that $\tilde{m}_j = 2$.

If $j_\tau \in \mathscr{C}_\tau$, then, since the inequality $4 = 2\tilde{m}_k \leq I^{(k)}$ holds, we have

$$\Gamma^{\mathrm{odd}}(\{j \mid p_{j_\tau} \preceq p_j\}) = 0 \leq \frac{I^{(k)}}{2} - 2. \tag{5.55}$$

When we put $W = \{j \mid p_{j_\tau} \prec p_j\}$, from Lemma 5.8 and (5.55), the assumption of Claim 5.9 is satisfied, and we obtain (5.16).

If $j_\tau \in \mathscr{D}_\tau$, then we have

$$\Gamma^{\mathrm{odd}}(\{j \mid p_{j_\tau} \preceq p_j\}) = 1 \leq \frac{I^{(k)}}{2} - 1 \tag{5.56}$$

and

$$\delta(\gamma_k) \leq \frac{\tilde{m}_k}{2} - 1. \tag{5.57}$$

Therefore, from (5.56) and (5.57), when we put $W = \{j \mid p_{j_\tau} \preceq p_j\} \cup \{k\}$, the assumption of Claim 5.9 is satisfied. Hence we obtain (5.16).

In case of (3), since the condition (a) is not satisfied and (d) is satisfied, $p_{b_\tau+1}$ is a singularity of type D with $\tilde{m}_{b_\tau+1} = 3$. We put

$$j_\tau = \max\{j \mid j \in \mathscr{C}_\tau \cup \mathscr{D}_\tau, p_{b_\tau+1} \preceq p_j, \tilde{m}_j = 3\}.$$

We can easily see that p_{j_τ} is a singularity of type D. Since the condition (c) is satisfied, $p_{j_\tau+1}$ is necessarily a singularity of type D. Furthermore, since the condition (b) is not satisfied, we have $\tilde{m}_j \le 1$ for any $p_{j_\tau+1} \prec p_j$. Hence we have

$$\Gamma^{\mathrm{odd}}(\{j \mid p_{j_\tau} \preceq p_j\}) = 2. \tag{5.58}$$

Now we have

$$I^{(j_\tau)} = \sum_{p \in (\mathrm{Sing}\, B_{j_\tau+1}) \cap E_{j_\tau}^{(j_\tau+1)}} (E^{(j_\tau+1)} \cdot \tilde{B}_{j_\tau+1})_p + \tilde{m}_{j_\tau}. \tag{5.59}$$

From the relation $p_{j_\tau+1} \in \mathrm{Sing}\, B_{j_\tau+1}$ and $\tilde{m}_{j_\tau+1} = 2$, we have

$$\sum_{p \in (\mathrm{Sing}\, B_{j_\tau+1}) \cap E_{j_\tau}^{(j_\tau+1)}} (E^{(j_\tau+1)} \cdot \tilde{B}_{j_\tau+1})_p \ge 5. \tag{5.60}$$

From (5.59) and (5.60), we obtain

$$I^{(j_\tau)} \ge 8. \tag{5.61}$$

From Lemma 5.8, (5.58) and (5.61), the assumption of Claim 5.9 for $W = \{j \mid p_{j_\tau} \preceq p_j\}$ is satisfied, we obtain (5.16).

In case of (4), since neither (a) nor (d) are satisfied and the condition (c) is satisfied, $p_{b_\tau+1}$ is necessarily a singularity of type D with $\tilde{m}_{b_\tau+1} = 2$. Since the condition (b) is not satisfied, the inequality $\tilde{m}_j \le 1$ holds for any $p_{b_\tau+1} \prec p_j$. Hence the equality

$$\Gamma^{\mathrm{odd}}(\{j \mid p_{b_\tau+1} \preceq p_j\}) = 1 \tag{5.62}$$

follows.

In case of $I_{b_\tau}^{(b_\tau+1)} = 2$, there does not exist a singularity of type B on $E_{b_\tau}^{(b_\tau+1)}$. In fact, if there exists such a singularity on $E_{b_\tau}^{(b_\tau+1)}$, necessarily $\mu = \tau + 1$, and since $I_{b_\tau}^{(b_\mu)}$ is an even number from the equality

$$\begin{aligned} I_{b_\tau}^{(b_\mu)} &= E_{b_\tau}^{(b_\tau+1)} \cdot \tilde{B}_{b_\tau+1} - I_{b_\tau}^{(b_\tau+1)} \\ &= \tilde{m}_{b_\tau} - 2, \end{aligned} \tag{5.63}$$

p_{b_μ} is not of type B^{I}. This contradicts the assumption of the proposition. Hence there exists no singularity of $B_{b_\tau+1}$ on $E_{b_\tau}^{(b_\tau+1)}$ except $p_{b_\tau+1}$. From the equality $\delta(\gamma_{b_\tau}) = 0$ and (5.62), we have

$$\Gamma^{\mathrm{odd}}(\{j \mid p_{b_\tau} \preceq p_j\}) = \Gamma^{\mathrm{odd}}(\{j \mid p_{b_\tau+1} \preceq p_j\}) = 1. \tag{5.64}$$

Because $4 = 2\tilde{m}_{b_\tau+1} \leq I^{(b_\tau+1)}$ and (5.64), we have

$$\Gamma^{\mathrm{odd}}(\{j \mid p_{b_\tau} \preceq p_j\}) \leq \frac{I^{(b_\tau+1)}}{2} - 1. \tag{5.65}$$

From (5.65) and Lemma 5.8, we obtain (5.16).

In case that $I_{b_\tau}^{(b_\tau+1)}$ is an even number and not two, by the same way as above, there exists no singularity of $B_{b_\tau+1}$ on $E_{b_\tau}^{(b_\tau+1)}$ except $p_{b_\tau+1}$. From the inequality $I^{(b_\tau+1)} \geq 6$ and Lemma 5.8, the assumption of Claim 5.9 for $W = \{j \mid p_{b_\tau+1} \preceq p_j\}$ is satisfied. Hence we obtain (5.16).

In case that $I_{b_\tau}^{(b_\tau+1)}$ is an odd number, since $\tilde{m}_{b_\tau}$ is an even number, there exists a singularity of type B of $B_{b_\tau+1}$ on $E_{b_\tau}^{(b_\tau+1)}$. Therefore, by the assumption of the proposition, $\mu = \tau + 1$ and p_{b_μ} is a singularity of type B^{I}. From $I^{(b_\tau+1)} \geq 5$, we have

$$\Gamma^{\mathrm{odd}}(\{j \mid p_{b_\tau+1} \preceq p_j\}) \leq \frac{I^{(b_\tau+1)}}{2} - \frac{3}{2}. \tag{5.66}$$

From (5.66), Lemma 5.8 and (1) of Proposition 5.3, the assumption of Claim 5.9 for $W = \{j \mid p_{b_\tau+1} \preceq p_j, b_\tau < j < b_{\tau+1}\} \cup \{j \mid p_{b_\mu} \preceq p_j\}$ is satisfied. Hence we obtain (5.16).

In case of (5), since the condition of (a) is not satisfied and (d) is satisfied, $p_{b_\tau+1}$ is a singularity of type D with $\tilde{m}_{b_\tau+1} = 3$. We put

$$j_\tau = \max\{j \mid j \in \mathscr{C}_\tau \cup \mathscr{D}_\tau, p_{b_\tau+1} \preceq p_j, \tilde{m}_j = 3\}.$$

Since the condition (c) is not satisfied, there does not exist a singularity p of type D with $\mathrm{mult}_p \tilde{B}_{j_\tau+1} = 2$ on $E_{j_\tau}^{(j_\tau+1)}$. Since the condition (e) is satisfied, $p_{j_\tau+1}$ is a singularity of type D with $\tilde{m}_{j_\tau+1} = 1$. Furthermore there does not exist a singularity p of type B on $E_{j_\tau}^{(j_\tau+1)}$ with $\mathrm{mult}_p \tilde{B}_{j_\tau+1} = 2$. In fact, if there exist such a singularity p, then p is not of type B^{I}. This contradicts the assumption of the proposition. Therefore we have $\tilde{m}_j \leq 1$ for any $p_{j_\tau} \prec p_j$. It follows the equality

$$\Gamma^{\mathrm{odd}}(\{j \mid p_{j_\tau} \preceq p_j\}) = \delta(\gamma_{j_\tau}) = 1. \tag{5.67}$$

From $6 = 2\tilde{m}_{j_\tau} \leq I^{(j_\tau)}$, (5.67) and Lemma 5.8, the assumption of Claim 5.9 for $W = \{j \mid p_{j_\tau} \preceq p_j\}$ is satisfied. Therefore we obtain (5.16).

In case of (6), since the condition of (a) is not satisfied and (d) is satisfied, $p_{b_\tau+1}$ is a singularity of type D with $\tilde{m}_{b_\tau+1} = 3$. We put

$$j_\tau = \max\{j \mid j \in \mathscr{C}_\tau \cup \mathscr{D}_\tau, p_{b_\tau+1} \preceq p_j, \tilde{m}_j = 3\}.$$

Since neither (c) nor (e) are satisfied, there does not exist a singularity p of type D on $E_{j_\tau}^{(j_\tau+1)}$ with $\mathrm{mult}_p \tilde{B}_{j_\tau+1} \geq 1$. Therefore there exists a singularity of type B on $E_{j_\tau}^{(j_\tau+1)}$. By the assumption of the proposition, $\mu = \tau + 1$ and p_{b_μ} is a singularity of type B^{I} with $\tilde{m}_{b_\mu} = 3$.

In case of $j_\tau \geq b_\tau + 3$, we have

$$\delta(\gamma_j) = 1 \leq \frac{\tilde{m}_j}{2} - \frac{1}{2} \tag{5.68}$$

for any $b_\tau + 1 \leq j \leq b_\tau + 3$. From (1) of Proposition 5.3 and (5.68), the assumption of Claim 5.9 on $W = \{j \mid b_\tau + 1 \leq j \leq b_\tau + 3\} \cup \{j \mid b_\mu \leq j\}$ is satisfied. Hence we obtain (5.16).

In case of $j_\tau = b_\tau + 2$, we can easily see that $I_{b_\tau}^{(b_\tau+1)} = 3$ or 6. Now there exists no singularity of $B_{b_\tau+1}$ on $E_{b_\tau}^{(b_\tau+1)}$ except $p_{b_\tau+1}$. Hence the equality $I_{b_\tau}^{(b_\tau+1)} = \tilde{m}_{b_\tau}$ follows. By $E_{b_\tau}^{(b_\tau+1)} \subset B_{b_\tau+1}$, $\tilde{m}_{b_\tau}$ is necessarily an even number. Therefore we have $\tilde{m}_{b_\tau} = 6$. It follows $\delta(\gamma_{b_\tau}) = 0$. Since $\delta(\gamma_j) \leq \tilde{m}_j/2 - 1/2$ for any $b_\tau + 1 \leq j \leq b_\tau + 2$, we have

$$\Gamma^{\mathrm{odd}}(\{b_\tau\} \cup \mathscr{C}_\tau \cup \mathscr{D}_\tau) = \Gamma^{\mathrm{odd}}(\mathscr{C}_\tau \cup \mathscr{D}_\tau) \leq \sum_{j \in \mathscr{C}_\tau \cup \mathscr{D}_\tau} \frac{\tilde{m}_j}{2} - 1. \tag{5.69}$$

From (1) of Proposition 5.3 for $\alpha = \mu$ and (5.69), we have

$$\Gamma^{\mathrm{odd}}(\{j \mid p_{b_\tau} \preceq p_j\}) \leq \sum_{\substack{b_\tau \leq j \\ p_j:\text{type C or D}}} \frac{\tilde{m}_j}{2} + \sum_{b_\tau \leq j} \frac{N_8(J_j)}{2} - \frac{3}{2}.$$

Hence we obtain (5.16).

The case of $j_\tau = b_\tau + 1$ does not occur. If $j_\tau = b_\tau + 1$, then we have $\tilde{m}_{b_\tau} = I_{b_\tau}^{(b_\tau+1)} = 3$. This contradicts that $E_{b_\tau}^{(b_\tau+1)} \subset B_{b_\tau+1}$.

In case of (7), since none of (a), (c) and (d) are satisfied and the condition (e) is satisfied, $p_{b_\tau+1}$ is a singularity of type D with $\tilde{m}_{b_\tau+1} = 1$. Hence we have

$$\Gamma^{\mathrm{odd}}(\{j \mid p_{b_\tau+1} \preceq p_j\}) = 0. \tag{5.70}$$

Now we have $I^{(b_\tau+1)} \geq 2\tilde{m}_{b_\tau+1} = 2$ and $p_{b_\tau+1} \preceq p_{b_\mu}$. If $I^{(b_\tau+1)} = 2$, then, since there exists no singularity of $B_{b_\tau+1}$ on $E_{b_\tau}^{(b_\tau+1)}$ except $p_{b_\tau+1}$, we have $\tilde{m}_{b_\tau} =$

$I_{b_\tau}^{(b_\tau+1)} = 1$. This contradicts that $E_{b_\tau}^{(b_\tau+1)} \subset B_{b_\tau+1}$. Therefore $I^{(b_\tau+1)} \neq 2$. If $I^{(b_\tau+1)} = 3$, then we have $\tilde{m}_{b_\tau} \leq 2$. Hence the equality $\delta(\gamma_{b_\tau}) = 0$ holds. Therefore, from (5.70), we have

$$\Gamma^{\mathrm{odd}}(\{j \mid b_\tau \leq j\}) = 0 = \frac{I^{(b_\tau+1)}}{2} - \frac{3}{2}.$$

If $I^{(b_\tau+1)} \geq 4$, then we have

$$\Gamma^{\mathrm{odd}}(\{j \mid b_\tau \leq j\}) \leq 1 \leq \frac{I^{(b_\tau+1)}}{2} - 1.$$

In any case, from Lemma 5.8, we obtain (5.16).

The case of (8) does not occur. To show that, we assume that the case does. Since none of (a), (c), (d) and (e) are satisfied, $p_{b_\tau+1}$ is a singularity of type D with $\tilde{m}_{b_\tau+1} = 0$. Hence there exists a singularity of type B. By the assumption of the proposition, $\mu = \tau + 1$ and p_{b_μ} is a singularity of type B^{I} on $E_{b_\tau}^{(b_\mu)}$. Therefore $\tilde{m}_{b_\tau} = I_{b_\tau}^{(b_\mu)} = \tilde{m}_{b_\mu}$ is an odd number. This contradicts that $E_{b_\tau}^{(b_\tau+1)} \subset B_{b_\tau+1}$.

Q.E.D.

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